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CAT^1-POLYGROUPS AND PULLBACK CAT^1-POLYGROUPS

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Abstract. In this paper, we give the notions of crossed polymodule and cat^1-polygroup as a generalization of Loday's definition. Then, we define the pullback cat^1-polygroup and we obtain some results in this respect. Specially, we prove that by a pullback cat^1-polygroup we can obtain a cat^1-group.

Keywords: polygroup, crossed polymodule, cat^1-group, cat^1-polygroup, pullback cat^1-polygroup.
MSC(2010): Primary: 20N20; Secondary: 18D35.

1. Introduction

Crossed module was presented by Whitehead in [24]. So many applications of crossed module have been made by mathematicians. A very important application of crossed module is cat^1-group structure. Loday showed that the category of crossed module is equivalent to the category of cat^1-group in [21]. This application gave the new direction to crossed module. So many applications of cat^1-groups have been found by several mathematicians. After defining cat^1-group structure mathematicians have tried to study these categories. Important calculation examples of these categories were given by Brown and Wensley in [6] and [7]. The other important application of crossed module is defining pullback crossed module. Pullback crossed module was defined by Brown and Wensley in [6] and [7]. They gave many examples and applications of pullback crossed module in their work. Other cat^1-groups application is

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Pullback cat^1-group which was defined by Alp using the equivalence between the category of crossed module and the category of cat^1-groups in [2]. GAP [17] program calculations of these categories were presented by Alp and Wensley in [3]. Crossed polymodule and its application derivation and actor crossed module were presented by Alp and Davvaz. In this paper, we use the same idea to define cat^1-polygroups and pullback cat^1-polygroups in Loday and Alp's way. We study the connections between crossed polymodules and cat^1-polygroups. We present some basic definitions and results of polygroups and crossed polymodules in Section 2. In Section 3, we give the definition of cat^1-polygroup and some properties of cat^1-polygroups. In the last section, we define the concept of pullback cat^1-polygroup and we obtain some results in this respect. Specially, we prove that by a pullback cat^1-polygroup we can obtain a cat^1-group.

2. **Polygroups and crossed polymodules**

The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many areas, such as geometry, lattice theory, combinatorics and color schemes. There exists a rich bibliography: publications appeared within 2012 can be found in "Polygroup Theory and Related Systems" by Davvaz [12]. This book contains the principal definitions endowed with examples and the basic results of the theory. Applications of hypergroups appear in special subclasses like polygroups that they were studied by Comer [8], also see [12, 13, 14]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup. We recall the following definition from [8]. A *polygroup* is a multi-valued system $\mathcal{M} =< P, \circ, e, (\ )^{-1} >$, with $e \in P$, $(\ )^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all x, y, z in P: (1) $(x \circ y) \circ z = x \circ (y \circ z)$, (2) $e \circ x = x \circ e = x$, (3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$. In this definition, $\mathcal{P}^*(P)$ is the set of all non-empty subsets of P, and if $x \in P$ and A, B are non-empty subsets of P, then $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$. The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$ and $(x^{-1})^{-1} = x$. In the rest of this section we present the facts about polygroups that underlie the subsequent material. For further discussion of polygroups, we refer the readers to Davvaz's book [12]. Many important examples of polygroups are collected in [12] such as Double coset algebra, Prenowitz algebras, Conjugacy class polygroups, Character polygroups, Extension of polygroups, and Chromatic polygroups. Clearly, every group is a polygroup. If K is a non-empty subset of P, then K is called a *subpolygroup* of P if $e \in K$ and $< K, \circ, e, (\ )^{-1} >$ is a polygroup. The subpolygroup N of P is said to be *normal* in P if $a^{-1} \circ N \circ a \subseteq N$, for every $a \in P$. There are several kinds of homomorphisms between polygroups [12]. In this paper, we apply only the notion of strong homomorphism. Let $< P, \circ, e, (\ )^{-1} >$ and $< P', \star, e, (\ )^{-1} >$ be two polygroups. A mapping ϕ from P into P' is said to be a *strong homomorphism* if $\phi(e) = e$ and for all $a, b \in P$, $\phi(a \circ b) = \phi(a) \star \phi(b)$, for all $a, b \in P$. A strong homomorphism ϕ is said to be an *isomorphism* if ϕ is one to one and onto. Two polygroups P and P' are said to be *isomorphic* if there is an isomorphism from P to P'. The defining condition for a strong homomorphism is also valid for sets, i.e., if A, B are non-empty subsets of P, then it follows that $f(A \circ B) = f(A) \star f(B)$. By using the concept of generalized permutation, in [10], Davvaz defined permutation polygroups and action of a polygroup on a set. For the definition of crossed polymodule, we need the notion of polygroup action.

