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SOME RESULTS ON p-BEST APPROXIMATION IN VECTOR SPACES

H. MAZAHERI* AND S. M. MOSHTAGHIOUN

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Abstract. The purpose of this paper is to introduce and to discuss the concept of p-approximation and p-orthogonality in vector spaces, and to obtain some results on p- orthogonality in vector spaces similar to some well known results on the orthogonality in normed spaces. We also discuss the concept of p-extension of linear functionals on a vector space, and give a characterization of linear functionals on a subspace having a unique p-extension Hahn-Banach to the whole vector space.

1. Introduction

Here, all normed spaces under consideration are real. A seminorm is a function $p: X \to [0,\infty)$ such that $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$, for all $x, y \in X$ and $\alpha \in \mathbb{R}$. It is clear that for every seminorm p, $p(0) = 0$. Also, the seminorm p is a norm, if $p(x) = 0$ implies $x = 0$.

Many authors have introduced the concept of orthogonality in different ways (see [1-4], [6]). In [1], Birkhoff modified the concept of orthogonality. By his definition, if X is a normed linear space and $x, y \in X$, x is said to be orthogonal to y and is denoted by $x \perp y$ if and only if

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*Corresponding author

$\|x\| \leq \|x + \alpha y\|$, for all scalars α. Note that this orthogonality is not symmetric in general [2].

Let X be such a vector space, $x, y \in X$ and p be a fixed seminorm. We say that x is p-orthogonal to y if $x = 0$ or else,

$$p(x) \neq 0,\ p(x) = \inf{}_{\alpha} p(x + \alpha y),$$

in which case we write $x \perp^p y$. If M_1 and M_2 are subsets of X, then we say that M_1 is p-orthogonal to M_2 p if $g_1 \perp^p g_2$, for all $g_1 \in M_1,\ g_2 \in M_2$. If M_1 is p-orthogonal to M_2, then we write $M_1 \perp^p M_2$.

First we state the following lemma of Hahn- Banach which is needed in the proof of the main results.

Lemma 1.1. *[5] Let M be a subspace of a vector space X, p be a seminorm on X, and let f be a linear functional on M such that*

$$|f(x)| \leq p(x)\ (x \in M).$$

Then, f extends to a linear functional Λ on X which satisfies:

$$|\Lambda(x)| \leq p(x)\ (x \in X).$$

Suppose p is a seminorm on X. For $x \in X$, let

$$M_x^p = \{\Lambda :\ X \overset{linear}{\rightarrow} \mathbb{R} :\ \Lambda(x) = p(x),\ |\Lambda(z)| \leq p(z), \forall z \in X\}.$$

For $x \in X$, if we let $M =< x >$ ($< x >$ is the subspace of X generated by x) and define $f(\alpha x) = \alpha p(x)$, then by Lemma 1.1, the linear functional f extends to a linear functional $\Lambda \in M_x^p$. Therefore, M_x^p is nonempty.

Here, we are concerned with the concepts of p-best approximation and p-orthogonality in a vector space. The concept of approximation in normed linear spaces was defined by I. Singer [6].

2. **Orthogonality in vector spaces**

In this section, we state and prove our main results for vector spaces. Also, we obtain results related to p-orthogonality on vector spaces.

Theorem 2.1. *Let X be a vector space, G be a subspace of X, p be a seminorm on X, $x \in X \backslash G$ and $p(x) \neq 0$. Then, the following statements are equivalent:*
a) $x \perp^p G$.

b) There is a linear functional Λ *on* X *such that* $\Lambda \in {M^p}_x$ *and* $\Lambda|_G = 0$.

