ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 3, pp. 639–655

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 3, pp. 639–655
Online ISSN: 1735-8515

ON THE CATEGORY OF GEOMETRIC SPACES AND THE CATEGORY OF (GEOMETRIC) HYPERGROUPS

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(Communicated by Ali Reza Ashrafi)

Abstract. In this paper first we define the morphism between geometric spaces in two different types. We construct two categories of $\mathbb{U}-\mathbb{GESP}$ and $\mathbb{L}-\mathbb{GESP}$ from geometric spaces then investigate some properties of the two categories, for instance $\mathbb{U}-\mathbb{GESP}$ is topological. The relation between hypergroups and geometric spaces is studied. By constructing the category $\mathbf{SNS}-\mathbf{H_v}$ of H_v-groups we answer the question that which construction of hyperstructures on the category of sets has free object in the sense of universal property. At the end we define the category of geometric hypergroups and we study its relation with the category of hypergroup.

Keywords: Geometric hypergroups, H_v-groups, geometric spaces, topological categories.
MSC(2010): Primary: 20N20; Secondary: 18A40, 55U40.

1. Introduction

According to [7], a geometric space is a pair $(S, \mathfrak{B})$ such that S is a non-empty set, that its elements are called point and $\mathfrak{B}$ is a non-empty family of subsets of S, which its elements are called blocks. If C is a subset of S then it is called $\mathfrak{B}$-part of S if for every $B \in \mathfrak{B}$,

$$B \bigcap C \neq \emptyset \Rightarrow B \subseteq C.$$

Article electronically published on June 17, 2014.
Received: 25 September 2012, Accepted: 9 may 2013.
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For a subset X of S, we denote by $\Gamma_{\mathfrak{B}}(X)$ the intersection of all $\mathfrak{B}$-parts of S containing X.

Proposition 1.1. *(See [7]) Let $(S,\mathfrak{B})$ be a geometric space. For every $n \in \mathbb{N}$ and for every pair (X,Y) of subsets of S we have*

(i) $X \subseteq \Gamma_{\mathfrak{B}}(X)$;
(ii) $X \subseteq Y \Rightarrow \Gamma_{\mathfrak{B}}(X) \subseteq \Gamma_{\mathfrak{B}}(Y)$;
(iii) $\Gamma_{\mathfrak{B}}(\Gamma_{\mathfrak{B}}(X)) = \Gamma_{\mathfrak{B}}(X)$;
(iv) $\Gamma_{\mathfrak{B}}(X) = \bigcup_{x \in X} \Gamma_{\mathfrak{B}}(x)$, *where* $\Gamma_{\mathfrak{B}}(x) = \Gamma_{\mathfrak{B}}(\{x\})$.

For all subsets X of S, we can associate an ascending chain of subsets $(\Gamma^n_{\mathfrak{B}}(X))_{n\in\mathbb{N}}$ called cone of X, defined by the following conditions
$\Gamma^0_{\mathfrak{B}}(X) = X$;
and for every integer $n \geq 0$,

$$\Gamma^{n+1}_{\mathfrak{B}}(X) = \Gamma^n_{\mathfrak{B}}(X) \cup \Big[\bigcup \{B \in \mathfrak{B} \mid B \cap \Gamma^n_{\mathfrak{B}}(X) \neq \emptyset\}\Big].$$

Proposition 1.2. *(See [7]) Let $(S,\mathfrak{B})$ be a geometric space. For every $n \in \mathbb{N}$ and for every pair (X,Y) of subsets of S we have*

(i) $X \subseteq Y \Rightarrow \Gamma^n_{\mathfrak{B}}(X) \subseteq \Gamma^n_{\mathfrak{B}}(Y)$;
(ii) $\Gamma^n_{\mathfrak{B}}(X) = \bigcup_{x \in X} \Gamma^n_{\mathfrak{B}}(x)$, *where* $\Gamma^n_{\mathfrak{B}}(x) = \Gamma^n_{\mathfrak{B}}(\{x\})$;
(iii) $\Gamma^n_{\mathfrak{B}}(\Gamma^m_{\mathfrak{B}}(X)) = \Gamma^{n+m}_{\mathfrak{B}}(X)$;
(iv) $\Gamma_{\mathfrak{B}}(X) = \bigcup_{n \in \mathbb{N}} \Gamma^n_{\mathfrak{B}}(X)$;
(v) *If the family $\mathfrak{B}$ is a covering of S, i.e., $S = \bigcup_{B \in \mathfrak{B}} B$, then*

$$\Gamma^{n+1}_{\mathfrak{B}}(X) = \bigcup \{B \in \mathfrak{B} \mid B \cap \Gamma^n_{\mathfrak{B}}(X) \neq \emptyset\}.$$

Proposition 1.3. *(See [7]) For every pair (A,B) of blocks of a geometric space $(S,\mathfrak{B})$ and for every $n \in \mathbb{N}$, the following conditions are equivalent:*

(i) $A \cap B \neq \emptyset,\ x \in B \Rightarrow \exists C \in \mathfrak{B} : \quad (A \cup \{x\}) \subseteq C$;
(ii) $A \cap B \neq \emptyset,\ x \in \Gamma^n_{\mathfrak{B}}(B) \Rightarrow \exists C \in \mathfrak{B} : \quad (A \cup \{x\}) \subseteq C$;
(iii) $A \cap \Gamma^n_{\mathfrak{B}}(B) \neq \emptyset,\ x \in \Gamma^n_{\mathfrak{B}}(B) \Rightarrow \exists C \in \mathfrak{B} : \quad (A \cup \{x\}) \subseteq C$.

A geometric space $(S,\mathfrak{B})$ is strongly transitive if the family $\mathfrak{B}$ is a covering of S moreover, one of the three equivalent conditions of previous Proposition is satisfied.

Definition 1.4. *The geometric space $(S,\mathfrak{B})$ is finer than $(S,\mathfrak{B}')$ whenever for each $x \in S$ and each blocks $B' \in \mathfrak{B}'$ containing x, there is a block $B \in \mathfrak{B}$ such that $x \in B \subseteq B'$.*

Definition 1.5. *The geometric spaces* $(S, \mathfrak{B})$ *and* $(S', \mathfrak{B}')$ *are called equal, if*

(i) $S = S'$;
(ii) $(S, \mathfrak{B})$ *is finer than* $(S, \mathfrak{B}')$ *and* $(S, \mathfrak{B}')$ *is finer than* $(S, \mathfrak{B})$.