Definition 2.1. [10] Let $\mathcal{P} =< P, \circ, e, (\ )^{-1} >$ be a polygroup and Ω be a non-empty set. A map $\alpha : P \times \Omega \to \mathcal{P}^*(\Omega)$, where $\alpha(g, \omega) := {}^{g}\omega$ is called a *(left) polygroup action* on Ω if the following axioms hold:

(1) ${}^{e}\omega = \omega$,
(2) ${}^{h}({}^{g}\omega) = {}^{h\circ g}\omega$, where ${}^{g}A = \bigcup\limits_{a \in A} {}^{g}a$ and ${}^{B}\omega = \bigcup\limits_{b \in B} {}^{b}\omega$, for all $A \subseteq \Omega$ and $B \subseteq P$,
(3) $\bigcup\limits_{\omega \in \Omega} {}^{g}\omega = \Omega$,
(4) for all $g \in P$, $a \in {}^{g}b \Rightarrow b \in {}^{g^{-1}}a$.

Example 2.2. Suppose that $< P, \circ, e, (\ )^{-1} >$ is a polygroup. Then, P acts on itself by conjugation. Indeed, if we consider the map $\alpha : P \times P \to \mathcal{P}^*(P)$ by $\alpha(g, x) = {}^{g}x := g \circ x \circ g^{-1}$, then

(1) ${}^{e}x = x$,
(2) ${}^{h}({}^{g}x) = {}^{h}(g \circ x \circ g^{-1}) = h \circ g \circ x \circ g^{-1} \circ h^{-1} = (h \circ g) \circ x \circ (h \circ g)^{-1} = \bigcup\limits_{b \in h \circ g} (b \circ x \circ b^{-1}) = \bigcup\limits_{b \in h \circ g} {}^{b}x = {}^{h\circ g}x$,

(3) $\bigcup\limits_{x\in P} {}^{g}x = \bigcup\limits_{x\in P} g\circ x\circ g^{-1} = P$,
(4) if $a \in {}^{g}b = g\circ b\circ g^{-1}$, then $g \in a\circ g\circ b^{-1}$ and hence $b^{-1} \in g^{-1}\circ a^{-1}\circ g$. This implies that $b \in g^{-1}\circ a\circ g$.

Note that the above definition is a generalization of the group action. Let G be a group and Ω be a non-empty set. A *(left) group action* is a binary operator from $G\times\Omega$ to Ω that satisfies the following two axioms: ${}^{gh}\omega = {}^{g}({}^{h}\omega)$ and ${}^{e}\omega = \omega$, for all $g,h \in G$ and $\omega \in \Omega$. Now, we present the notion of crossed polymodule and main results about fundamental relation on polygroups and fundamental crossed polymodule..

Definition 2.3. A *crossed polymodule* $\mathcal{X} = (C,P,\partial,\alpha)$ consists of polygroups $< C,\star,e,(\ )^{-1} >$ and $< P,\circ,e,(\ )^{-1} >$ together with a strong homomorphism $\partial : C \to P$ and a (left) action $\alpha : P\times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:
(1) $\partial({}^{p}c) = p\circ\partial(c)\circ p^{-1}$, for all $c \in C$ and $p \in P$,
(2) ${}^{\partial(c)}c' = c\star c'\star c^{-1}$, for all $c,c' \in C$.