Proof. *a)* $\Rightarrow$ *b)*. Suppose $x \perp^p G$. Consider $M = < x > \bigoplus G$. We define a linear functional f on M by $f(\alpha x + y) = \alpha p(x)$, where $y \in G$ and $\alpha \in \mathbb{R}$. It is clear that $f(y) = 0$, for every $y \in G$, and $f(x) = p(x)$. Now, suppose $z = \alpha x + y \in M$. Then,

$$\begin{aligned} |f(z)| &= |f(\alpha x + y)| \\ &= |\alpha| p(x) \\ &\leq |\alpha| p(x + \frac{1}{\alpha} y) \\ &= p(\alpha x + y) \\ &= p(z). \end{aligned}$$

From Lemma 1.1, there exists a linear functional Λ on X such that

$$\Lambda(x) = p(x),\ \Lambda|_G = 0,\ |\Lambda(z)| \leq p(z)\ for\ all\ z \in X.$$

b) $\Rightarrow$ *a)*. Suppose that there exists a linear functional Λ on X such that $\Lambda \in {M^P}_x$ and $\Lambda|_G = 0$. For every $\alpha \in \mathbb{R}$ and $y \in G$, we have,

$$p(x + \alpha y) \geq |\Lambda(x + \alpha y)| = |\Lambda(x)| = p(x).$$

Therefore, $\inf_\alpha p(x + \alpha y) = p(x)$, and hence $x \perp^p y$. Thus, $x \perp^p G$. $\square$

Now, we shall obtain from Theorem 2.1 various corollaries on p-orthogonality.

Corollary 2.2. *Let* X *be a vector space,* p *be a seminorm on* X *and* $x, y \in X$. *If* $x \perp^p y$, *then* $< x > \cap < y >= \{0\}$.

Corollary 2.3. *Let* X *be a vector space,* A *be a nonempty subset of* X, p *be a seminorm on* X *such that* $p(y) \neq 0$, *for all* $y \in A$ *and* $x \in X \setminus < A >$. *Then, the following two statements are equivalent:*
a) $A \perp^p x$.
b) For every $y \in A$, *there exists a linear functional* Λ *on* X *with* $\Lambda \in M_y^p$ *and* $\Lambda(x) = 0$.

Definition 2.4. Let X be a vector space and p be a seminorm on X. The element $x \in X$ is called a p-normal element if there exists only one linear functional Λ_x on X such that $\Lambda_x \in M_x^p$; i.e., M_x^p is a singleton.

Corollary 2.5. *Let X be a vector space, G be a linear subspace of X, p be a seminorm on X and $p(x) \neq 0$. If $x \in X$ is a p-normal element associated with p on X, then the following statements are equivalent:*
a) $x \perp^p G$.
b) There exists a unique linear functional Λ on X such that $\Lambda \in {M^p}_x$ and $\Lambda|_G = 0$.

Let G be a subspace of the space X equipped with a seminorm p. Define,

$$\hat{G}_p = \{x \in X : \ x \perp^p G\},$$

and

$$\breve{G}_p = \{x \in X : \ G \perp^p x\}.$$

Corollary 2.6. *Let X be a vector space, G be a subspace of X and p be a seminorm on X. Then,*
a) $G \cap \hat{G}_p = \{0\}$
b) $G \cap \breve{G}_p = \{0\}$.
c) $\alpha x \in \hat{G}_p$, *if* $x \in \hat{G}_p$ *and* $\alpha \in \mathbb{R}$.
d) $\alpha x \in \breve{G}_p$, *if* $x \in \breve{G}_p$ *and* $\alpha \in \mathbb{R}$.

Proof. The statements (c) and (d) are consequences of the definition of p-orthogonality. Suppose $x \in G \cap \hat{G}_p$ (resp. $x \in G \cap \breve{G}_p$). Then, $x \perp G$ w.r.t. p (resp. $G \perp x$ w.r.t. p) and $x \in G$. Therefore, $x \perp^p x$, and form Corollary 2.2, $x = 0$. □

3. p-Best approximation and linear functional

Here, we shall introduce and discuss the concept of p-extension of linear functionals on a vector space, and show that a linear functional on a subspace has a unique p-extension to the whole vector space if and only if $G^\perp$ has some properties.