Example 1.6. *Let* $\mathbb{R}$ *be the set of real numbers and* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *are the families of open and closed intervals in* $\mathbb{R}$*, respectively. The two strongly transitive geometric space* $(\mathbb{R}, \mathfrak{B})$ *and* $(\mathbb{R}, \mathfrak{B}')$ *are equal.*

A hyperstructure (or hypergroupoid) is a non-empty set H with a hyperoperation $\circ$ defined on H, that is a mapping of $H \times H$ into the family of non-empty subsets of H [9]. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y \mid x \in A, y \in B\}$. The notation $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. A hyperstructure $(H, \circ)$ is called a *hypergroup* in the sense of Marty, if for all $(x, y, z) \in H^3$ the following two conditions hold: (i) $x \circ (y \circ z) = (x \circ y) \circ z$, (ii) $x \circ H = H \circ x = H$. The second condition is called the reproduction axiom, it means that for every $(x, y) \in H^2$ there exists $(u, v) \in H^2$ such that $y \in x \circ u$ and $y \in v \circ x$. If $(H, \circ)$ satisfies only the first axiom, then it is called a semi-hypergroup. Let $(H, \circ)$ be a hypergroup an element e in H is called a scalar identity whenever for every $x \in H$, $x \circ e = e \circ x = \{x\}$. Let $(H, \circ)$ and $(H', \circ')$ be two hyperstructures. A function $f : H \longrightarrow H'$ is called a *homomorphism,* if $f(a \circ b) \subseteq f(a) \circ' f(b)$, for every a and b in H.

An exhaustive review updated to 1992 of hypergroup theory appears in [3]; also see [5], [6] and [8]. A recent book [4] contains a wealth of applications.

The pair $(H, \circ)$ is called H_v-group, whenever

(i) for each $(a, b, c) \in H^3$, $a \circ (b \circ c) \cap (a \circ b) \circ c \neq \emptyset$;
(ii) for each $a \in H$, $a \circ H = H \circ a = H$.

In the following we give some categorical notions that we use in this article.

Definition 1.7. *(See* [1]*) Let* $\mathcal{A}$ *and* $\mathcal{C}$ *be categories and* $\mathbf{G} : \mathcal{A} \longrightarrow \mathcal{C}$ *be a functor. A source* $\mathcal{S} = (A \xrightarrow{f_i} A_i)_I$ *in* $\mathcal{A}$ *is called* $\mathbf{G}$*-initial provided that for each source* $\mathcal{T} = (C \xrightarrow{g_i} A_i)_I$ *in* $\mathcal{A}$ *with the same codomain as* $\mathcal{S}$ *and each* $\mathcal{C}$*-morphism* $\mathbf{G}C \xrightarrow{h} \mathbf{G}A$ *with* $\mathbf{G}\mathcal{T} = \mathbf{G}\mathcal{S} \circ h$

there exists a unique $\mathcal{A}$-morphism $C \xrightarrow{\bar{h}} A$ *with* $\mathcal{T} = \mathcal{S} \circ \bar{h}$ *and* $h = \mathbf{G}\bar{h}$.

Definition 1.8. *(See* [1]*) A functor* $\mathcal{A} \xrightarrow{\mathbf{G}} \mathcal{C}$ *is called topological provided that every* $\mathbf{G}$*-structured source* $(C \xrightarrow{f_i} \mathbf{G}(A_i))_I$ *has a unique* $\mathbf{G}$*-initial lift* $(A \xrightarrow{\bar{f}_i} A_i)_I$.

Definition 1.9. *(See* [1]*) A concrete category* $(\mathcal{A}, \mathbf{U})$ *is called topological provided that* $\mathbf{U}$ *is topological.*

2. The category of u-geometric spaces

In this section we introduce the category $\mathbb{U} - \mathbb{GESP}$ and we prove that its construction is topological.

Definition 2.1. *The pair* $(S, \mathfrak{B})$ *is called u-geometric space whenever* $\mathfrak{B}$ *is a covering of* S *and for every non-empty family* $\{B_i\}_{i\in I}$ *of elements of* $\mathfrak{B}$ *we have* $\bigcap_{i\in I} B_i \in \mathfrak{B}$.

Example 2.2. $(\mathbb{R}, \mathfrak{B})$ *is u-geometric space, where* $\mathfrak{B}$ *is close sets in* $\mathbb{R}$.

Definition 2.3. *Suppose that* $(S, \mathfrak{B})$ *and* $(S', \mathfrak{B}')$ *are two u-geometric spaces. A function* $f : S \longrightarrow S'$ *is called a u-geometric homomorphism whenever for each* $B' \in \mathfrak{B}'$ *and each* $x \in f^{-1}(B')$ *there is a* $B \in \mathfrak{B}$ *such that* $x \in B$ *and* $B \subseteq f^{-1}(B')$.

For simplicity u-geometric homomorphism is called u-morphism.

The collection of u-geometric spaces together with u-morphisms forms a category which is denoted by $\mathbb{U} - \mathbb{GESP}$.

Proposition 2.4. *The morphism* $(S, \mathfrak{B}) \xrightarrow{f} (S', \mathfrak{B}')$ *in* $\mathbb{U} - \mathbb{GESP}$ *is monomorphism if and only if its underlying map* $S \xrightarrow{f} S'$ *is an injective.*

Proof. Let f be a u-monomorphism and the diagram

$$S'' \underset{h}{\overset{g}{\rightrightarrows}} S \xrightarrow{f} S'$$

in **Set** (the category of sets) such that $fg = fh$ be given. Define

$$\mathfrak{B}'' :\overset{\text{def}}{=} \{B'' \mid B'' = g^{-1}(B_1) \cap h^{-1}(B_2) \text{ for some } B_1, B_2 \in \mathfrak{B}\},$$

it is easy to see that $(S'', \mathfrak{B}'')$ is a u-geometric space. Since $\mathfrak{B}$ cover S, g and h are u-morphisms. Also $fg = fh$ and f is monomorphism so we have $g = h$. □

Theorem 2.5. $\mathbb{U} - \mathbb{GESP}$ *construction is topological.*

Proof. Let $\mathcal{U} = (S \xrightarrow{f_i} \mathbf{U}(S_i, \mathfrak{B}_\mathfrak{i}))_I$ be a **U**-structured source, where

$$\mathbf{U} : \mathbb{U} - \mathbb{GESP} \longrightarrow \mathbf{Set}\ ,$$

is a forgetful functor. Define

$$\mathfrak{B} :\overset{\text{def}}{=} \{B \subseteq S \mid B = \bigcap_{i \in I} f_i^{-1}(B_i),\ \text{where } B_i \in \mathfrak{B}_\mathfrak{i}\}.$$