When we wish to emphasize the codomain P, we call $\mathcal{X}$ a *crossed P-polymodule*. The strong homomorphism $\partial : C \to P$ is called the *boundary* homomorphism.

Example 2.4. A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In particular, for any polygroup P the identity map $Id_P : P \to P$ is a crossed polymodule with the action of P on itself by conjugation. Indeed, there are two canonical ways in which a polygroup P may be regarded as a crossed polymodule: via the identity map or via the inclusion of the trivial subpolygroup.

Example 2.5. If C is a P-polymodule, then there is a well defined action α of P on C. This together with the zero homomorphism yields a crossed polymodule $(C,P,0,\alpha)$.

Example 2.6. The direct product of $\mathcal{X}_1\times\mathcal{X}_2$ of two crossed polymodules has source $C_1\times C_2$, range $P_1\times P_2$ and boundary homomorphism $\partial_1\times\partial_2$ with $P_1\times P_2$ acting obviously on $C_1\times C_2$.

Note that the above definition is a generalization of the notion of crossed module. We recall that a *crossed module* $X = (M,G,\partial,\tau)$ consists of groups M and G together with a homomorphism $\partial : M \to G$ and a (left) action $\tau : G\times M \to M$ on M, satisfying the conditions:

$\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$, $g \in G$, and ${}^{\partial(m)}m' = mm'm^{-1}$, for all $m, m' \in M$.

Theorem 2.7. *Every crossed module is a crossed polymodule.*

Proof. Since every group is a polygroup, the proof is straightforward. $\square$

Definition 2.8. Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule and $\iota : Q \to P$ be a morphism of polygroups. Then $\iota^{\bullet}\mathcal{X} = (\iota^{\bullet}C, Q, \partial^{\bullet}, \alpha^{\bullet})$ is the pullback of $\mathcal{X}$ by ι, where $\iota^{\bullet}C = \{(q,c) \in Q \times C \mid \iota(q) = \partial(c)\}$ and $\partial^{\bullet}(q,c) = q$. The polygroup action of Q on $\iota^{\bullet}C$ is given by

$$
{}^{q}(q_1, c) \quad = \quad \{(x,y) \mid (x,y) \in (q \circ q_1 \circ q^{-1}, {}^{\iota q}c)\}.
$$

$$
\begin{tikzcd}
\iota^{\bullet}C \arrow[r, "h"] \arrow[d, "\partial^{\bullet}"'] & C \arrow[d, "\partial"] \\
Q \arrow[r, "\iota"'] & P
\end{tikzcd}
$$

Theorem 2.9. $\iota^{\bullet}\mathcal{X} = (\iota^{\bullet}C, Q, \partial^{\bullet}, \alpha^{\bullet})$ *is a crossed polymodule.*

Proof. The verification of crossed polymodule axioms is similar to the crossed module axioms in [6]. $\square$

Let $< P, \circ, e, (\ )^{-1} >$ be a polygroup. We define the relation β_P^* as the smallest equivalence relation on P such that the quotient P/β_P^*, the set of all equivalence classes, is a group. In this case β_P^* is called the *fundamental equivalence relation* on P and P/β_P^* is called the *fundamental group*. The product $\odot$ in P/β_P^* is defined as follows: $\beta_P^*(x) \odot \beta_P^*(y) = \beta_P^*(z)$, for all $z \in \beta_P^*(x) \circ \beta^*(y)$. This relation is introduced by Koskas [18] and studied mainly by Corsini [9], Leoreanu-Fotea et al. [19, 20] and Freni [15, 16] concerning hypergroups, Vougiouklis [23] for H_v-groups, Davvaz for polygroups [11, 22], and many others. We consider the relation β_P as follows:

$$
x \ \beta_P \ y \ \Leftrightarrow \ \text{there exist } z_1, \ldots z_n \text{ such that } \{x,y\} \subseteq \prod_{i=1}^{n} z_i.
$$

Freni in [15] proved that for hypergroups $\beta = \beta^*$. Since polygroups are certain subclass of hypergroups, we have $\beta_P^* = \beta_P$. The kernel of the *canonical map* $\varphi_P : P \longrightarrow P/\beta_P^*$ is called the *core* of P and is denoted by ω_P. Here we denote by ω_P the unit of P/β_P^*. It is easy to prove the following statements: $\omega_P = \beta_P^*(e)$ and $\beta_P^*(x)^{-1} = \beta_P^*(x^{-1})$, for all $x \in P$.

