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ON INVERSE PROBLEM FOR SINGULAR STURM-LIOUVILLE OPERATOR WITH DISCONTINUITY CONDITIONS

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ABSTRACT. In this study, properties of spectral characteristic are investigated for singular Sturm-Liouville operators in the case where an eigen parameter not only appears in the differential equation but is also linearly contained in the jump conditions. Also Weyl function for considering operator has been defined and the theorems which related to uniqueness of solution of inverse problem according to Weyl function and two spectra have been proved.
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1. **Introduction**

We consider the boundary value problem L for the equation:

$$(1.1) \qquad \ell(y) := -y'' + \frac{C}{x}y + q(x)y = \lambda y, \ \lambda = k^2$$

on the interval $0 < x < \pi$, where λ is spectral parameter; $C \in \mathbb{R}$.

Let us define the boundary value problem L for the equation 1.1 with the boundary conditions

$$(1.2) \qquad\qquad y(0) = 0, \ y(\pi) = 0$$

and with the jump conditions

(1.3) $\qquad \begin{cases} y\left(d+0\right)=\alpha y\left(d-0\right) \\ y'\left(d+0\right)=\alpha^{-1}y'\left(d-0\right)+2k\beta y\left(d-0\right) \end{cases}$

where $\alpha, \beta \in \mathbb{R}, \alpha \neq 1, \alpha > 0, d \in \left(\dfrac{\pi}{2}, \pi\right), q\left(x\right)$ is a real valued bounded function and $q\left(x\right) \in L_2\left(0, \pi\right).$

The boundary value problems that contain the spectral parameter in boundary conditions linearly were investigated in [8]-[11] . In [4],[6]-[11], [14]-[16] ,[30] an operator-theoretic formulation of the problems (1.1)-(1.3) has been given

The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disiplines ranging from engineering to the geo-sciences. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [25, 23]. After reducing corresponding mathematical model we come to boundary value problem L where $q\left(x\right)$ must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium[18, 27]. Boundary value problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth [3, 19]. Here, the main discontinuity is caused by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behavior of solutions for such nonlinear equations. We also note that inverse problem considered here appear in mathematics for investigating spectral properties of some classes of differential, integro-differential and integral operators.

When $q\left(x\right)$ is a first order singular generalized function, singular Sturm-Liouville operator which has a potential as $q = u'$ by using concept of generalized derivative shuch that $u \in L_2\left(0,1\right)$ has been defined in [28, 29]. Also the equation

(1.4) $\qquad \ell_a\left(y\right) := -y''\left(x\right) + \dfrac{C}{x^a}y\left(x\right) + q\left(x\right)y\left(x\right),\ 0 < x < \pi$

was considered, where C is a real number, $q(x)$ is a real valued bounded function. The self-adjoint extensions of differential operators generated by the differential expression $\ell_a(y)$ which has a potential $q = u'$ such that $u \in L_2(0, 1)$ was studied. When $a \neq 2, 4, 6, \ldots$ the generalized functions are in correspondence with to the functions $|x|^{-a} sgnx$ by using the method of canonical regularization [10]. When $a < 3/2$, the generalized functions which are so obtained can be shown as generalized derivative of functions from the space L_2. Therefore the Sturm-Liouville operator which is given by the differential equation $\ell_a(y)$ is defined such that it has a potential like $q(x) = |x|^{-a} sgnx$. In particular, it has shown in [12] that if $q(x)$ is known a priori on $[0, \pi/2]$ then $q(x)$ is uniquely determined on $[\pi/2, \pi]$ by the eigenvalues.

In this study, the case of $a = 1$ has been investigated. Then $u(x) = C \ln x$ and $(\Gamma y)(x) = y'(x) - u(x)y(x)$. The jump conditions (1.3) are different from [2] and [31]. Because the eigen parameter appears not only in the differential equations, but it also appears in the jump conditions. In section 3, properties of characteristic function of L_0 and asymptotic behaviors of spectral characteristics of considering operator have been given such that the remaining parts are in the space ℓ_2 as in [2].

In section 4, Weyl function for considering operator has been defined and the theorem which is related to uniqueness of solution of inverse problem according to Weyl function has been proved.

In section 5, it has been proved that the system of the eigenfunctions of the boundary value problem L is complete and forms an orthogonal basis in $L_2(0, \pi)$.

2. Representation for the solution

We define $y_1(x) = y(x), y_2(x) = (\Gamma y)(x) = y'(x) - u(x)y(x), u(x) = C \ln x$ and let us write the expression of left hand side of equation (1.1) as follows:

$$(2.1) \quad \ell(y) = -[(\Gamma y)(x)]' - u(x)(\Gamma y)(x) - u^2(x)y + q(x)y = k^2 y$$

then equation (1.1) reduces to the system;

$$(2.2) \qquad \begin{cases} y_1' - y_2 = u(x)y_1 \\ y_2' + k^2 y_1 = -u(x)y_2 - u^2(x)y_1 + q(x)y_1 \end{cases}$$

with the boundary conditions

$$(2.3) \qquad\qquad\qquad y_1(0) = 0, y_1(\pi) = 0$$

and with the jump conditions

$$(2.4) \quad \begin{cases} y_1\,(d+0) = \alpha y_1\,(d-0) \\ y_2\,(d+0) = \alpha^{-1}y_2\,(d-0) + 2\,[k\beta - \alpha^{-}u\,(d)]\,y_1\,(d-0)\,. \end{cases}$$

Matrix form of system (2.2) is

$$(2.5) \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -k^2 - u^2 + q & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or $y' = Ay$ such that $A = \begin{pmatrix} u\,(x) & 1 \\ -k^2 - u^2\,(x) + q\,(x) & -u\,(x) \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$

$x = 0$ is a regular singular end point for equation (2.5) and Theorem 2 in [26] (see Remark 1-2, p.56) extends to interval $[0, \pi]$.

Now let us consider the following theorem in [28].

Theorem 2.1 ([28]). *Let $A\,(x)$ be $n \times n$ matrix whose entires are functions belonging to the space $L_1\,(0,1)$, and let $f \in [L_1\,(0,1)]^n$ be a vector-function. Then for every $c \in [0,1]$ the equation*

$$y' = A\,(x)\,y + f, \quad y\,(c) = \xi, \xi \in \mathbb{C}^n,$$

has a unique solution $y\,(x)$ and $y'\,(x)$ is absolutely continuous on $[0,1]$. If a sequence of matrices $A_\epsilon\,(x)$ with entires from the space $L_1\,(0,1)$, is such that $\|A_\epsilon\,(x) - A\,(x)\|_{L_1} \to 0$ as $\epsilon \to 0$, then the solutions of the equations

$$y_\epsilon' = A_\epsilon\,(x)\,y_\epsilon + f, \quad y_\epsilon\,(c) = \xi \in \mathbb{C}^n,$$

converge uniformly on $[0,1]$ (and even in the form of the space $W_1^1\,[0,1]$) to the function $y\,(x)$. Moreover, the estimate

$$|y\,(x) - y_\epsilon\,(x)| \leq C\,\|f\|_{L_1}\,\|A_\epsilon\,(x) - A\,(x)\|_{L_1}$$

holds, where the constant C does not depend on f and ϵ.

If we apply this theorem to our equation, there exists only one solution of the system (2.2) which satisfies the initial conditions $y_1\,(\xi) = v_1, y_2\,(\xi) = v_2$ for each $\xi \in [0, \pi]$, $v = (v_1, v_2)^T \in C^2$, especially the initial conditions $y_1\,(0) = 1, y_2\,(0) = ik$.