$(S, \mathfrak{B})$ is a u-geometric space. Moreover, we show that for each $i \in I$, f_i is a u-geometric homomorphism for this reason suppose $j \in I$, $B_j \in \mathfrak{B}_\mathfrak{j}$ and $x \in S$ such that $x \in f_j^{-1}(B_j)$ are given. Since for each $i \in I$ S_i is covered by $\mathfrak{B}_\mathfrak{i}$, there exists a $B_i \in \mathfrak{B}_\mathfrak{i}$ such that $f_i(x) \in B_i$, for each $i \in I$ such that $i \neq j$. Therefore $x \in \bigcap_{i \in I} f_i^{-1}(B_i) \subseteq f_j^{-1}(B_j)$ and $\bigcap_{i \in I} f_i^{-1}(B_i) \in \mathfrak{B}$.

In the following we show that

$$\mathcal{S} := \big(\ (S, \mathfrak{B}) \xrightarrow{f_i} (S_i, \mathfrak{B}_\mathfrak{i})\ \big)_{i \in I}$$

is an **U**-initial source. Suppose the source

$$\mathcal{T} = \big(\ (S', \mathfrak{B}') \xrightarrow{g_i} (S_i, \mathfrak{B}_\mathfrak{i})\ \big)_{i \in I}$$

and the map $\mathbf{U}(S', \mathfrak{B}') \xrightarrow{h} \mathbf{U}(S, \mathfrak{B})$ such that $\mathbf{U}(\mathcal{S})h = \mathbf{U}(\mathcal{T})$ are given. If $B \in \mathfrak{B}$ and $x \in h^{-1}(B)$, then $B = \bigcap_{i \in I} f_i^{-1}(B_i)$, where for each $i \in I$, $B_i \in \mathfrak{B}_\mathfrak{i}$ and hence $x \in \bigcap_{i \in I} h^{-1}(f_i^{-1}(B_i)) = \bigcap_{i \in I} g_i^{-1}(B_i)$. Therefore for each $i \in I$, there is a $B'_i \in \mathfrak{B}'$ such that $x \in B'_i \subseteq g_i^{-1}(B_i)$ hence $x \in \bigcap_{i \in I} B'_i \subseteq \bigcap_{i \in I} g_i^{-1}(B_i) = h^{-1}(B)$. Since $\bigcap_{i \in I} B'_i \in \mathfrak{B}'$, we conclude that h is a u-geometric homomorphism such that $\mathbf{U}(h) = h$. Because **U** is forgetful $\mathcal{S}$ is an **U**-initial source and $\mathbf{U}(\mathcal{S}) = \mathcal{U}$.

Now let the $\mathbf{U}$-initial source $\mathcal{S}' = \big((T,\mathfrak{D}) \xrightarrow{f_i'} (S_i,\mathfrak{B}_i) \big)_{i\in I}$ such that $\mathbf{U}(\mathcal{S}') = \mathcal{U}$ be given. So $T = S$ and $f_i' = f_i$, for all $i \in I$. We prove $(T,\mathfrak{D}) = (S,\mathfrak{B})$. Consider the identity map $\mathbf{U}(S,\mathfrak{B}) \xrightarrow{1_S} \mathbf{U}(S,\mathfrak{D})$. Since $\mathcal{S}$ is an $\mathbf{U}$-initial source, there exists a unique u-morphism

$$\bar{h} : (S,\mathfrak{B}) \longrightarrow (S,\mathfrak{D})$$

such that $\bar{h} = \mathbf{U}(\bar{h}) = 1_S$. Suppose $D \in \mathfrak{D}$ and $x \in S$ such that $x \in \bar{h}^{-1}(D)$ are given. So there exists $B \in \mathfrak{B}$ such that $x \in B \subseteq \bar{h}^{-1}(D) = D$. Therefore $(S,\mathfrak{B})$ is finer than $(S,\mathfrak{D})$. Similarly $(S,\mathfrak{D})$ is finer than $(S,\mathfrak{B})$. Thus $(T,\mathfrak{D}) = (S,\mathfrak{B})$. □

3. The category of l-geometric spaces

In this section we introduce the category $\mathbb{L} - \mathbb{GESP}$ and we investigate some of its categorical properties.

Definition 3.1. *Suppose that $(S,\mathfrak{B})$ and $(S',\mathfrak{B}')$ are two geometric spaces. A function $f : S \longrightarrow S'$ is called l-geometric homomorphism whenever for each $B \in \mathfrak{B}$ there exists $B' \in \mathfrak{B}'$ such that $f(B) \subseteq B'$.*

For simplicity l-geometric homomorphism is called l-morphism.

Example 3.2. *Every open map between two topological spaces is l-morphism.*

The collection of geometric spaces together with l-morphisms forms a category which is denoted by $\mathbb{L} - \mathbb{GESP}$. We have the following proposition.

Proposition 3.3. *The morphism $(S,\mathfrak{B}) \xrightarrow{f} (S',\mathfrak{B}')$ in $\mathbb{L} - \mathbb{GESP}$*

(i) *is monomorphism if and only if its underlying map $S \xrightarrow{f} S'$ is an injective;*

(ii) *is epimorphism if and only if its underlying map $S \xrightarrow{f} S'$ is a surjective.*

Proof. (i) Suppose the monomorphism f and the diagram

$$S'' \underset{h}{\overset{g}{\rightrightarrows}} S \xrightarrow{f} S' ,$$

in **Set** such that $fg = fh$ are given. Define

$$\mathfrak{B}'' :\stackrel{\text{def}}{=} \{B'' \mid B'' = g^{-1}(B) \cap h^{-1}(B) \text{ for some } B \in \mathfrak{B}\},$$

g and h are l-morphisms . Since $fg = fh$ and f is monomorphism, we have $g = h$.