Lemma 2.10. [9] *ω_P is a subpolygroup of P.*

Lemma 2.11. [1] *Let ω_P, ω_Q and $\omega_{P\times Q}$ be the cores of P, Q and $P\times Q$, respectively. Then, $\omega_{P\times Q} = \omega_P \times \omega_Q$.*

Throughout the paper, for the polygroupos $< P, \circ, e, (\ )^{-1} >$, $< C, \star, e, (\ )^{-1} >$ and $< Q, \cdot, e, (\ )^{-1} >$, we denote the binary operations of the fundamental groups P/β_P^*, C/β_C^* and Q/β_Q^* by $\odot$, $\otimes$ and $\oslash$, respectively.

Proposition 2.12. [4] *Let $< C, \star, e, (\ )^{-1} >$ and $< P, \circ, e, (\ )^{-1} >$ be two polygroups and let $\partial : C \to P$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta_C^* \to P/\beta_P^*$ by setting*

$$\mathcal{D}(\beta_C^*(c)) = \beta_P^*(\partial(c)),\ \text{for all } c \in C.$$

We say the action of P on C is *productive*, if for all $c \in C$ and $p \in P$ there exist $c_1, \ldots, c_n$ in C such that ${}^{p}c = c_1 \star \ldots \star c_n$.

Example 2.13. The action defined in Example 2.2 is productive.

Let $< C, \star, e, (\ )^{-1} >$ and $< P, \circ, e, (\ )^{-1} >$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : P/\beta_P^* \times P/\beta_C^* \to \mathcal{P}^*(P/\beta_C^*)$ as follows:

$$\psi(\beta_P^*(p), \beta_C^*(c)) = \{\beta_C^*(x) \mid x \in \bigcup_{\substack{y \in \beta_C^*(c) \\ z \in \beta_P^*(p)}} {}^{z}y\}.$$

By definition of β_C^*, since the action of P on C is productive, we conclude that $\psi(\beta_P^*(p), \beta_C^*(c)$ is singleton, i.e., we have

$$\psi : P/\beta_P^* \times P/\beta_C^* \to P/\beta_C^*,$$
$$\psi(\beta_P^*(p), \beta_C^*(c)) = \beta_C^*(x),\ \text{for all } x \in \bigcup_{\substack{y \in \beta_C^*(c) \\ z \in \beta_P^*(p)}} {}^{z}y.$$

We denote $\psi(\beta_P^*(p), \beta_C^*(c)) = {}^{[\beta_P^*(p)]}[\beta_C^*(c)]$.

Proposition 2.14. [4] *Let $< C, \star, e, (\ )^{-1} >$ and $< P, \circ, e, (\ )^{-1} >$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. Then, ψ is an action of the group P/β_P^* on the group P/β_C^*.*

Theorem 2.15. [4] *Let* $\mathcal{X} = (C, P, \partial, \alpha)$ *be a crosed polymodule such that the action of* P *on* C *is productive. Then,* $\mathcal{X}_{\beta^*} = (C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ *is a crossed module.*

3. Cat^1-polygroups

Cat^1-groups are the first in a series of models for homotopy n-types introduced by Loday. According to [21], Loday's definition of a cat^1-group consists of groups G and S, an embedding $k : S \to G$ and epimorphisms $t, h : G \to S$ satisfying (1) $tk = hk = Id_S$, (2) $[kert, kerh] = \{1_G\}$. Now, we give a generalization of Loday's definition. First, we need the following definition of the kernel homomorphism of polygroups. Let $< P, \circ, e, (\ )^{-1} >$ and $< C, \star, e, (\ )^{-1} >$ be two polygroups and $\phi : P \to C$ be a strong homomorphism. The *core-kernel* of ϕ is defined by