Definition 2.2. *The first component of the solution of system (2.2) which satisfies the initial conditions $y_1\,(\xi) = v_1, y_2\,(\xi) = (\Gamma y)\,(\xi) = v_2$ is called the solution of equation (1.1) which satisfies the same initial conditions.*

It was shown in [1] by the successive approximations method that (see [24]) the following theorem is true.

Theorem 2.3 ([1])**.** *For each solution of system (2.2) which satisfies the initial conditions* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ ik \end{pmatrix}$ *and the jump conditions (2.4) the following expression is true: for* $x < d$

$$\begin{cases} y_1 = e^{ikx} + \int_{-x}^{x} K_{11}(x,t) e^{ikt} dt \\ y_2 = ike^{ikx} + b(x) e^{ikx} + \int_{-x}^{x} K_{21}(x,t) e^{ikt} dt + ik \int_{-x}^{x} K_{22}(x,t) e^{ikt} dt, \end{cases}$$

for $x > d$

$$\begin{cases} y_1 = (\alpha^+ + \beta) e^{ikx} + (\alpha^- - \beta) e^{ik(2d-x)} + \int -x^x K_{11}(x,t) e^{ikt} dt \\ y_2 = ik \left(\alpha^+ e^{ikx} - \alpha^- e^{ik(2d-x)} \right) + ik\beta \left(e^{ikx} + \alpha^- e^{ik(2d-x)} \right) \\ \qquad + b(x) \left[(\alpha^+ + \beta) e^{ikx} + (\alpha^- - \beta) e^{ik(2d-x)} \right] \\ \qquad + \int_{-x}^{x} K_{21}(x,t) e^{ikt} dt + ik \int_{-x}^{x} K_{22}(x,t) e^{ikt} dt \end{cases}$$

where

$$b(x) = -\frac{1}{2} \int_0^x \left[u^2(s) - q(s) \right] e^{-\frac{1}{2} \int_s^x u(t) dt} ds$$

$$K_{11}(x,x) = \frac{(\alpha^+ + \beta)}{2} u(x)$$

$$K_{21}(x,x) = b'(x) - \frac{1}{2} \int_0^x \left[u^2(s) - q(s) \right] K_{11}(s,s) ds - \frac{1}{2} \int_0^x u(s) K_{21}(s,s) ds$$

$$K_{22}(x,x) = -\frac{(\alpha^+ + \beta)}{2} \left[u(x) + 2b(x) \right]$$

$$K_{11}(x, 2d - x + 0) - K_{11}(x, 2d - x - 0) = \frac{(\alpha^- - \beta)}{2} u(x)$$

$$\frac{\partial K_{ij}(x,.)}{\partial x}, \frac{\partial K_{ij}(x,.)}{\partial t} \in L_2(0, \pi), i, j = 1, 2.$$

3. Properties of the spectrum

Let us denote problem L as L_0 in the case of $C = 0$ and $q(x) \equiv 0$. It is easily shown that when $C = 0$ and $q(x) \equiv 0$, the solution $\varphi_0(x,k)$ satisfying the initial conditions $\varphi_0(0,k) = 0, (\Gamma\varphi_0)(0,k) = k$ and the jump conditions (2.4) is shown as

$$\begin{aligned} \varphi_0(x,k) &= \frac{y_0(x,k) - \overline{y_0(x,k)}}{2i} \\ &= \begin{cases} \sin kx &, \text{ for } x < d \\ (\alpha^+ - i\beta) \sin kx + (\alpha^- + i\beta) \sin k(2d - x) &, \text{ for } x > d \end{cases} \end{aligned}$$

Let $\Delta_0(k)$ be a characteristic function of problem L_0. Then the characteristic equation of problem L_0 is of the form

$$(3.1) \qquad \Delta_0(k) = \left(\alpha^+ - i\beta\right)\sin k\pi + \left(\alpha^- + i\beta\right)\sin k(2d - \pi) = 0.$$

We denote characteristic function, eigenvalues sequence and normalizing constant sequence by $\Delta(k), \{k_n\}$ and $\{\alpha_n\}$, respectively. Denote

$$(3.2) \qquad \Delta(k) = \langle \psi(x,k), \varphi(x,k) \rangle,$$

where

$$\langle y(x), z(x) \rangle := y(x)(\Gamma z)(x) - (\Gamma y)(x) z(x).$$

According to the Liouville formula, $\langle \psi(x,k), \varphi(x,k) \rangle$ does not depend on x.

Definition 3.1. *The functions $y(x,\lambda), z(x,\mu) \in D(L)$ are called orthogonal, if following equality is valid:*

$$\int_0^x y(x,\lambda)\overline{z(x,\mu)}dx + \frac{2\alpha\beta}{(\lambda+\mu)}y(d-0,\lambda)\overline{z(d-0,\mu)} = 0.$$

Definition 3.2. *For $y(x,\lambda) \in D(L)$, the norming constants α_n are defined as follows:*

$$(3.3) \qquad \alpha_n = \int_0^x y^2(x,\lambda_n)\,dx + \frac{\alpha\beta}{(\lambda+\mu)}y^2(d-0,\lambda_n).$$

We shall assume that $\varphi(x,k)$, $C(x,k)$ and $\psi(x,k)$ are solutions of equation (1.1) under the following initial conditions:

$$\varphi(0,k) = 0, (\Gamma\varphi)(0,k) = k, \psi(\pi,k) = 0, (\Gamma\psi)(\pi,k) = -1,$$
$$C(0,k) = 1, (\Gamma C)(0,k) = 0$$

Clearly, for each x, $\langle \psi(x,k), \varphi(x,k) \rangle$ is entire in k and

$$(3.4) \qquad \Delta(k) = V(\varphi) = U(\psi) = \varphi(\pi,k) = \psi(0,k).$$

By using the representation of the function $y(x,k)$ for the solution $\varphi(x,k)$:

$$\varphi(x,k) = \varphi_0(x,k) + \int_0^x \widetilde{K}_{11}(x,t)\sin kt\,dt$$

is obtained. Therefore characteristic function of the problem L is obtained as

$$(3.5) \qquad \Delta(k) = \Delta_0(k) + \int_0^x \widetilde{K}_{11}(x,t)\sin kt\,dt$$

where $\widetilde{K}_{11}(x,t) = K_{11}(x,t) - K_{11}(x,-t)$.

Lemma 3.3.   *(Lagrange Fomula) Let $y, z \in D(L_0^*)$. Then*

$$\left(L_0^* y, z\right) = \int_0^\pi \ell(y)\, \overline{z}\, dx = \left(y, L_0^* z\right) + [y, \overline{z}]\left(\big|_0^{d-0} + \big|_{d+0}^\pi\right)$$

where
$$[y, z]\left(\big|_0^{d-0} + \big|_{d+0}^\pi\right) = \left[(\Gamma\overline{z})(x)\, y(x) - (\Gamma y)(x)\, \overline{z(x)}\right]\left(\big|_0^{d-0} + \big|_{d+0}^\pi\right).$$

Proof. We have
$$\left(L_0^* y, z\right) = -\int_0^\pi (y' - u\, y)'\, \overline{z}\, dx - \int_0^\pi u\, (y' - u\, y)\, \overline{z}\, dx - \int_0^\pi \left(u^2 - q(x)\right) y\overline{z}\, dx$$
$$= \int_0^\pi (y' - u\, y)(\overline{z}' - u\overline{z})\, dx - \int_0^\pi \left(u^2 - q(x)\right) y\overline{z}\, dx$$
$$- (\Gamma y)(x)\, \overline{z(x)}\left(\big|_0^{d-0} + \big|_{d+0}^\pi\right)$$
$$= \int_0^\pi y\ell(\overline{z})\, dx + [y, \overline{z}]\left(\big|_0^{d-0} + \big|_{d+0}^\pi\right) = \left(y, L_0^* z\right) + [y, \overline{z}]\left(\big|_0^{d-0} + \big|_{d+0}^\pi\right).$$
$$\square$$

Lemma 3.4. *The zeros $\{k_n\}$ of the characteristic function coincide with the eigenvalues of the boundary value problem L. The functions $\varphi(x, k_n)$ and $\psi(x, k_n)$ are eigenfunctions and there exists a sequence $\{\gamma_n\}$ such that*

(3.6) $\psi(x, k_n) = \gamma_n \varphi(x, k_n), \ \gamma_n \neq 0.$

Proof. 1) Let k_0 be a zero of the function $\Delta(k)$. Then by virtue of equation (3.2) and (3.3), $\psi(x, k_0) = \gamma_0 \varphi(x, k_0)$ and the functions $\psi(x, k_0)$, $\varphi(x, k_0)$ satisfy the boundary condition (1.2). Hence, k_0 is an eigenvalue and $\psi(x, k_0)$, $\varphi(x, k_0)$ are eigenfunctions related to k_0.