(ii) Suppose the epimorphism f and the diagram $S \xrightarrow{f} S' \underset{h}{\overset{g}{\rightrightarrows}} S''$ in **Set** such that $gf = hf$ are given. Define

$$\mathfrak{B}'' :\stackrel{\text{def}}{=} \{B'' \mid B'' = g(B') \cup h(B') \text{ for some } B' \in \mathfrak{B}'\},$$

g and h are l-morphisms. Since $gf = hf$ and f is an epimorphism we have $g = h$. □

Definition 3.4. *The source* $\mathcal{S} = \big((S, \mathfrak{B}) \xrightarrow{f_i} (S_i, \mathfrak{B}_\mathfrak{i}) \big)_{i \in I}$ *in the category* $\mathbb{L} - \mathbb{GESP}$ *is called l-initial whenever for each family* $\{B_i\}_{i\in I}$ *of blocks there exists* $B \in \mathfrak{B}$ *such that* $\bigcap_{i\in I} f_i^{-1}(B_i) \subseteq B$*, where for each* $i \in I$, $B_i \in \mathfrak{B}_\mathfrak{i}$.

Remark 3.5. *The class of l-initial Mono-Sources is closed under composition with isomorphisms, i.e., if* $\big((S, \mathfrak{B}) \xrightarrow{f_i} (S_i, \mathfrak{B}_\mathfrak{i}) \big)_{i \in I}$ *is an l-initial Mono-Source in* $\mathbb{L} - \mathbb{GESP}$ *and* $h : (S', \mathfrak{B}') \longrightarrow (S, \mathfrak{B})$ *is an isomorphism in* $\mathbb{L} - \mathbb{GESP}$*, then* $\big((S', \mathfrak{B}') \xrightarrow{f_i h} (S_i, \mathfrak{B}_\mathfrak{i}) \big)_{i \in I}$ *is an l-initial Mono-Source.*

Proposition 3.6. *Let* $\mathbf{G} : \mathbb{L} - \mathbb{GESP} \longrightarrow \mathbf{Set}$ *be the forgetful functor. Then we have,*

(i) $\mathbf{G}$ *creates isomorphism;*

(ii) *If the* $\mathbf{G}$*-structure morphism* $S' \xrightarrow{h} \mathbf{G}(S, \mathfrak{B}) = S$ *is generating then the map h is an* **Set***-epi;*

(iii) $\mathbf{G}$ *is a (Epi, l-initial Mono-Source)-functor;*

(iv) $\mathbf{G}$ *is an adjoint functor.*

Proof. (i) Suppose that the bijection $S' \xrightarrow{h} \mathbf{G}(S, \mathfrak{B}) = S$ is given. Define

$$\mathfrak{B}' :\stackrel{\text{def}}{=} \{h^{-1}(B) \mid B \in \mathfrak{B}\}.$$

So $(S', \mathfrak{B}')$ is a geometric space and $(S', \mathfrak{B}') \xrightarrow{h} (S, \mathfrak{B})$ is an isomorphism in $\mathbb{L} - \mathbb{GESP}$ such that $\mathbf{G}(h) = h$.

(ii) The proof is similar to the proof of Proposition 3.3(ii).

(iii) The desired factorizations of $\mathbf{G}$-structured sources of the form

$$\big(S \xrightarrow{f_i} \mathbf{G}(S_i, \mathfrak{B}_{\mathrm{i}})\big)_{i \in I},$$

can be obtained in two steps. First let $S \xrightarrow{f_i} S_i = S \xrightarrow{e} S' \xrightarrow{m_i} S_i$, be a (Epi , Mono-Sourse)-factorization in **Set**. Second, let

$$\mathfrak{B}' :\overset{\text{def}}{=} \{\bigcap_{i \in I} m_i^{-1}(B_i) \mid \text{ for some } B_i \in \mathfrak{B}_{\mathrm{i}}\},$$

so $\big(\,(S', \mathfrak{B}') \xrightarrow{m_i} (S_i, \mathfrak{B}_{\mathrm{i}})\,\big)_{i \in I}$ is a l-initial Mono-Source and provides a factorization with the desired properties.

(iv) The functor $\mathbf{Set} \xrightarrow{\mathbf{F}} \mathbb{L} - \mathbb{GESP}$, is defined by $\mathbf{F}(S) = (S, \mathfrak{B})$, where $\mathfrak{B} = \{\emptyset\}$ for each set S which is left adjoint to the $\mathbf{G}$. $\square$

Remark 3.7. *In the category* $\mathbb{L} - \mathbb{GESP}$ *we have*

(i) the coequalizer of the diagram $(S', \mathfrak{B}') \overset{f}{\underset{g}{\rightrightarrows}} (S'', \mathfrak{B}'')$, *is the pair* $(S, \mathfrak{B})$ *together with the map* c *such that* $S :\overset{def}{=} \frac{S''}{\sim}$ *and* $\mathfrak{B} :\overset{def}{=} \{\frac{B''}{\sim} \mid B'' \in \mathfrak{B}''\}$, *where* $\sim$ *is the smallest equivalence relation on* S'' *such that* $f(a) \sim g(a)$, *for all* $a \in S'$ *and* $\frac{B''}{\sim} = \{\frac{b''}{\sim} \mid b'' \in B''\}$ *and* $c : S'' \longrightarrow S$ *is defined by* $c(x) = \frac{x}{\sim}$*;*

(ii) the equalizer of the diagram $(S', \mathfrak{B}') \overset{f}{\underset{g}{\rightrightarrows}} (S'', \mathfrak{B}'')$ *is the pair* $(S, \mathfrak{B})$ *together with the map* e *such that* $S :\overset{def}{=} \{x \in S' \mid f(x) = g(x)\}$, $\mathfrak{B} :\overset{def}{=} \{B' \cap S \mid B' \in \mathfrak{B}'\}$ *and* e *is the inclusion map;*

(iii) the coproduct of the family $\{(S_\alpha, \mathfrak{B}_\alpha)\}_{\alpha \in I}$ *is the pair* $(S, \mathfrak{B})$ *together with the family* $(S_\alpha \xrightarrow{\iota_\alpha} S)_{\alpha \in I}$ *such that* $S :\overset{def}{=} \biguplus_{\alpha \in I} S_\alpha$ *and*

$$\mathfrak{B} :\overset{def}{=} \{\biguplus_{\beta \in J} B_\beta \subseteq S \mid J \subseteq I \text{ and } \forall \beta \in J, B_\beta \in \mathfrak{B}_\beta\},$$

where $\biguplus$ *is denoted the disjoint union of sets and* ι_α*'s are injection maps;*