Let X be a vector space and p be a seminorm on X. A point $g_0 \in G$ is said to be a p-best approximation for $x \in X$ if and only if $p(x - g_0) \neq 0$ and for all $g \in G$, $p(x - g_0) \leq p(x - g)$. The set of all p-best approximations of $x \in X$ in G is denoted by $P^p_G(x)$. In other words,

$$P^p_G(x) = \{g_0 \in G \ : \ p(x - g_0) \neq 0, \ p(x - g_0) \leq p(x - g) \ \forall g \in G\}.$$

If $P^p_G(x)$ is non-empty for every $x \in X$, then G is called a p-proximinal set. The set G is p-Chebyshev if $P^p_G(x)$ is a singleton for every $x \in X$.

Theorem 3.1. *Let X be a vector space, G be a subspace of X, p be a seminorm on X, $g_0 \in G$, $x \in X \backslash G$ and $p(x-g_0) \neq 0$. Then, the following statements are equivalent:*
a) $g_0 \in P^p_G(x)$
b) There exists a linear functional Λ on X such that $\Lambda \in M^p_{x-g_0}$ and $\Lambda|_G = 0$.

Proof. We know that $g_0 \in P^p_G(x)$ if and only if $x - g_0 \perp^p G$. Now, Apply Theorem 2.1.

Theorem 3.2. *Let X be a vector space, p be a seminorm on X and G be a p-proximinal subspace of X. If G_p is a convex set, then G is p-Chebyshev.*

Proof. If $x \in X$ and $g_1, g_2 \in P^p{}_G(x)$, then $x - g_1, x - g_2 \in \hat{G}_p$. Since $\hat{G}_p$ is convex, then it follows that $\frac{1}{2}(g_1 - g_2) \in \hat{G}_p$. Since $\frac{1}{2}(g_1 - g_2) \in G$, then Lemma 2.6 shows that $g_1 = g_2$. □

Theorem 3.3. *Let X be a vector space, G be a subspace of X, p be a seminorm on X, $g_0 \in G$, and $x \in X \backslash G$ $p(x-g_0) \neq 0$. Then,*

$$g_0 \in P^p_G(x) \Leftrightarrow p((x-g_0)|_{G^\perp}) = p(x-g_0),$$

where,

$$p((x-g_0)|_{G^\perp}) = \sup\{|\Lambda(x-g_0)| : \ \Lambda \in G^\perp, \ |\Lambda(z)| \leq p(z), \ \forall z \in X\},$$

and the annihilator of G is the set,

$$G^\perp = \{f : \ X \stackrel{linear}{\rightarrow} \boldsymbol{R} : \ f(x) = 0 \ for \ all \ x \in G\}.$$

Proof. Let $g_0 \in P^p_G(x)$. It follows from $p(x-g_0) \neq 0$ and Theorem 3.1 that there exists a linear functional Λ on X such that for all $z \in X$, $|\Lambda(z)| \leq p(z)$, $\Lambda(x-g_0) = p(x-g_0)$ and $\Lambda|_G = 0$. Therefore, $p(x-g_0) = |\Lambda(x-g_0)| \leq p((x-g_0)|_{G^\perp})$. Now, suppose $\Lambda \in G^\perp$ and $|\Lambda(z)| \leq p(z)$ for all $z \in X$. Then, $|\Lambda(x-g_0)| \leq p(x-g_0)$, and thus $p((x-g_0)|_{G^\perp}) \leq p(x-g_0)$.

Conversely, suppose $p((x-g_0)|_{G^\perp}) = p(x-g_0)$. Since $p((x-g)|_{G^\perp}) \leq p(x-g)$, then similarly we have,

$$\begin{aligned} p(x-g_0) &= p((x-g_0)|_{G^\perp}) \\ &= p((x-g)|_{G^\perp}) \\ &\leq p(x-g). \end{aligned}$$

That is, $g_0 \in P_G^p(x)$. □

Let X be a vector space and p be a seminorm on X. The dual space of X with respect to p is denoted by:

$$X_p^* = \{\Lambda : \ X \stackrel{linear}{\to} \mathbf{R} : \ p'(\Lambda) < \infty\},$$

where,

$$p'(\Lambda) = \sup\{|\Lambda(x)| : \ p(x) \leq 1, \ x \in X\}.$$

It is clear that p' is a seminorm on X_p^*. Similarly, we can define p'' on X_p^{**} and p''' on X_p^{***}(see [5]).