2) Let k_0 be an eigenvalue of L, and let y_0 be a corresponding eigenfunctions. Then $U(y_0) = V(y_0) = 0$. Clearly $y_0(0) = 0$. Without loss of generality we put $(\Gamma y_0)(0) = ik$. Hence $y_0(x) \equiv \varphi(x, k_0)$. Thus, from equation (3.3), $\Delta(k_0) = V(\varphi(x, k_0)) = V(y_0(x)) = 0$ is obtained. $\square$

Lemma 3.5. *The roots of characteristic equation $\Delta_0(k) = 0$ are separate i.e., $\inf\limits_{n \neq m} \left|k_n^0 - k_m^0\right| = a > 0.$*

Proof.   Let us assume that sequence $\left\{k_n^0\right\}$ has two subsequences $\left\{k_{n_p}^0\right\}$ and $\left\{\widetilde{k}_{n_p}^0\right\}$ such that $k_{n_p}^0 \neq \widetilde{k}_{n_p}^0$, $k_{n_p}^0$ and $\widetilde{k}_{n_p}^0 \to \infty$ as $p \to \infty$ and $\lim\limits_{p \to \infty} \left|k_{n_p}^0 - \widetilde{k}_{n_p}^0\right| = 0.$

If we use orthogonality of eigenfunctions $\varphi_0\left(x, k_{n_p}^0\right)$ and $\varphi_0\left(x, \widetilde{k}_{n_p}^0\right)$ of problem L_0 in space $L_2(0, \pi)$

$$
\begin{aligned}
0 \; = \; & \int_0^\pi \varphi_0\left(x,k_{n_p}^0\right)\overline{\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)}dx + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\varphi_0\left(d-0,\widetilde{k}_{n_p}^0\right)} \\
= \; & \int_0^\pi \varphi_0\left(x,k_{n_p}^0\right)\overline{\varphi_0\left(x,k_{n_p}^0\right)}dx + \int_0^\pi \varphi_0\left(x,k_{n_p}^0\right)\overline{\left[\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(x,k_{n_p}^0\right)\right]}dx \\
& + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\varphi_0\left(d-0,k_{n_p}^0\right)} \\
& + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\left[\varphi_0\left(d-0,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(d-0,k_{n_p}^0\right)\right]} \\
\geq \; & \int_0^d \varphi_0\left(x,k_{n_p}^0\right)\overline{\varphi_0\left(x,k_{n_p}^0\right)}dx + \int_0^\pi \varphi_0\left(x,k_{n_p}^0\right)\overline{\left[\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(x,k_{n_p}^0\right)\right]}dx \\
& + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\varphi_0\left(d-0,k_{n_p}^0\right)} \\
& + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\left[\varphi_0\left(d-0,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(d-0,k_{n_p}^0\right)\right]} \\
= \; & \int_0^d \sin^2 k_{n_p}^0 x\,dx + \int_0^\pi \varphi_0\left(x,k_{n_p}^0\right)\overline{\left[\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(x,k_{n_p}^0\right)\right]}dx \\
& + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\varphi_0\left(d-0,k_{n_p}^0\right)} \\
& + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\varphi_0\left(d-0,k_{n_p}^0\right)\overline{\left[\varphi_0\left(d-0,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(d-0,k_{n_p}^0\right)\right]} \\
= \; & \frac{d}{2} - \frac{\sin 2k_{n_p}^0 d}{2k_{n_p}^0} + \frac{2\alpha\beta}{\left(k_{n_p}^0+\widetilde{k}_{n_p}^0\right)}\sin^2 k_{n_p}^0 d
\end{aligned}
$$

(3.7)
$$
+ \int_0^\pi \varphi_0\left(x,k_{n_p}^0\right)\overline{\left[\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(x,k_{n_p}^0\right)\right]}dx
$$

From the representation of function $\varphi_0\left(x,k\right)$, we get that

$$
p\to\infty\lim\left|\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(x,k_{n_p}^0\right)\right| = 0,
$$

i.e., as $p\to\infty$, $\left|\varphi_0\left(x,\widetilde{k}_{n_p}^0\right)-\varphi_0\left(x,k_{n_p}^0\right)\right|$ uniformly converges to zero with respect to x in the interval $[0,\pi]$. For this reason, if we pass through the limit as $p\to\infty$ then inequality $\dfrac{d}{2}\leq 0$ is obtained.

This contradiction gives the proof of Lemma 3.5. $\qquad\square$

Lemma 3.6. *Eigenvalues of problem L are simple that is* $\dot{\Delta}\left(k_n\right)\neq 0$.

Proof. Since $\varphi(x, k)$ and $\psi(x, k)$ are solutions of equation (1.1)

$$-\psi''(x, k) + [u'(x) + q(x)]\psi(x, k) = k\psi(x, k)$$
$$-\varphi''(x, k_n) + [u'(x) + q(x)]\varphi(x, k_n) = k_n\varphi(x, k_n).$$

If the first equation is multiplied by $\varphi(x, k_n)$, the second equation is multiplied by $\psi(x, k)$ and subtracting them side by side and finally integrating over the interval $[0, \pi]$, the equalities

$$\frac{d}{dx}\langle\psi(x, k), \varphi(x, k_n)\rangle = (k - k_n)\psi(x, k)\varphi(x, k_n)$$

$$\langle\psi(x, k), \varphi(x, k_n)\rangle\left[|_0^{d-0} + |_{d+0}^{\pi}\right] = (k - k_n)\int_0^{\pi}\psi(x, k)\varphi(x, k_n)\, dx$$

are obtained.

If jump conditions (1.3) and (3.3), (3.6), are satisfied then

$$\int_0^{\pi}\psi(x, k_n)\varphi(x, k_n)\, dx = -\dot{\Delta}(k_n) \ \text{ as } \ k \to k_n \ \text{ is obtained.}$$

From Lemma 3.4, we get that

$$(3.8) \qquad\qquad\qquad \alpha_n\gamma_n = -\dot{\Delta}(k_n).$$

It is obvious that $\dot{\Delta}(k_n) \neq 0$. So the lemma is proved. $\qquad\square$

Now, consider the problems

$$L : \begin{cases} -y'' + [u'(x) + q(x)]y = \lambda y, \\ (\Gamma y)(0) - hy(0) = 0 \\ (\Gamma y)(\pi) + Hy(\pi) = 0 \\ y(d+0) = \alpha y(d-0) \\ (\Gamma y)(d+0) = \alpha^{-1}(\Gamma y)(d-0) + 2\left[\sqrt{\lambda}\beta - \alpha^- u(d)\right]y(d-0) \end{cases}$$

and

$$\widetilde{L} : \begin{cases} -y'' + [u'(x) + q(x)]y = \mu y, \\ (\Gamma y)(0) - hy(0) = 0 \\ (\Gamma y)(\pi) + \widetilde{H}y(\pi) = 0 \\ y(d+0) = \alpha y(d-0) \\ (\Gamma y)(d+0) = \alpha^{-1}(\Gamma y)(d-0) + 2\left[\sqrt{\mu}\beta - \alpha^- u(d)\right]y(d-0) \end{cases}$$

where $H \neq \widetilde{H}$. Let $\{\lambda_n\}_{n\geq 0}$ and $\{\mu_n\}_{n\geq 0}$ be eigenvalues of the problems $L(q(x), h, H)$ and $\widetilde{L}\left(q(x), h, \widetilde{H}\right)$.