(iv) the product of the family $\{(S_\alpha, \mathfrak{B}_\alpha)\}_{\alpha\in I}$ *is the pair* $(S, \mathfrak{B})$ *together with the family* $(S_\alpha \xrightarrow{pr_\alpha} S)_{\alpha\in I}$ *such that* $S :\stackrel{def}{=} \prod_{\alpha\in I} S_\alpha$, $\mathfrak{B} :\stackrel{def}{=} \{pr_\alpha^{-1}(B_\alpha) \mid B_\alpha \in \mathfrak{B}_\alpha, \text{ for all } \alpha \in I\}$ *and* $pr_\alpha : \prod_{\alpha\in I} S_\alpha \longrightarrow S_\alpha$ *are projection maps.*

Corollary 3.8. *The category* $\mathbb{L} - \mathbb{GESP}$ *is complete and cocomplete.*

4. Relation between the categories of H_v-group and geometric spaces

In this section we prove that for every set X there exists a free object on X, in the category $\mathbf{SNS} - \mathbf{H_v}$. Although in [10] we have shown that the categories of semi-hypergroups and hypergroups have no free objects.

Definition 4.1. *(i) The geometric space* $(S, \mathfrak{B})$ *is called* 2*-geometric space whenever*

$$\forall (s,t) \in S^2 \ \exists B \in \mathfrak{B} \text{ such that } \{s,t\} \subseteq B;$$

(ii) The l*-morphism* $f : (S, \mathfrak{B}) \to (S', \mathfrak{B}')$ *of* 2*-geometric spaces is called* l_2*-morphism whenever*

$$\forall (s,t) \in S^2 \text{ such that } s \neq t, \ \ f(\bigcap_{\{s,t\}\subseteq B\in\mathfrak{B}} B) \subseteq \bigcap_{\{f(s),f(t)\}\subseteq B'\in\mathfrak{B}'} B'.$$

Remark 4.2. *Every* 2*-geometric space is complete.*

The collection of 2-geometric spaces together with l_2-morphisms forms a category which is denoted by $\mathbb{L}_\mathbf{2}-\mathbb{GESP}$.

Suppose that $\mathbf{U}_2 : \mathbb{L}_\mathbf{2}-\mathbb{GESP} \longrightarrow \mathbf{Set}$ is a forgetful functor. We have the following proposition.

Proposition 4.3. $\mathbf{U}_2$ *is an adjoint functor.*

Proof. Consider the map $\mathbf{F}_2 : \mathbf{Set} \longrightarrow \mathbb{L}_\mathbf{2}-\mathbb{GESP}$ such that for each set X is defined by $\mathbf{F}_2(X) :\stackrel{\text{def}}{=} (X, \mathfrak{B}_X^2)$, where $\mathfrak{B}_X^2 :\stackrel{\text{def}}{=} \{\{a,b\} \mid (a,b) \in X^2\}$. Therefore $\mathbf{F}_2(X)$ is a 2-geometric space. Let $f : X \longrightarrow Y$ be a map, so $(X, \mathfrak{B}_X^2) \xrightarrow{f} (Y, \mathfrak{B}_Y^2)$ is a l-morphism. Suppose $(s,t) \in X^2$ such that $s \neq t$ and $f(x) \in f(\bigcap_{\{s,t\}\subseteq B\in\mathfrak{B}_X^2} B)$ are given. From $\{s,t\} \subseteq B \in$

$\mathfrak{B}^2_X$ we have $B = \{s,t\}$ hence $f(x) \in \{f(s), f(t)\}$. Therefore $\mathbf{F}_2(f)$ is a l_2-morphism and so $\mathbf{F}_2$ is a functor. It is easy to see that $\mathbf{F}_2$ is left adjoint to the $\mathbf{U}_2$. □

Definition 4.4. *The H_v-group $(H, \circ)$ is called*

(i) *strong if for each $(a,b) \in H^2$, $\{a,b\} \subseteq a \circ b$;*
(ii) *near strong if $a \circ b \subseteq \bigcap_{\{a,b\} \subseteq a' \circ b'} a' \circ b'$.*

The collection such that whose elements are both strong and near strong H_v-groups together with homomorphisms forms a category which is denoted by $\mathbf{SNS - H_v}$.

Example 4.5. *Let $H = \{e,a,b\}$ and the hyperopereation $\circ$ be as follows:*

$\circ$	e	a	b
e	e,a	e,a	e,a,b
a	e,a	a	a,b
b	e,a,b	a,b	b

It is easy to see that $(H, \circ)$ are both strong and near strong H_v-group.

Example 4.6. *Let $H = \{e,a,b\}$ and the hyperopereation $\circ'$ be as follows:*

$\circ'$	e	a	b
e	e,a	e,a	e,a,b
a	e,a	e,a	a,b
b	e,a,b	a,b	a,b

In this case $(H, \circ')$ is a strong H_v-group which is not a hypergroup.

Proposition 4.7. *In $\mathbf{SNS - H_v}$ the morphism $(H, \circ) \xrightarrow{f} (H', \circ')$ is monomorphism if and only if $f : H \longrightarrow H'$ is an injective map.*

Proof. Suppose the monomorphism f and the diagram

$$H'' \underset{h}{\overset{g}{\rightrightarrows}} H \xrightarrow{f} H' ,$$

in **Set** such that $fg = fh$ are given. For each $(a,b) \in H''^2$, $a \circ'' b :\overset{\text{def}}{=} g^{-1}(g(a) \circ g(b)) \cap h^{-1}(h(a) \circ h(b))$. Thus $(H'', \circ'')$ is an object in $\mathbf{SNS}-\mathbf{H_v}$ and g and h are morphisms in $\mathbf{SNS}-\mathbf{H_v}$. Since $fg = fh$ and f is an monomorphism, $g = h$. □

Let $(H,\circ)$ be a H_v-group. Define the geometric space

$$(H, P^2_\circ(H))$$

whose points are elements of H and whose blocks are the hyperproducts as the form $a \circ b$ of elements of H. If $B \in P^2_\circ(H)$, then there exists a 2-tuple $(h_1, h_2) \in H^2$ such that $B = h_1 \circ h_2$.