It is clear that if X is a vector, p is a seminorm on X, $\Lambda \in X_p^*$, $x \in X$ and $p(x) \neq 0$, then,

$$p(x) = \sup\{|\Lambda(x)| : \ p'(\Lambda) \leq 1, \ \Lambda \in X_p^*\}.$$

Lemma 3.4. *Let X be a vector space, and p be a seminorm on X. For each $x \in X$, define the linear functional $\widehat{x}$ on X_p^* by $\widehat{x}(\varphi) = \varphi(x)$, for $\varphi \in X_p^*$. Then, for all $x \in X$,*

$$p''(\widehat{x}) = p(x).$$

Proof. We have,

$$\begin{aligned} p''(\widehat{x}) &= \sup\{|\widehat{x}(\varphi)| : \ p'(\varphi) \leq 1, \ \varphi \in X_p^*\} \\ &= \sup\{|\varphi(x)| : \ p'(\varphi) \leq 1, \ \varphi \in X_p^*\} \\ &= p(x). \end{aligned}$$

□

Definition 3.5. Let X be a vector space, p be a seminorm on X, $\Lambda \in X_p^*$ and G be a subspace of X. Λ is called a p-extension of the linear functional $f : G \longrightarrow \mathbf{R}$, if $\Lambda|_G = f$ and $p'(\Lambda) = p'(f)$. $p'(\Lambda)$ and $p'(f)$ are computed relative to the domains of Λ and f, explicitly as:

$$p'(\Lambda) = \sup\{|\Lambda(x)| : \ p(x) \leq 1, \ x \in X\}$$

and

$$p'(f) = \sup\{|f(x)| : \ p(x) \leq 1, \ x \in G\}.$$

Theorem 3.6. *Let X be a vector space, G be a subspace of X, p be a seminorm on X and $f \in X_p^* \backslash G^{\perp}$. Then, there is an one-to-one correspondence between the set $P_{G^{\perp}}^{p'}(f)$ and the set of all $g \in X_p^*$ such that $\widehat{g}$ is a p-extension of $\widehat{f}|_{G^{\perp\perp}}$, given by $g \mapsto f - g$.*

Proof. If $g \in P_{G^{\perp}}^{p'}(f)$, then it is clear that $\widehat{f}|_{G^{\perp\perp}} = \widehat{f-g}|_{G^{\perp\perp}}$, and from Theorem 3.2 and Corollary 3.4 we have,

$$\begin{aligned} p'''(\widehat{f-g}) &= p'(f-g) \\ &= p'((f-g)|_{G^{\perp\perp}}) \\ &= p'''(\widehat{(f-g)}|_{G^{\perp\perp}}) \\ &= p'''(\widehat{f}|_{G^{\perp\perp}}). \end{aligned}$$

Therefore, $\widehat{f-g}$ is a p-extension of $\widehat{f}|_{G^{\perp\perp}}$. We show that this map is onto. Suppose $\widehat{h}$ is a p-extension of $\widehat{f}|_{G^{\perp\perp}}$. Let $g = f - h$. We show that $g \in P_{G^{\perp}}^{p'}(f)$. For this, since $\widehat{f-g}$ is a p-extension of $\widehat{f}|_{G^{\perp\perp}}$, then $\widehat{f}|_{G^{\perp\perp}} = \widehat{f-g}|_{G^{\perp\perp}}$ and $p'''(\widehat{f}|_{G^{\perp\perp}}) = p'''(\widehat{f-g})$. Now, we have,

$$\begin{aligned} p'((f-g)|_{G^{\perp\perp}}) &= p'''(\widehat{(f-g)}|_{G^{\perp\perp}}) \\ &= p'''(\widehat{f}|_{G^{\perp\perp}}) \\ &= p'''(\widehat{f-g}) \\ &= p'(f-g). \end{aligned}$$