Lemma 3.7. *The eigenvalues of the problems* L *and* $\widetilde{L}$ *are interlace, i.e.,*

(3.9)

$$\lambda_n < \mu_n < \lambda_{n+1}, \ if \ \widetilde{H} > H \ and \ \mu_n < \lambda_n < \mu_{n+1} \ , if \ H > \widetilde{H}, \ n \geq 0$$

Proof. As in the proof of Lemma 3.6, we get that

$$\frac{d}{dx} \langle \varphi(x,\lambda), \varphi(x,\mu) \rangle = (\lambda - \mu) \varphi(x,\lambda) \varphi(x,\mu)$$

and so

$$
\begin{aligned}
(\lambda - \mu) \int_0^\pi \varphi(x,\lambda) \varphi(x,\mu) \, dx &= \langle \varphi(x,\lambda), \varphi(x,\mu) \rangle \left[|_0^{d-0} + |_{d+0}^\pi \right] \\
&= \varphi(\pi,\lambda)(\Gamma\varphi)(\pi,\mu) - (\Gamma\varphi)(\pi,\lambda)\varphi(\pi,\mu) \\
&\quad + 2\alpha\beta \left(\sqrt{\lambda} - \sqrt{\mu} \right) \varphi(d-0,\lambda)\varphi(d-0,\mu) \\
&= \frac{1}{\widetilde{H} - H} \left[\widetilde{\Delta}(\lambda)\Delta(\mu) - \widetilde{\Delta}(\mu)\Delta(\lambda) \right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\lambda - \mu) \int_0^\pi & \varphi(x,\lambda) \varphi(x,\mu) \, dx \\
&= \frac{1}{\widetilde{H} - H} \left[\frac{\widetilde{\Delta}(\lambda) - \widetilde{\Delta}(\mu)}{\lambda - \mu} \Delta(\mu) - \frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} \widetilde{\Delta}(\mu) \right] \\
&\quad + 2\alpha\beta \left(\sqrt{\lambda} - \sqrt{\mu} \right) \varphi(d-0,\lambda)\varphi(d-0,\mu)
\end{aligned}
$$

As $\mu \to \lambda$

(3.10)

$$\int_0^\pi \varphi^2(x,\lambda) \, dx + \frac{\alpha\beta}{\sqrt{\lambda}} \varphi^2(d-0,\lambda) = \frac{1}{\widetilde{H} - H} \left[\dot{\widetilde{\Delta}}(\lambda)\Delta(\lambda) - \dot{\Delta}(\lambda)\widetilde{\Delta}(\lambda) \right]$$

where $\dot{\Delta}(\lambda) = \dfrac{d}{d\lambda}\Delta(\lambda)$, $\dot{\widetilde{\Delta}}(\lambda) = \dfrac{d}{d\lambda}\widetilde{\Delta}(\lambda)$. From equation (3.10) $-\infty < \lambda < \infty$, if $\widetilde{\Delta}(\lambda) \neq 0$

$$\frac{1}{\widetilde{\Delta}^2(\lambda)} \left[\int_0^\pi \varphi^2(x,\lambda) \, dx + \frac{\alpha\omega}{\sqrt{\lambda}} \varphi^2(d-0,\lambda) \right] = -\frac{1}{\widetilde{H} - H} \ \frac{d}{d\lambda} \left(\frac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)} \right)$$

is obtained.

If $\widetilde{H} > H$ then $\dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}$ is monotonically decreasing in the set of $R \setminus$

$\{\mu_n, n \geq 0\}$ Thus it is obvious that $\lim\limits_{\lambda \to \mu_n^{\pm 0}} \dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)} = \pm\infty$.

When $H > \widetilde{H}$ if we write the equality (3.10) as

$$\frac{1}{\Delta^2(\lambda)} \left[\int_0^\pi \varphi^2(x,\lambda)\, dx + \frac{\alpha\beta}{\sqrt{\lambda}} \varphi^2(d-0,\lambda) \right] = -\frac{1}{H - \widetilde{H}} \frac{d}{d\lambda} \left(\frac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)} \right),$$

$-\infty < \lambda < \infty, \Delta(\lambda) \neq 0$, we get that the function $\dfrac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)}$ is monotoni-

cally decreasing in $R \setminus \{\lambda_n, n \geq 0\}$ and it is clear that $\lim\limits_{\lambda \to \lambda_n^{\pm 0}} \dfrac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)} = \pm\infty$

From here we obtain (3.9). $\square$

Lemma 3.8. *The eigenvalues of problem L have the following asymptotic behavior*

(3.11) $$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{\delta_n}{k_n^0}$$

where $\delta_n \in \ell_2$ and

$$d_n = \frac{(\alpha^+ + \beta)\cos\left(k_n^0 + \varepsilon_n\right)\pi - (\alpha^- - \beta)\cos\left(k_n^0 + \varepsilon_n\right)(2d - \pi)}{2\dot{\Delta}_0\left(k_n^0\right)} u(\pi)$$

is a bounded sequence.

Proof. Denote

$$G_n = \left\{ k : |k| = \left|k_n^0\right| + \frac{\sigma}{2}, \ n = 0, \pm 1, \pm 2, ... \right\}$$
$$G_\delta = \left\{ k : \left|k - k_n^0\right| \geq \delta, \ n = 0, \pm 1, \pm 2, .., \delta > 0. \right\}$$

where δ is sufficiently small positive number $\left(\delta \ll \dfrac{\sigma}{2}\right)$.

As shown in [5] that for $k \in \overline{G_\delta}$, $|\Delta_0(k)| \geq C_\delta e^{|Imk|\pi}$.
On the other hand [[24], Lemma 1.3.1], since

$$\lim\limits_{|k| \to +\infty} e^{-|Imk|\pi}(\Delta(k) - \Delta_0(k)) = \lim\limits_{|k| \to +\infty} \int_0^\pi \widetilde{K}_{11}(\pi, t)\sin kt\, dt = 0,$$

for sufficiently large values of n and $k \in G_n$, we get

$$|\Delta(k) - \Delta_0(k)| < \frac{C_\delta}{2} e^{|Imk|\pi}$$

Thus, for $k \in G_n$,

$$|\Delta_0(k)| > C_\delta e^{|Imk|\pi} > \frac{C_\delta}{2} e^{|Imk|\pi} > |\Delta(k) - \Delta_0(k)|.$$

such that n is a sufficiently large natural number.

It follows from that for sufficiently large values of n, functions $\Delta_0(k)$ and $\Delta_0(k) + (\Delta(k) - \Delta_0(k)) = \Delta(k)$ have the same number of zeros counting multiplicities inside the contour G_n, according to Rouche's theorem. That is, they have the $(n+1)$ number of zeros: $k_0, k_1, ..., k_n$.