Proposition 4.8. *The mapping* $\mathbf{SNS}-\mathbf{H_v} \xrightarrow{\mathcal{F}_2} \mathbb{L}_\mathbf{2}-\mathbb{GESP}$ *, which is defined by* $\mathcal{F}_2\big((H,\circ)\big) = (H, P^2_\circ(H))$ *is a functor.*

Proof. Since $(H,\circ)$ is strong, $(H, P^2_\circ(H))$ is an object in $\mathbb{L}_\mathbf{2}-\mathbb{GESP}$. Consider $f : (H,\circ) \longrightarrow (H', \circ')$ is a morphism in $\mathbf{SNS}-\mathbf{H_v}$ and $h'_1 \circ' h'_2$ is an arbitrary block in $P^2_{\circ'}(H')$ such that $\{f(a), f(b)\} \subseteq h'_1 \circ' h'_2$. Therefore $f(a) \circ' f(b) \subseteq h'_1 \circ' h'_2$. Since

$$\bigcap_{\{a,b\}\subseteq h_1 \circ h_2} f(h_1) \circ' f(h_2) \subseteq f(a) \circ' f(b),$$

we have

$$f(\bigcap_{\{a,b\}\subseteq h_1 \circ h_2} h_1 \circ h_2) \subseteq \bigcap_{\{f(a),f(b)\}\subseteq h'_1 \circ' h'_2} h'_1 \circ' h'_2.$$

So $f : (H, P^2_\circ(H)) \longrightarrow (H', P^2_{\circ'}(H'))$ is a morphism in $\mathbb{L}_\mathbf{2}-\mathbb{GESP}$ and hence $\mathcal{F}_2$ is a functor. □

Theorem 4.9. *The mapping* $\mathcal{G}_2 : \mathbb{L}_\mathbf{2}-\mathbb{GESP} \longrightarrow \mathbf{SNS}-\mathbf{H_v}$ *which is defined by*

$$\mathcal{G}_2\big(S, \mathfrak{B}\big) = (S, \star_\mathfrak{B}),$$

where for all $(s,t) \in S^2$, $s \star_\mathfrak{B} t :\overset{def}{=} \bigcap_{\{s,t\}\subseteq B \in \mathfrak{B}} B$ *is a right adjoint to the* $\mathcal{F}_2$.

Proof. It is easy to see that $(S,\star_{\mathfrak{B}})$ is a strong H_v-group. Now let $(a,b)\in S^2$ be given such that $\{s,t\}\subseteq a\star_{\mathfrak{B}} b$. Let $B\in\mathfrak{B}$ be an arbitrary block such that $\{a,b\}\subseteq B$, so $a\star_{\mathfrak{B}} b\subseteq B$. Thus $\{s,t\}\subseteq B$ and hence $s\star_{\mathfrak{B}} t\subseteq B$. Therefore $s\star_{\mathfrak{B}} t\subseteq \bigcap_{\{a,b\}\subseteq B\in\mathfrak{B}} B = a\star_{\mathfrak{B}} b$ and consequently $(S,\star_{\mathfrak{B}})$ is an object in $\mathbf{SNS}-\mathbf{H_v}$. Let $f:(S,\mathfrak{B})\longrightarrow(S',\mathfrak{B}')$ be a morphism in $\mathbb{L}_{\mathbf{2}}-\mathbb{GESP}$. We have

$$\begin{aligned} f(a\star_{\mathfrak{B}} b) &= f(\bigcap_{\{a,b\}\subseteq B\in\mathfrak{B}} B) \\ &\subseteq \bigcap_{\{f(a),f(b)\}\subseteq B'\in\mathfrak{B}'} B' \\ &= f(a)\star_{\mathfrak{B}'} f(b). \end{aligned}$$

Therefore $\mathcal{G}_2$ is a functor.

Now we prove that $\mathcal{F}_2$ is left adjoint to $\mathcal{G}_2$. For this reason we show that there is a natural isomorphism as follows:

$$\theta:\mathbb{L}_{\mathbf{2}}-\mathbb{GESP}\Big(\mathcal{F}_2(H,\circ),(S,\mathfrak{B})\Big)\longrightarrow \mathbf{SNS}-\mathbf{H_v}\Big((H,\circ),\mathcal{G}_2(S,\mathfrak{B})\Big).$$

Let the l_2-morphism $f:(H,P^2_\circ(H))\longrightarrow(S,\mathfrak{B})$ be given. We have

$$\begin{aligned} f(a\circ b) &\subseteq f(\bigcap_{\{a,b\}\subseteq h_1\circ h_2} h_1\circ h_2) \\ &\subseteq \bigcap_{\{f(a),f(b)\}\subseteq B\in\mathfrak{B}} B \\ &= f(a)\star_{\mathfrak{B}} f(b). \end{aligned}$$

Therefore $f:(H,\circ)\longrightarrow(S,\star_{\mathfrak{B}})$ is a morphism in $\mathbf{SNS}-\mathbf{H_v}$.

Conversely let $f:(H,\circ)\longrightarrow(S,\star_{\mathfrak{B}})$ be a morphism in $\mathbf{SNS}-\mathbf{H_v}$. Let $s\in f(\bigcap_{\{h_1,h_2\}\subseteq P\in P^2_\circ(H)} P)$ and $B\in\mathfrak{B}$ such that $\{f(h_1),f(h_2)\}\in B$ be given. Since $\{h_1,h_2\}\subseteq h_1\circ h_2\in P^2_\circ(H)$, $s\in f(h_1\circ h_2)$. Therefore we have $s\in f(h_1)\star_{\mathfrak{B}} f(h_2)=\bigcap_{\{f(h_1),f(h_2)\}\subseteq B\in\mathfrak{B}} B$. Thus $(H,P^2_\circ(H))\xrightarrow{f}(S,\mathfrak{B})$ is a morphism in $\mathbb{L}_{\mathbf{2}}-\mathbb{GESP}$ and hence $\theta(f)\overset{\text{def}}{:=} f$ is a natural isomorphism. □

Remark 4.10. *By Proposition 4.3 and Theorem 4.9 we have the following adjoint pairs*

$$\mathbf{Set} \underset{\mathbf{F}_2}{\overset{\mathbf{U}_2}{\leftrightarrows}} \mathbb{L}_{\mathbf{2}}-\mathbb{GESP} \underset{\mathcal{F}_2}{\overset{\mathcal{G}_2}{\rightleftarrows}} \mathbf{SNS}-\mathbf{H_v}\ .$$

Define $\mathfrak{U} :\overset{def}{=} \mathbf{U}_2 \circ \mathcal{F}_2$ *and* $\mathfrak{F} :\overset{def}{=} \mathcal{G}_2 \circ \mathbf{F}_2$. *Therefore we have the following adjoint pair*

$$\mathbf{SNS}-\mathbf{H_v} \underset{\mathfrak{F}}{\overset{\mathfrak{U}}{\rightleftarrows}} \mathbf{Set}\ .$$

Corollary 4.11. *For every set X, there exists a free object on X in* $\mathbf{SNS}-\mathbf{H_v}$.