$$ker^*\phi = \{x \in P \mid \phi(x) \in \omega_C\}.$$

Definition 3.1. A cat^1-polygroup $\mathcal{C} = (k; t, h : P \to C)$ consists of polygroups P and C, two strong epimorphisms $t, h : P \to C$ and an embedding $k : C \to P$ satisfying

$$\begin{array}{ll} \text{CAT-P-1}: & tk = hk = Id_C, \\ \text{CAT-P-2}: & [x, y] \subseteq w_P, \forall x \in ker^*t, \forall y \in ker^*h, \end{array}$$

where $[x, y] = \{z \mid z \in x \circ y \circ x^{-1} \circ y^{-1}\}$.

The maps t, h are called the *source* and *target*.

Lemma 3.2. *Condition CAT-P-2 is equivalent to, for all* $x, y \in P$,

$$[\beta^*_P(x), \beta^*_P(y)] = w_P = 1_{P/_{\beta^*_P}}.$$

Proof. $[x, y] \subseteq w_P$ iff $x \circ y \circ x^{-1} \circ y^{-1} \subseteq w_P$ iff $\beta^*_P(x \circ y \circ x^{-1} \circ y^{-1}) = w_P$ iff $\beta^*_P(x) \otimes \beta^*_P(y) \otimes \beta^*_P(x^{-1}) \otimes \beta^*_P(y^{-1}) = w_P$ iff $\beta^*_P(x) \otimes \beta^*_P(y) \otimes \beta^*_P(x)^{-1} \otimes \beta^*_P(y)^{-1} = w_P$. □

Theorem 3.3. *A cat*^{1}*-group is a cat*^{1}*-polygroup.*

Proof. If P and C are groups, then $\omega_P = \{e\}$, $ker^*t = kert$ and $ker^*h = kerh$. □

Theorem 3.4. *If* $\mathcal{X} = (C, P, \partial, \alpha)$ *is a crossed polymodule, then* $(k; t, h : P/_{\beta^*_P} \ltimes C/_{\beta^*_C} \longrightarrow P/_{\beta^*_P})$ *is a cat*^{1}*-group.*

Proof. According to Theorem 2.15, we know $(C/_{\beta^*_C}, P/_{\beta^*_P}, \mathcal{D}, \psi)$ is a crossed module. Now, we can consider

$$\begin{tikzcd} P/_{\beta^*_P} \ltimes C/_{\beta^*_C} \arrow[r, shift left, "h"] \arrow[r, shift right, "t"'] & P/_{\beta^*_P} \arrow[l, bend left, "k"] \end{tikzcd}$$

where

$$h(\beta^*_P(p), \beta^*_C(c)) = \mathcal{D}(\beta^*_C(c)) \odot \beta^*_P(p),$$
$$t(\beta^*_P(p), \beta^*_C(c)) = \beta^*_P(p),$$
$$k(\beta^*_P(p)) = (\beta^*_P(p), w_C).$$

Then

$$h_{|_{P/_{\beta^*_P}}} = t_{|_{P/_{\beta^*_P}}} = Id_P$$

and $[kerh, kert] = 1_{P/_{\beta^*_P} \ltimes C/_{\beta^*_C}}$. Therefore we obtain a cat^1-group. □

Lemma 3.5. *For a cat^1-polygroup $\mathcal{C} = (k; t, h : P \to C)$,*

$$P/_{\beta^*_P} \cong \ker t^* \ltimes C/_{\beta^*_C},$$

where $t^* : P/_{\beta^*_P} \to C/_{\beta^*_C}$, $t^*(\beta^*_P(p)) = \beta^*_C(t(p))$ and $k^* : C/_{\beta^*_C} \to P/_{\beta^*_P}, k^*(\beta^*_C(c)) = \beta^*_P(k(c))$