Analogously, it is shown by Rouche's theorem that for sufficiently large values of n, $\Delta(k)$ has a unique zero inside each circle $\left|k - k_n^0\right| < \delta$.

Since δ is a sufficiently small number, the presentation $k_n = k_n^0 + \varepsilon_n$ is obtained where $\lim\limits_{n \to \infty} \varepsilon_n = 0$.

Since k_n's are zeros of the characteristic function $\Delta(k)$,

$$\Delta(k_n) = \Delta_0\left(k_n^0 + \varepsilon_n\right) + \int_0^\pi \widetilde{K}_{11}(\pi, t) \sin\left(k_n^0 + \varepsilon_n\right) t\, dt = 0.$$

On the other hand,

$$\Delta_0(k) = \alpha^+ \sin k\pi + \alpha^- \sin k(2d - \pi) = 0,$$

$$\Delta_0\left(k_n^0 + \varepsilon_n\right) = \dot{\Delta}_0\left(k_n^0\right)\varepsilon_n + o(\varepsilon_n).$$

In that case the equality

$$(3.12) \qquad \left(\dot{\Delta}_0\left(k_n^0\right) + o(1)\right)\varepsilon_n + \int_0^\pi \widetilde{K}_{11}(\pi, t) \sin\left(k_n^0 + \varepsilon_n\right) t\, dt = 0$$

is obtained.

Since $\Delta_0(k)$ is of type "sine" [[20] ,p. 119], the number $\gamma_\delta > 0$ exists such that for all n, $\left|\dot{\Delta}_0\left(k_n^0\right)\right| \geq \gamma_\delta > 0$.

if we use conclusions on [33] (see also [17]) is used then we get that $k_n^0 = n + h_n$ where $\sup\limits_n |h_n| < M$.

Hence, when we apply certain methods [24, p. 67] in equality (3.11), we get that $\varepsilon_n \in \ell_2$. If we use the expression of ε_n and Theorem 2.3, then

$$\varepsilon_n = \frac{\left(\alpha^+ + \beta\right)\cos\left(k_n^0 + \varepsilon_n\right)\pi - \left(\alpha^- - \beta\right)\cos\left(k_n^0 + \varepsilon_n\right)(2d - \pi)}{2\dot{\Delta}_0\left(k_n^0\right)k_n^0} u(\pi) + \frac{\widetilde{\delta}_n}{k_n^0}, \quad \widetilde{\delta}_n \in \ell_2.$$

So for the eigenvalues k_n of the problem L, asymptotic formula (3.11) is true. Therefore the lemma is proved. $\qquad \square$

Lemma 3.9. *The normalized numbers of problem L have the asymptotic behaviour $\alpha_n = \alpha_n^0 + \delta_n$ where $\delta_n \in \ell_2$.*

Proof. Since

$$\Delta(k) = \Delta_0(k) + \int_0^\pi \widetilde{K}_{11}(\pi, t) \sin kt\, dt$$

$$\dot{\Delta}(k_n) = \dot{\Delta}_0(k_n) + \int_0^\pi t\widetilde{K}_{11}(\pi, t) \cos k_n t\, dt$$

it is clear that

$$\dot{\Delta}_0(k_n) = \dot{\Delta}_0(k_n^0 + \varepsilon_n) = \dot{\Delta}_0(k_n^0) + O(\varepsilon_n), \quad \cos k_n t = \cos k_n^0 t + O(\varepsilon_n t),$$

where $\varepsilon_n \in \ell_2$. Then,

$\alpha_n \gamma_n = -\dot{\Delta}(k_n) = -\dot{\Delta}_0(k_n^0) - \int_0^\pi t\widetilde{K}_{11}(\pi, t) \cos k_n^0 t\, dt$
$-O(\varepsilon_n t) \int_0^\pi t\widetilde{K}_{11}(\pi, t) \cos k_n^0 t\, dt + O(\varepsilon_n)$

Since $\varepsilon_n \in \ell_2$, $\widetilde{K}_{11}(\pi, .) \in L_2(0, \pi)$ and $k_n^0 = n + h_n$,

$$\delta_n = -\int_0^\pi t\widetilde{K}_{11}(\pi, t) \cos k_n^0 t\, dt - O(\varepsilon_n t) \int_0^\pi t\widetilde{K}_{11}(\pi, t) \cos k_n^0 t\, dt + O(\varepsilon_n) \in \ell_2,$$

$$\alpha_n \gamma_n = \alpha_n^0 \gamma_n^0 + \delta_n \text{ where } \delta_n \in \ell_2.$$

$\square$

4. Inverse problem

Let $\Phi(x, k) = \begin{pmatrix} \Phi_1(x, k) \\ \Phi_2(x, k) \end{pmatrix}$ be solution of (2.2) under the conditions $U(\Phi) = \Phi_1(0, k) = 1$ and $V(\Phi) = \Phi_1(\pi, k) = 0$ and under the jump conditions (2.4). We set $M(k) := \Phi_2(0, k)$. The functions $\Phi(x, k)$ and $M(k)$ are respectively called the Weyl solution and the Weyl function for the boundary value problem L.

Denote

$$(4.1) \qquad\qquad M(k) = \frac{\delta(k)}{\Delta(k)}$$

where $\delta(k) = \psi_2(0, k)$. Clearly,

$$(4.2) \qquad\qquad \Phi(x, k) = M(k)\varphi(x, k) + C(x, k)$$

Weyl solution and the Weyl function are meromorphic functions of a parameter k with poles on the spectrum of the problem L.

It follows from (4.1) and (4.2) that

$$(4.3) \qquad \Phi(x, k) = \frac{\psi(x, k)}{\Delta(k)} \text{ and } \Phi_2(0, k) = \frac{\psi_2(0, k)}{\Delta(k)} = M(k)$$

where

(4.4) $$\psi(x,k) = \psi_2(0,k)\varphi(x,k) + \Delta(k)C(x,k)$$

Note that, by virtue of equalities $\langle C(x,k), \varphi(x,k) \rangle \equiv 1$, (4.2) and (4.3) we have,

(4.5) $$\langle \Phi(x,k), \varphi(x,k) \rangle \equiv 1, \quad \langle \psi(x,k), \varphi(x,k) \rangle \equiv \Delta(k)$$

for $x < d$ and $x > d$.

Theorem 4.1. *The following representation holds;*

(4.6) $$M(k) = \frac{1}{\alpha_0(k-k_0)} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\alpha_n(k-k_n)} + \frac{1}{\alpha_n^0 k_n^0} \right\}$$

Proof. Let's write a representation solution $\psi(x,k)$ as $\varphi(x,k)$:
for $x > d$

$$\begin{cases} \psi_1(x,k) = -\sin k(\pi-x) + \int_0^{\pi+x} \widetilde{N}_{11}(x,t)\sin kt dt \\ \psi_2(x,k) = k\cos k(\pi-x) - b(x)\sin k(\pi-x) + \int_0^{\pi+x} \widetilde{N}_{21}(x,t)\sin kt dt \\ + \int_0^{\pi+x} k\widetilde{N}_{22}(x,t)\cos kt dt \end{cases}$$

for $x < d$

$$\begin{cases} \psi_1(x,k) = (-\alpha^+ + \beta)\sin k(\pi-x) + (\alpha^- - \beta)\sin k(x+\pi-2d) \\ + \int_0^{\pi+x} \widetilde{N}_{11}(x,t)\sin kt dt \\ \psi_2(x,k) = k(\alpha^+ - \beta)\cos k(\pi-x) + k(\alpha^- + \beta)\cos k(x+\pi-2d) \\ + b(x)[(-\alpha^+ + \beta)\sin k(\pi-x) + (\alpha^- - \beta)\sin k(x+\pi-2d)] \\ + \int_0^{\pi+x} \widetilde{N}_{21}(x,t)\sin kt dt + \int_0^{\pi+x} k\widetilde{N}_{22}(x,t)\cos kt dt \end{cases}$$