5. Relation between the category of hypergroups and the category of geometric hypergroups

The category of hypergroups **HypGrp** is a category whose objects are hypergroups and whose morphisms are homomorphisms. **EHypGrp** is a full subcategory of **HypGrp**, where whose objects are hypergroups with scalar identity.

Let $(H, \circ)$ be a hypergroup. Construct a geometric space $(H, P_\circ(H))$ whose points are elements of H and whose blocks are hyperproducts of elements of H. If $B \in P_\circ(H)$, then there exists a n-tuple $(h_1, h_2, ..., h_n) \in H^n$ such that $B = h_1 \circ h_2 \circ ... \circ h_n$.

Definition 5.1. *(i)* $(H, \circ, \mathfrak{B})$ *is called geometric hypergroup whenever* $(H, \circ)$ *is a hypergroup,* $(H, \mathfrak{B})$ *is a geometric space and for each elements* $(x_1, ..., x_n) \in H^n$, *there exists* $B \in \mathfrak{B}$ *such that* $\prod_{i=1}^{n} x_i \subseteq B$;

(ii) Let the geometric hypergroups $(H, \circ, \mathfrak{B})$ *and* $(H', \circ', \mathfrak{B}')$ *be given. We say* $(H, \circ, \mathfrak{B}) \xrightarrow{f} (H', \circ', \mathfrak{B}')$ *is a l-geometrical homomorphism or for simplicity we say l-geometrical, whenever* $(H, \circ) \xrightarrow{f} (H', \circ')$ *is a homomorphism of hypergroups and* $(H, \mathfrak{B}) \xrightarrow{f} (H', \mathfrak{B}')$ *is a l-morphism.*

The collection of geometric hypergroups together with l-geometrical morphisms forms a category which is denoted by $\mathbf{L}-\mathbf{GeHypGrp}$.

Proposition 5.2. *The mapping* $\mathbb{U} : \mathbf{L-GeHypGrp} \longrightarrow \mathbf{HypGrp}$ *such that for each l-geometrical* $f : (H, \circ, \mathfrak{B}) \longrightarrow (H', \circ', \mathfrak{B}')$ *,*

$$\mathbb{U}(f) :\stackrel{def}{=} f : (H, \circ) \longrightarrow (H', \circ')$$

is a homomorphism of hypergroups defines a functor.

Proposition 5.3. *The mapping* $\mathbb{F} : \mathbf{HypGrp} \longrightarrow \mathbf{L-GeHypGrp}$ *, where for each hypergroup* $(H, \circ)$*,* $\mathbb{F}\big((H, \circ)\big) :\stackrel{def}{=} (H, \circ, P_\circ(H))$ *is a geometric hypergroup defines a functor which is a left adjoint to* $\mathbb{U}$*.*

Proof. Let $(H, \circ) \xrightarrow{f} (H', \circ')$ be an arbitrary homomorphism of hypergroups and $B \in P_\circ(H)$ be given. So $B = \prod_{i=1}^{n} x_i$ and hence

$$f(B) \subseteq \prod_{i=1}^{n} f(x_i) \in P_{\circ'}(H').$$

Thus $f : (H, \circ, \mathfrak{B}) \longrightarrow (H', \circ', \mathfrak{B}')$ is a morphism in $\mathbf{L-GeHypGrp}$. Therefore $\mathbb{F}$ is a functor and it is left adjoint to the $\mathbb{U}$. □

Definition 5.4. *(i) The geometric hypergroup* $(H, \circ, \mathfrak{B})$ *is called I-geometric hypergroup or for simplicity is called IG-hypergroup, whenever* $(H, \circ)$ *is a hypergroup with scaler identity and for each non-empty family* $\{B_i\}_{i \in I}$ *of elements of* $\mathfrak{B}$*, we have* $\bigcap_{i \in I} B_i \in \mathfrak{B}$*;*

(ii) The l-geometrical $(H, \circ, \mathfrak{B}) \xrightarrow{f} (H', \circ', \mathfrak{B}')$ *of IG-hypergroups is called LI-geometrical homomorphism or for simplicity is called LI-geometrical, whenever* $B \in \mathfrak{B}'$ *and* $\prod_{i=1}^{n} f(x_i) \subseteq B'$*, then there exists* $B \in \mathfrak{B}$ *such that* $\prod_{i=1}^{n} x_i \subseteq B$ *and* $f(B) \subseteq B'$*.*

The collection of *IG*-hypergroups together with *LI*-geometrical morphisms forms a category. We denoted it by $\mathbf{LI-GeHypGrp}$. We have the following theorem.

Theorem 5.5. *The mapping* $\mathbb{G} : \mathbf{LI-GeHypGrp} \longrightarrow \mathbf{EHypGrp}$ *, where for each LI-geometrical* $f : (H, \circ, \mathfrak{B}) \longrightarrow (H', \circ', \mathfrak{B}')$ *,*

$$\mathbb{G}(f) :\stackrel{def}{=} f : (H, \circ) \longrightarrow (H', \circ')$$

is homomorphism of hypergroups defines a functor which is topological.

Proof. Let $\mathcal{U} = ((H, \circ) \xrightarrow{f_i} \mathbb{G}(H_i, \circ_i, \mathfrak{B}_{\mathfrak{i}}))_{i\in I}$ be a $\mathbb{G}$-structured source. Define $\mathfrak{B} :\overset{\text{def}}{=} \{B{\subseteq}H \mid B = \bigcap\limits_{i\in I} f_i^{-1}(B_i),\ \text{where } B_i \in \mathfrak{B}_{\mathfrak{i}} \text{ for all } i \in I\}$. Let $\{B_\gamma\}_{\gamma\in\Gamma}$ be a non-empty family of elements of $\mathfrak{B}$, so $B_\gamma = \bigcap\limits_{i\in I} B_{i\gamma}$, where $B_{i\gamma} \in \mathfrak{B}_{\mathfrak{i}}$, for all $i \in I$. Thus $\bigcap\limits_{\gamma\in\Gamma} B_\gamma \in \mathfrak{B}$. Let $\prod\limits_{n=1}^{m} x_n{\subseteq}H$ be given. Since $(H_i, \circ_i, \mathfrak{B}_{\mathfrak{i}})$ is I-geometric hypergroups, for all $i \in I$, there exists $B_i \in \mathfrak{B}_{\mathfrak{i}}$ such that $\prod\limits_{n=1}^{m} f(x_n){\subseteq}B_i$. Therefore $\prod\limits_{n=1}^{m} x_n{\subseteq} \bigcap\limits_{i\in I} f_i^{-1}(B_i) \in \mathfrak{B}$ and hence $(H, \circ, \mathfrak{B})$ is an I-geometric hypergroups.