Proof. We define $f : P/_{\beta^*_P} \to kert^* \ltimes C/_{\beta^*_C}$ by

$$f(\beta^*_P(p)) = (k^*t^*(\beta^*_P(p)) \otimes \beta^*_P(p), t^*(\beta^*_P(p)))$$

and $g : kert^* \ltimes C/_{\beta^*_C} \to P/_{\beta^*_P}$ by

$$g(\beta^*_P(p), \beta^*_C(c)) = k^*(\beta^*_P(p)) \otimes \beta^*_C(c)).$$

It is not difficult to see that f, g are homomorphisms and f is the inverse of g. □

Note that in the previous lemma, since $kert^* \trianglelefteq P/_{\beta^*_P}$ and $k^*(C/_{\beta^*_C}) \leq P/_{\beta^*_P}$ there is an action of $k^*(C/_{\beta^*_C})$ on $kert^*$ by conjugation. Hence, the semi-direct product $\ker t^* \ltimes C/_{\beta^*_C}$ is defined.

Theorem 3.6. *If $\mathcal{C} = (k; t, h : P \to C)$ a cat^1-polygroup, then by putting $S = kert^*$ and $\mathcal{D} = h^*_{|\, kert^*}$, we obtain a crossed module.*

Proof. The action of $C/_{\beta_C^*}$ on S is conjugation in $P/_{\beta_P^*}$. Now, if $\beta_P^*(x) \in kert^*$ and $\beta_P^*(y) \in kerh^*$, then

$$\beta_P^*(x) = (w_C, \beta_P^*(a)),\ \beta_P^*(y) = (\mathcal{D}(\beta_P^*(b)), \beta_P^*(b^{-1})),$$

for all $\beta_P^*(a), \beta_P^*(b) \in S$. Thus,

$$\begin{aligned}
\beta_P^*(x) \odot \beta_P^*(y) &= (w_C, \beta_P^*(a)) \odot (\mathcal{D}(\beta_P^*(b)), \beta_P^*(b^{-1})) \\
&= (\mathcal{D}(\beta_P^*(b)), {}^{\mathcal{D}(\beta_P^*(b))}\beta_P^*(a) \odot \beta_P^*(b^{-1}))
\end{aligned}$$

$$\begin{aligned}
\beta_P^*(y) \odot \beta_P^*(x) &= (\mathcal{D}(\beta_P^*(b)), \beta_P^*(b^{-1})) \odot (w_C, \beta_P^*(a)) \\
&= (\mathcal{D}(\beta_P^*(b)), {}^{w_C}\beta_P^*(b^{-1}) \odot \beta_P^*(a)) \\
&= (\mathcal{D}(\beta_P^*(b)), \beta_P^*(b^{-1}) \odot \beta_P^*(a))
\end{aligned}$$

Thus, the equality $\beta_P^*(x) \odot \beta_P^*(y) = \beta_P^*(y) \odot \beta_P^*(x)$ is equivalent to ${}^{\mathcal{D}(\beta_P^*(b))}\beta_P^*(a) = \beta_P^*(b^{-1}) \odot \beta_P^*(a) \odot \beta_P^*(b)$. □

Corollary 3.7. The following diagram shows all the results obtained and thus gives their relations.

$$\begin{tikzcd}
\mathrm{Cat}^1-\text{groups} \arrow[rr, "Inc"] \arrow[dr, shift left] & & \mathrm{Cat}^1-\text{polygroups} \arrow[dl, "\varphi_{\beta^*}"] \\
& \text{Crossed modules} \arrow[ul, shift left, "\cong"] \arrow[d, shift left, "Inc"] & \\
& \text{Crossed polymodules} \arrow[u, shift left, "\varphi_{\beta^*}"] &
\end{tikzcd}$$

4. Pullback cat^1-polygroups

In this section, we define the pullback cat^1-polygroup and we obtain some results in this respect. Specially, we prove that by a pullback cat^1-polygroup we can obtain a cat^1-group.