where $\widetilde{N}_{ij}(x,t) = N_{ij}(x,t) - N_{ij}(x,-t), i,j = 1,2.$ In the case $C = 0$ and $q(x) \equiv 0$ the solution are $\psi_{01}(x,k)$ and $\psi_{02}(x,k)$, so we have

$$\begin{cases} \psi_1(x,k) = \Psi_{01}(x,k) + f_1 \\ \psi_2(x,k) = \Psi_{02}(x,k) + f_2 \end{cases}$$

where
$f_1 = \int_0^{\pi+x} \widetilde{N}_{11}(x,t)\sin kt dt$
$f_2 = b(x)[(-\alpha^+ + \beta)\sin k(\pi-x) + (\alpha^- - \beta)\sin k(x+\pi-2d)]$
$\quad + \int_0^{\pi+x} \widetilde{N}_{21}(x,t)\sin kt dt + \int_0^{\pi+x} k\widetilde{N}_{22}(x,t)\cos kt dt$
On the other hand we can write
$$M(k) - M_0(k) = \frac{\psi_2(0,k)}{\psi_1(0,k)} - \frac{\psi_{02}(0,k)}{\psi_{01}(0,k)} = \frac{f_2}{\Delta(k)} - \frac{f_1}{\Delta(k)}M_0(k).$$

In view of $\lim\limits_{|k|\to\infty} e^{-|Imk\pi|}\,|f_i(k)| = 0$, $k \in G_\delta$ and $\Delta(k) > C_\delta e^{|Imk|\pi}$,

$$(4.7) \qquad \limsup\limits_{|k|\to\infty\ k\in G_\delta} |M(k) - M_0(k)| = 0.$$

from (3.5) that the Weyl function $M(k)$ is meromorphic with poles k_n. Using (3.5), (4.2) and Lemma 3.4, we calculate that

$$\mathop{Res}\limits_{k=k_n} M(k) = \frac{\psi_2(0,k_n)}{\dot\Delta(k_n)} = \frac{1}{\dot\Delta(k_n)\,\varphi_2(\pi,k_n)} = -\frac{1}{\alpha_n}$$

$$\mathop{Res}\limits_{k=k_n^0} M_0(k) = \frac{\psi_{02}(0,k_n^0)}{\dot\Delta(k_n^0)} = \frac{1}{\dot\Delta_0(k_n^0)\,\varphi_{02}(\pi,k_n^0)} = -\frac{1}{\alpha_n^0}.$$

Consider the contour integral

$$I_n(k) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{M(\mu) - M_0(\mu)}{k-\mu}\,d\mu \ , \ k \in int\Gamma_n$$

By virtue of $\limsup |M(k) - M_0(k)| = 0$, $\lim\limits_{n\to\infty} I_n(k) = 0$. On the other hand, the residue theorem and (4.8) yield

$$I_n(k) = -M(k) + M_0(k) + \sum\limits_{k_n\in int\Gamma_n} \frac{1}{\alpha_n(k-k_n)} - \sum\limits_{k_n^0\in int\Gamma_n} \frac{1}{\alpha_n^0(k_n^0-k)}$$

and theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us formulate a theorem on the uniqueness of a solution of the inverse problem with the use of the Weyl function. For this purpose, parallel with L, we consider the boundary-value problem $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}(x)$. It is assumed in what follows that if a certain symbol α denotes an object related to the problem L, then $\widetilde{\alpha}$ denotes the corresponding object related to the problem $\widetilde{L}$.

Theorem 4.2. *If $M(k) = \widetilde{M}(k)$ then $L = \widetilde{L}$ Thus the specification of the Weyl function uniquely determines the operator.*

Proof. Since

$$\psi^{(v)}(x,k) = O\left(|k|^{v-1}\exp\left(|Imk|(\pi-x)\right)\right)$$

$$(4.8) \qquad |\Delta(k)| \geq C_\delta\,|k|\exp\left(|Imk|\pi\right), \ k \in \widetilde{G}_\delta, \ v = 0,1$$

then it follows from (4.9) and (4.5) that

$$(4.9) \qquad \left|\Phi^{(v)}(x,k)\right| \leq C_\delta\,|k|^{v-1}\exp\left(-|Imk|\pi\right), \ k \in G_\delta.$$

Let us define the matrix $P(x, k) = [P_{jk}(x, k)]_{j,k=1,2}$ by the formula

$$(4.10) \qquad P(x, k) \begin{pmatrix} \widetilde{\varphi}_1 & \widetilde{\Phi}_1 \\ \widetilde{\varphi}_2 & \widetilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 & \Phi_1 \\ \varphi_2 & \Phi_2 \end{pmatrix}.$$

Using (4.11) and (4.5) we calculate

$$(4.11) \quad \begin{cases} P_{11}(x, k) = \varphi_1(x, k) \widetilde{\Phi}_2(x, k) - \Phi_1(x, k) \widetilde{\varphi}_2(x, k) \\ P_{12}(x, k) = \Phi_1(x, k) \widetilde{\varphi}_1(x, k) - \varphi_1(x, k) \widetilde{\Phi}_1(x, k) \\ P_{21}(x, k) = \varphi_2(x, k) \widetilde{\Phi}_2(x, k) - \Phi_2(x, k) \widetilde{\varphi}_2(x, k) \\ P_{22}(x, k) = \Phi_2(x, k) \widetilde{\varphi}_1(x, k) - \varphi_2(x, k) \widetilde{\Phi}_1(x, k) \end{cases}$$

and

$$(4.12) \quad \begin{cases} \varphi_1(x, k) = P_{11}(x, k) \widetilde{\varphi}_1(x, k) + P_{12}(x, k) \widetilde{\varphi}_2(x, k) \\ \varphi_2(x, k) = P_{21}(x, k) \widetilde{\varphi}_1(x, k) + P_{22}(x, k) \widetilde{\varphi}_2(x, k) \\ \Phi_1(x, k) = P_{11}(x, k) \widetilde{\Phi}_1(x, k) + P_{12}(x, k) \widetilde{\Phi}_2(x, k) \\ \Phi_2(x, k) = P_{21}(x, k) \widetilde{\Phi}_1(x, k) + P_{22}(x, k) \widetilde{\Phi}_2(x, k) \end{cases} .$$

It follows from (4.11), (4.1) and (4.4)

$$P_{11}(x, k) = 1 + \frac{1}{\Delta(k)} \left[\Psi_1(x, k) (\widetilde{\varphi}_2(x, k) - \varphi_2(x, k)) \right.$$
$$\left. -\varphi_1(x, k) \left(\widetilde{\Psi}_2(x, k) - \Psi_2(x, k) \right) \right],$$

$$P_{12}(x, k) = \frac{1}{\Delta(k)} \left[\varphi_1(x, k) \widetilde{\Psi}_1(x, k) - \Psi_1(x, k) \widetilde{\varphi}_1(x, k) \right],$$

$$P_{21}(x, k) = \frac{1}{\Delta(k)} \left[\Psi_2(x, k) \widetilde{\varphi}_2(x, k) - \varphi_1(x, k) \widetilde{\Psi}_2(x, k) \right],$$

$$P_{22}(x, k) = 1 + \frac{1}{\Delta(k)} \left[\varphi_2(x, k) \left(\widetilde{\Psi}_1(x, k) - \widetilde{\Psi}_2(x, k) \right) \right.$$
$$\left. -\Psi_2(x, k) (\widetilde{\varphi}_1(x, k) - \widetilde{\varphi}_2(x, k)) \right].$$