Now we show that

$$\mathcal{S} = \big(\ (H, \circ, \mathfrak{B}) \xrightarrow{f_i} (H_i, \circ_i, \mathfrak{B}_{\mathfrak{i}})\ \big)_{i\in I}$$

is a $\mathbb{G}$-initial source. To do this let $i \in I$ be fixed and $\prod\limits_{n=1}^{m} f_i(x_n){\subseteq}B_i$, where $B_i \in \mathfrak{B}_{\mathfrak{i}}$. For all $j \in I$ such that $j \neq i$ there exists $B_j \in \mathfrak{B}_{\mathfrak{j}}$ such that $\prod\limits_{n=1}^{m} f_j(x_n){\subseteq}B_j$. Define

$$D_j = \begin{cases} B_i, & \text{if } j = i; \\ B_j, & \text{if } j \neq i. \end{cases}$$

Therefore $\prod\limits_{n=1}^{m} x_n{\subseteq} \bigcap\limits_{j\in I} f_j^{-1}(D_j) \in \mathfrak{B}$ and $f_i(\bigcap\limits_{j\in I} f_j^{-1}(D_j)){\subseteq}f_i f_i^{-1}(B_i){\subseteq}B_i$. Thus f_i is a LI-geometrical. Let

$$\mathcal{T} = \big(\ (H', \circ', \mathfrak{B}') \xrightarrow{g_i} (H_i, \circ_i, \mathfrak{B}_{\mathfrak{i}})\ \big)_{i\in I},$$

be a $\mathbb{G}$-structured source and $h : (H', \circ') \longrightarrow (H, \circ)$ be a homomorphism of hypergroups such that $\mathbb{G}(\mathcal{S})h = \mathbb{G}(\mathcal{T})$. Let $B' \in \mathfrak{B}'$. Thus for each $i \in I$ there exists $B_i \in \mathfrak{B}_{\mathfrak{i}}$ such that $g_i(B'){\subseteq}B_i$ and hence $B'{\subseteq} \bigcap\limits_{i\in I} g_i^{-1}(B_i) = h^{-1}(\bigcap\limits_{i\in I} f_i^{-1}(B_i))$. So $h(B'){\subseteq} \bigcap\limits_{i\in I} f_i^{-1}(B_i) \in \mathfrak{B}$. Now let $\prod\limits_{n=1}^{m} h(x_n){\subseteq}B$, where $B \in \mathfrak{B}$. Therefore $B = \bigcap\limits_{i\in I} f_i^{-1}(B_i)$ and hence $\prod\limits_{n=1}^{m} g_i(x_n){\subseteq}B_i$, for all $i \in I$. Thus for all $i \in I$, there exists $B'_i \in \mathfrak{B}'$

such that $\prod_{n=1}^{m} x_n \subseteq B'_i$ and $g(B'_i) \subseteq B_i$. Therefore $\prod_{n=1}^{m} x_n \subseteq \bigcap_{i \in I} B'_i \in \mathfrak{B}'$ and $h(\bigcap_{i \in I} B'_i) \subseteq \bigcap_{i \in I} f_i^{-1}(B_i) = B$ and hence

$$h : (H', \circ', \mathfrak{B}') \longrightarrow (H, \circ, \mathfrak{B})$$

is a LI-geometrical. So $\mathcal{S}$ is a $\mathbb{G}$-initial source and $\mathbb{G}(\mathcal{S}) = \mathcal{U}$.

Now let $\mathcal{S}' = \big((G, \star, \mathfrak{D}) \xrightarrow{\bar{f}_i} (H_i, \circ_i, \mathfrak{B}_i) \big)_{i \in I}$ be another $\mathbb{G}$-initial source such that $\mathbb{G}(\mathcal{S}') = \mathcal{U}$. Thus for all $i \in I$, $\bar{f}_i = f_i$ and $(G, \star) = (H, \circ)$. We need to show that $(G, \star, \mathfrak{D}) = (H, \circ, \mathfrak{B})$. For this reason consider the identity homomorphism

$$id : \mathbb{G}(H, \circ, \mathfrak{B}) \longrightarrow \mathbb{G}(H, \circ, \mathfrak{D}) \ .$$

Since $\mathcal{S}'$ is a $\mathbb{G}$-initial source, there exists a unique LI-geometrical $s : (H, \circ, \mathfrak{B}) \longrightarrow (H, \circ, \mathfrak{D})$ such that $\mathbb{G}(s) = id$ and hence $s = id$. Suppose $D \in \mathfrak{D}$ and $x \in h$ such that $x \in D$ are given. Let e be the scalar identity of H. So $x \circ e \subseteq D$. Since $s(x) \circ s(e) = x \circ e \subseteq D$, there exists $B \in \mathfrak{B}$ such that $x \circ e \subseteq B$ and $s(B) \subseteq D$. Therefore $x \in B \subseteq D$ and hence $(H, \mathfrak{B})$ is finer than $(H, \mathfrak{D})$. Similarly we can show that $(H, \mathfrak{D})$ is finer than $(H, \mathfrak{B})$. Therefore $(G, \star, \mathfrak{D}) = (H, \circ, \mathfrak{B})$ and the proof is complete. □

6. **Conclusions**

In this paper we have extended some notions of categories to hypergroup theory. We construct two categories of $\mathbb{U} - \mathbb{GESP}$ and $\mathbb{L} - \mathbb{GESP}$ from geometric spaces then investigate some properties of the two categories. The category of geometric hypergroups was introduced and studied its relation with the category of hypergroup. It seems interesting to define the notion geometric hyperrings and extend some notion of categories to hyperrings.

Acknowledgments

The authors wish to thank the referee for giving them many valuable comments and useful suggestions which helps to improve the presentation of this paper. This research has been supported financially by Mahani Mathematical Research Center.

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