Definition 4.1. A pullback cat^1-polygroup is defined as follows.

$$\begin{tikzcd}
\iota^{\bullet\bullet}P \arrow[r, "h^{\bullet\bullet}"] \arrow[d, "t^{\bullet\bullet}"'] \arrow[dr, "\pi"] & Q \arrow[l, bend right, "k^{\bullet\bullet}"'] \arrow[dr, "\iota"] & \\
Q \arrow[u, bend left, "k^{\bullet\bullet}"] \arrow[dr, "\iota"'] & P \arrow[r, "h"] \arrow[d, "t"] & C \arrow[l, bend right, "k"'] \\
& C \arrow[u, bend left, "k"] &
\end{tikzcd}$$

Let $\mathcal{C} = (k; t, h : P \to C)$ be a cat^1-polygroup and let $\iota : Q \to C$ be a strong homomorphism. Define $\iota^{\bullet\bullet}\mathcal{C} = (k^{\bullet\bullet}; t^{\bullet\bullet}, h^{\bullet\bullet} : \iota^{\bullet\bullet}P \to Q)$ to be the pullback of P, where

$$\iota^{\bullet\bullet}P = \{(q_1, p, q_2) \in Q \times P \times Q \ \mid \ \iota(q_1) = t(p),\ \iota(q_2) = h(p)\},$$

$t^{\bullet\bullet}(q_1, p, q_2) = q_1$, $h^{\bullet\bullet}(q_1, p, q_2) = q_2$ and $k^{\bullet\bullet}(q) = (q, k\iota(q), q)$. Multiplication in $\iota^{\bullet\bullet}P$ is componentwise. The pair (π, ι) is a morphism of cat^1-polygroups, where $\pi : \iota^{\bullet\bullet}P \to P$, $(q_1, p, q_2) \mapsto p$.

Theorem 4.2. *By a pullback cat^1-polygroup, we have a cat^1-polygroup.*

Proof. We verify the cat^1-polygroup axioms. For the first axiom, we have

$$t^{\bullet\bullet}k^{\bullet\bullet}(q) = t^{\bullet\bullet}(q, k\iota(q), q) = q,$$

$$h^{\bullet\bullet}k^{\bullet\bullet}(q) = h^{\bullet\bullet}(q, k\iota(q), q) = q.$$

Thus, $t^{\bullet\bullet}k^{\bullet\bullet} = h^{\bullet\bullet}k^{\bullet\bullet} = Id_Q$ and CAT-P-1 is satisfied.

In order to prove the second condition, suppose that $x = (q'_1, p_1, q_1) \in \ker^* t^{\bullet\bullet}$, $y = (q_2, p_2, q'_2) \in \ker^* h^{\bullet\bullet}$. Then, $t^{\bullet\bullet}(q'_1, p_1, q_1) = q'_1 \in \omega_Q$ and $h^{\bullet\bullet}(q_2, p_2, q'_2) = q'_2 \in \omega_Q$. By Lemma 2.10, ω_Q is a subpolygroup of Q. We show that it is also normal. Suppose that $b \in Q$ and $a \in \omega_Q$ are arbitrary. For each $z \in b \cdot a \cdot b^{-1}$, we have

$$\begin{aligned} \beta_Q^*(z) &= \beta_Q^*(b) \oslash \beta_Q^*(a) \oslash \beta_Q^*(b^{-1}) \\ &= \beta_Q^*(b) \oslash \omega_Q \oslash \beta_Q^*(b^{-1}) \\ &= \beta_Q^*(y) \oslash \beta_Q^*(b^{-1}) \\ &= \beta_Q^*(b \cdot b^{-1}) \\ &= \beta_Q^*(e) = \omega_Q. \end{aligned}$$

So, $z \in \omega_Q$. Therefore, we conclude that

$$q_1' \cdot q_2 \cdot q_1'^{-1} \cdot q_2^{-1} \subseteq \omega_Q \text{ and } q_1 \cdot q_2' \cdot q_1^{-1} \cdot q_2'^{-1} \subseteq \omega_Q.$$

On the other hand, by the definition of $\iota^{\bullet\bullet}$, we obtain

$$\iota(q_1') = t(p_1) \in \iota(\omega_Q) \text{ and } \iota(q_2') = h(p_2) \in \iota(\omega_Q).$$