Therefore, by Theorem 3.2, $g \in P_{G^{\perp}}^{p'}(f)$. □

Theorem 3.7. *Let X be a vector space, G be a subspace of X, p be a seminorm on X, $f \in G_p^*$ and let $\widetilde{f} \in X_p^*$ be a p-extension of f. Then, there is a one-to-one correspondence between the set of all p-extensions of f to X and the set $P_{G^{\perp}}^{p'}(\widetilde{f})$, given by $g \longrightarrow \widetilde{f} - g$.*

Proof. Suppose $g \in X_p^*$ is a p-extension of f. Then, $g|_G = f$ and $p'(f) = p'(g)$. For all $\varphi \in G^{\perp\perp}$,

$$\widehat{g}(\varphi) = \varphi(g) = \varphi(\widetilde{f}) = \widehat{\widetilde{f}}(\varphi),$$

since $g|_G = \widetilde{f}|_G$. Also $\widetilde{f}|_G = f$, and thus $\widehat{\widetilde{f}}|_{G^{\perp\perp}} = \widehat{f}$ and

$$p'''(\widehat{g}) = p'(g) = p'(f) = p'''(\widehat{f}) = p'''(\widehat{\widetilde{f}}|_{G^{\perp\perp}}).$$

So by Theorem 3.6, $\widetilde{f} - g \in P^{p'}_{G^{\perp}}(\widetilde{f})$.

Now, suppose that $g \in P^{p'}_{G^{\perp}}(\widetilde{f})$. Then, $\widehat{\widetilde{f} - g}$ is a p-extension of $\widehat{\widetilde{f}}|_{G^{\perp\perp}}$. We know,

$$(\widetilde{f} - g)(x) = f(x),$$

for all $x \in G$, and

$$p'(\widetilde{f} - g) = p'''(\widehat{\widetilde{f} - g}) = p'''(\widehat{\widetilde{f}}|_{G^{\perp\perp}}) = p'''(\widehat{f}) = p'(f).$$

That is, $\widetilde{f} - g$ is a p-extension of f. □

Corollary 3.8. *Let X be a vector space, G be a subspace of X, and p be a seminorm on X. Then, the following statements are equivalent:*
a) Every non-zero $f \in G^*_p$ has a unique p-extension of X.
b) For each $f \in G^*_p \backslash G^{\perp}$, there is a unique $g \in X^*_p$ such that $\widehat{g}$ is a p-extension of $\widehat{f}|_{G^{\perp\perp}}$.
c) $G^{\perp}$ is a p'-Chebyshev subspace of X^*_p.

Example 3.9. Let f be a linear functional on a real vector space X. Then, $p(x) = |f(x)|$ gives a seminorm on X. If f is nonzero and $dim(X) > 1$, then p is seminorm that is not a norm. Now, let $G_1 = kerf$ and $G_2 = X \backslash kerf$. Then, G_1 is a subspace of X. For $y \in G_1$, we have $f(y) = 0$, and therefore for all $x \in X \backslash G_1$, $x \perp^p G_1$. Also, since for every $y \in G_2$, $f(y) \neq 0$, then,

$$\inf{}_{\alpha} p(x + \alpha y) \leq |f(x) + (-\frac{f(x)}{f(y)}) f(y)| = 0.$$

Therefore, for all $x \in X \backslash G_2$, $x \perp^p G_2$.

Example 3.10. The elements of $X = L^2$ are the real Lebesgue measurable functions f on $[0, 1]$. We can define a seminorm p on X by:

$$p(f) = \{\int_0^1 |f(t)|^2 dt\}^{\frac{1}{2}}, \ f \in X.$$

It is clear that p is a seminorm on X and is not a norm. Now, we can apply Theorem 3.7 and Theorem 3.6 to this example.

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H. Mazaheri and S. M. Moshtaghioun

Department of Mathematics, University of Yazd, P.O. Box 89195-741, Yazd, Iran

`Email: hmazaheri@yazduni.ac.ir`

`Email: moshtagh@yazduni.ac.ir`