According to (4.12) and (4.1), for each fixed x, the functions $P_{jk}(x, k)$ are meromorphic in k with poles in the points k_n and $\widetilde{k}_n$. Denote $G_\delta^0 = G_\delta \cap \widetilde{G}_\delta$. By virtue of $\varphi^{(v)}(x, k) = O\left(|k|^v \exp(|Imk| x)\right)$, (4.8) and (4.9) this yields

$$(4.13) \quad |P_{11}(x, k) - 1| \le \frac{C_\delta}{|k|}, \quad |P_{12}(x, k)| \le \frac{C_\delta}{|k|}, \quad k \in G_\delta^0, \ |k| \ge k^*,$$

$$(4.14) \quad |P_{22}(x, k) - 1| \le \frac{C_\delta}{|k|}, \quad |P_{21}(x, k)| \le \frac{C_\delta}{|k|}, \quad k \in G_\delta^0, \ |k| \ge k^*$$

According to (4.1) (4.2) and (4.9) we have

$$P_{11}(x,k) = \varphi_1(x,k)\widetilde{C}_2(x,k) - C_1(x,k)\widetilde{\varphi}_2(x,k)$$
$$+ \left(\widetilde{M}(k) - M(k)\right)\varphi_1(x,k)\widetilde{\varphi}_2(x,k),$$

$$P_{12}(x,k) = \widetilde{\varphi}_1(x,k)\widetilde{C}_2(x,k) - \widetilde{C}_1(x,k)\varphi_1(x,k)$$
$$+ \left(M(k) - \widetilde{M}(k)\right)\varphi_1(x,k)\widetilde{\varphi}_1(x,k),$$

$$P_{21}(x,k) = \varphi_2(x,k)\widetilde{C}_2(x,k) - C_2(x,k)\widetilde{\varphi}_2(x,k)$$
$$+ \left(\widetilde{M}(k) - M(k)\right)\varphi_2(x,k)\widetilde{\varphi}_2(x,k),$$

$$P_{22}(x,k) = \widetilde{\varphi}_1(x,k)C_2(x,k) - \widetilde{C}_1(x,k)\varphi_2(x,k)$$
$$+ \left(M(k) - \widetilde{M}(k)\right)\varphi_2(x,k)\widetilde{\varphi}_1(x,k).$$

Thus if $M(k) = \widetilde{M}(k)$ then the functions $P_{jk}(x,k)$ are entire in k for each fixed x. Together with (4.13) and (4.14), (4.15) this yields,

$$P_{11}(x,k) \equiv 1, P_{12}(x,k) \equiv 0, P_{21}(x,k) \equiv 0, P_{22}(x,k) \equiv 1$$

Substituting into (4.12) we get

$$\varphi_1(x,k) \equiv \widetilde{\varphi}_1(x,k), \varphi_2(x,k) \equiv \widetilde{\varphi}_2(x,k),$$
$$\Phi_1(x,k) \equiv \widetilde{\Phi}_1(x,k), \Phi_2(x,k) \equiv \widetilde{\Phi}_2(x,k)$$

for all x and k. Consequently $L = \widetilde{L}$. $\square$

Theorem 4.3. *If* $k_n = \widetilde{k}_n$, $\alpha_n = \widetilde{\alpha}_n$, $n \geq 0$ *then* $L = \widetilde{L}$. *Thus, the specification of the spectral data* $\{k_n, \alpha_n\}_{n \geq 0}$ *uniquely determines the operator.*

Proof.

$$(4.15) \qquad M(k) = \frac{1}{\alpha_0(k - k_0)} + \sum_{n=-\infty}^{\infty}\left\{\frac{1}{\alpha_n(k - k_n)} + \frac{1}{\alpha_n^0 k_n^0}\right\}$$

$$\widetilde{M}(k) = \frac{1}{\widetilde{\alpha}_0\left(\widetilde{k} - \widetilde{k}_0\right)} + \sum_{n=-\infty}^{\infty}\left\{\frac{1}{\widetilde{\alpha}_n\left(\widetilde{k} - \widetilde{k}_n\right)} + \frac{1}{\widetilde{\alpha}_n^0 \widetilde{k}_n^0}\right\}$$

Under the hypothesis of the theorem and in view of (4.15), we get that $M(k) = \widetilde{M}(k)$ and consequently by Theorem 4.2, $L = \widetilde{L}$. $\square$

Theorem 4.4. : *If* $k_n = \widetilde{k}_n$, $\mu_n = \widetilde{\mu}_n$,$n \geq 0$, *then* $L = \widetilde{L}$.

Proof. By the properties of functions $\Delta(k)$ and $\widetilde{\Delta}(k)$, it is clear that $\lim\limits_{k\to\infty} \dfrac{\Delta(k)}{\widetilde{\Delta}(k)} = 1$. Under the hypothesis $k_n = \widetilde{k}_n$ and $\Delta(k)$ and $\widetilde{\Delta}(k)$ functions are entire we get that $\Delta(k) = \widetilde{\Delta}(k)$. From Lemma 3.4, we have $\widetilde{\psi}\left(x, \widetilde{k}_n\right) = \widetilde{\gamma}_n\varphi\left(x, \widetilde{k}_n\right) = \widetilde{\gamma}_n\widetilde{\varphi}(x, k_n)$ and $\widetilde{\Psi}\left(x, \widetilde{k}_n\right) = \widetilde{\Psi}(x, k_n) = \gamma_n\widetilde{\varphi}(x, k_n)$. It follows $\gamma_n = \widetilde{\gamma}_n$ and so $\alpha_n = \widetilde{\alpha}_n$. Consequently by Theorem 4.3, $L = \widetilde{L}$. $\square$

5. Properties of eigenfunctions

In this section, properties of eigenfunctions of problem L is considered. We have representation of eigen functions as follows: for $x < d$

$$\varphi(x, k_n) = \sin k_n x + \int_0^x \widetilde{K}_{11}(x, t) \sin k_n t dt,$$

for $x > d$

$$\varphi(x, k_n) =$$
$$\left(\alpha^+ + \beta\right)\sin k_n x + \left(\alpha^- - \beta\right)\sin k_n(2d - x) + int_0^x \widetilde{K}_{11}(x, t)\sin k_n t dt.$$

Theorem 5.1. *(i) The system of eigenfunctions $\{\varphi(x, k_n)\}_{n\geq 0}$ of the boundary value problem L is complete in $L_2(0, \pi)$.*

(ii) Let $f(x)$, $x \in [0, d) \cup (d, \pi]$ be an absolutely continuous function and satisfy the jump conditions:

$$\begin{cases} f(d + 0) = \alpha f(d - 0) \\ f'(d + 0) = \alpha^{-1}f'(d - 0) + 2k\beta f(d - 0) \end{cases}$$

Then

$$(5.1) \qquad f(x) = \sum_{n=0}^{\infty} a_n\varphi(x, k_n), \ a_n = \frac{1}{\alpha_n}\int_0^\pi \varphi(t, k_n) f(t) dt$$

and the series converges uniformly on $[0, d) \cup (d, \pi]$.

Proof. *(i)* From Theorem 1, 3, 8 of [13] the system of eigenfunctions $\{\varphi(x, k_n)\}_{n\geq 0}$ of problem L is a Riesz Bazis in $L_2(0, \pi)$. Thus (i) is proved.
(ii) Denote

$$G(x, t, k) = -\frac{1}{\Delta(k)}\begin{cases} \varphi(x, k)\psi(t, k), \ x \leq t \\ \varphi(t, k)\psi(x, k), \ x \geq t \end{cases}$$

and consider the function

$$Y(x,k) = \int_0^\pi G(x,t,k) f(t)\, dt$$

$$= -\frac{1}{\Delta(k)} \psi(x,k) \int_0^x \varphi(t,k) f(t)\, dt$$

$$- \frac{1}{\Delta(k)} \varphi(x,k) \left\{ \int_x^d \psi(t,k) f(t)\, dt + \int_d^\pi \psi(t,k) f(t)\, dt \right\}.$$

The function $G(x,t,k)$ is called *Green's function* for L. $G(x,t,k)$ is the kernel of the inverse operator for the Sturm-Liouville operator, i.e. $Y(x,k)$ is the solution of the boundary value problem

(5.2) $$\ell Y - \lambda Y = f(x),\, \lambda = k^2$$

$$U(Y) = 0, V(Y) = 0$$

and satisfies the jump condition:

(5.3) $$\begin{cases} Y(d+0) = \alpha Y(d-0) \\ Y'(d+0) = \alpha^{-1} Y'(d-0) + 2k\beta Y(d-0) \end{cases}$$

this easily verified by differentation.

Let now $f, x \in [0, d) \cup (d, \pi]$ be an arbitrary absolutely continuous function. Since $\varphi(x,k)$ and $\psi(x,k)$ are solutions of (1.1), we transform $Y(x,k)$ as follows:

$$Y(x,k) = -\frac{1}{k^2 \Delta(k)} \left\{ \psi(x,k) \int_0^x \left[-\varphi''(t,k) + \left(u'(t) + q(t) \right) \varphi(t,k) \right] f(t)\, dt \right.$$

(5.4)

$$+ \varphi(x,k) \int_x^d \left[-\psi''(t,k) + \left(u'(t) + q(t) \right) \psi(t,k) \right] f(t)\, dt$$

$$\left. + \varphi(x,k) \int_d^\pi \left[-\psi''(t,k) + \left(u'(t) + q(t) \right) \psi(t,k) \right] f(t)\, dt \right\}$$

Integrating of the terms containing second derivates by parts, yields in view of (3.2),

(5.5) $$Y(x,k) = \frac{f(x)}{k^2} - \frac{1}{k^2} \left\{ Z_1(x,k) + Z_2(x,k) \right\}$$

where

$$Z_1(x,k) = \frac{1}{\Delta(k)} \left\{ \psi(x,k) \int_0^x \varphi'(t,k) g(t)\, dt + \varphi(x,k) \int_x^d \psi'(t,k) g(t)\, dt \right.$$
$$\left. + \varphi(x,k) \int_d^\pi \psi'(t,k) g(t)\, dt \right\}$$

$$+\frac{1}{\Delta(k)}\left[f'(0)\,\psi(x,k)+\psi(\pi,k)\,f'(\pi)\,\varphi(x,k)\right],$$

$$g(t):=f'(t),$$

$$Z_2(x,k)=\frac{1}{\Delta(k)}\varphi(x,k)\,\psi'(\pi,k)\,f(\pi)$$

$$+\frac{1}{\Delta(k)}\psi(x,k)\int_0^x\left(u'(t)+q(t)\right)\varphi(t,k)\,f(t)\,dt$$

$$+\frac{1}{\Delta(k)}\varphi(x,k)\int_x^d\left(u'(t)+q(t)\right)\psi(t,k)\,f(t)\,dt$$

$$+\frac{1}{\Delta(k)}\varphi(x,k)\int_d^\pi\left(u'(t)+q(t)\right)\psi(t,k)\,f(t)\,dt.$$

Using (4.8), we get for a fixed $\delta > 0$ and sufficiently large $k^* > 0$:

$$(5.6)\qquad \max_{0\le x\le\pi}|Z_2(x,k)|\le\frac{C}{|k|},\quad k\in G_\delta,\ \ |k|\ge k^*.$$

Let us show that

$$(5.7)\qquad \lim_{\substack{|k|\to\infty\\ k\in G_\delta}}\max_{0\le x\le\pi}|Z_1(x,k)|=0.$$

First we assume that $g(x)$ is absolutely continuous on $[0,d)\cup(d,\pi]$. In this case another integration by parts yields of $Z_1(x,k)$ we infer

$$\max_{0\le x\le\pi}|Z_1(x,k)|\le\frac{C}{|k|},\quad k\in G_\delta,\ \ |k|\ge k^*.$$

Let now $g(t)\in L[0,\pi]$. Fix $\varepsilon>0$ and an absolutely continuous function $g_\varepsilon(t)$ such that

$$\int_0^\pi|g(t)-g_\varepsilon(t)|\,dt<\frac{\varepsilon}{2C^+}$$

where

$$C^+=\max_{0\le x\le\pi}\sup_{k\in G_\delta}\frac{1}{|\Delta(k)|}\left\{|\psi(x,k)|\int_0^x\left|\varphi'(t,k)\right|dt+|\varphi(x,k)|\right.$$

$$\left.\left(\int_x^d\left|\psi'(t,k)\right|dt+\int_d^\pi\left|\psi'(t,k)\right|dt\right)\right\}.$$

Then, for $k\in G_\delta,\ |k|\ge k^*$, we have

$$\max_{0\le x\le\pi}|Z_1(x,k)|\le\max_{0\le x\le\pi}|Z_1(x,k;g_\varepsilon)|+\max_{0\le x\le\pi}|Z_1(x,k;g-g_\varepsilon)|\le\frac{\varepsilon}{2}+\frac{C(\varepsilon)}{|k|}.$$

Hence, there exists k^0 such that $\max_{0\le x\le\pi}|Z_1(x,k)|\le\varepsilon$ for $|k|>k^0$. Since $\varepsilon>0$ is arbitrary, we arrive at (5.9).

Consider the contour integral

$$I_N(x) = \frac{1}{2\pi i} \int_{\Gamma_N} Y(x,k)\, dk,$$

where

$$\Gamma_n = \left\{ k : |k| = |k_n^0| + \frac{\sigma}{2}, \; n = 0, \; \pm 1, \; \pm 2, ... \right\}.$$

It follows from (5.7)-(5.9) that

(5.8) $I_N(x) = f(x) + \varepsilon_N(x), \; \lim_{N \to \infty} \max_{0 \le x \le \pi} |\varepsilon_N(x)| = 0.$

On the other hand, we can calculate $I_N(x)$, with the help of the residue theorem. Taking (3.4) into account and using Lemma 3.6 we calculate

$$\operatorname*{Res}_{k=k_n} Y(x,k) = -\frac{1}{\dot{\Delta}(k_n)} \psi(x,k_n) \int_0^x \varphi(t,k_n) f(t)\, dt$$

$$-\frac{1}{\dot{\Delta}(k_n)} \varphi(x,k_n) \left\{ \int_x^d \psi(t,k_n) f(t)\, dt + \int_d^\pi \psi(t,k_n) f(t)\, dt \right\}$$

$$= -\frac{\gamma_n}{\dot{\Delta}(k_n)} \varphi(x,k_n) \int_0^\pi \varphi(t,k_n) f(t)\, dt,$$

by (3.7),

(5.9) $\operatorname*{Res}_{k=k_n} Y(x,k) = \frac{1}{\alpha_n} \varphi(x,k_n) \int_0^\pi \varphi(t,k_n) f(t)\, dt.$

And by (5.6),

$$I_N(x) = \sum_{n=0}^N a_n \varphi(x,k_n), \; a_n = \frac{1}{\alpha_n} \int_0^\pi \varphi(t,k_n) f(t)\, dt.$$

Comparing this with (5.11) we arrive at (5.1), where the series converges uniformly on $[0,d) \cup (d,\pi]$, i.e. (ii) is proved.   □

Acknowledgments

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