Now, we show that $\iota(\omega_Q) \subseteq \omega_C$. Since $e \in \omega_Q$, $\iota(e) \in \omega_C$. Now, suppose that there exists $a \in \omega_Q$ such that $\iota(a) \in \omega_C$. Since $a, e \in \omega_Q$, $\beta_C^*(\iota(a)) \neq \omega_C$. On the other hand, $\iota(e) = e \in \omega_C$ and so $\beta_C^*(\iota(e)) = \omega_C$. Thus, $\beta_C^*(\iota(e)) \neq \beta_C^*(\iota(a))$. This implies that $\iota^*(\beta_Q^*(e)) \neq \iota^*(\beta_Q^*(a))$, which is a contradiction. Hence, $t(p_1) \in \omega_C$ and $h(p_2) \in \omega_C$. Thus,

$$p_1 \in ker^*t \text{ and } p_2 \in ker^*h.$$

Now, we have

$$\begin{aligned} &[x, y] = x \boxdot y \boxdot x^{-1} \boxdot y^{-1} \\ &= \{(q, p, q') \mid q \in q_1' \cdot q_2 \cdot q_1'^{-1} \cdot q_2^{-1},\ p \in [p_1, p_2],\ q' \in q_1 \cdot q_2' \cdot q_1^{-1} \cdot q_2'^{-1}\} \\ &\subseteq \omega_Q \times \omega_Q \times \omega_Q. \end{aligned}$$

Therefore, CAT-P-2 is also satisfied. □

Theorem 4.3. If $\iota^\bullet \mathcal{X}$ is the pullback of the crossed polymodule $\mathcal{X}$ over $\iota : Q \to P$ and if $\mathcal{A}, \mathcal{B}$ are the cat^1-groups obtained from $\mathcal{X}, \iota^\bullet \mathcal{X}$ respectively, then $\mathcal{B} \cong \iota^{**}\mathcal{A}$.

Proof.

$$\begin{tikzcd} \iota^\bullet C \arrow[r] \arrow[d, "\partial^\bullet"'] & C \arrow[d, "\partial"] \\ Q \arrow[r, "\iota"'] & P \end{tikzcd} \qquad \begin{tikzcd} \iota^\bullet C/\beta^*_{\iota^\bullet C} \arrow[r] \arrow[d, "\mathcal{D}^\bullet"'] & C/\beta_C^* \arrow[d, "\mathcal{D}"] \\ Q/\beta_Q^* \arrow[r, "\iota^*"'] & P/\beta_P^* \end{tikzcd}$$

Starting with the pullback crossed polymodule $\iota^\bullet\mathcal{X} = (\iota^\bullet, Q, \partial^\bullet, \alpha^\bullet)$, where $\partial^\bullet : \iota^\bullet C \to Q$), the source polygroup of $\mathcal{B}$ is defined as the semi-direct product $Q/\beta^*_Q \ltimes \iota^\bullet C/\beta^*_{\iota^\bullet C}$.

$$\begin{tikzcd}
Q/\beta^*_Q \ltimes \iota^\bullet C/\beta^*_{\iota^\bullet C} \arrow[r] \arrow[d, shift right, "t^\bullet"'] \arrow[d, shift left, "h^\bullet"] & P/\beta^*_P \ltimes C/\beta^*_C \arrow[d, shift right, "t"'] \arrow[d, shift left, "h"] \\
Q/\beta^*_Q \arrow[r, "\iota^*"'] & P/\beta^*_P
\end{tikzcd}$$

The target, source and embedding of $\mathcal{B}$ are respectively given by

$$t^\bullet(\beta^*_Q(q'), \beta^*_{\iota^\bullet C}(q,c)) = \beta^*_Q(q'),$$

$$\begin{aligned}
h^\bullet(\beta^*_Q(q'), \beta^*_{\iota^\bullet C}(q,c)) \quad &= \mathcal{D}^\bullet(\beta^*_{\iota^\bullet C}(q,c)) \oslash \beta^*_Q(q') \\
&= \beta^*_Q(q) \oslash \beta^*_Q(q') \\
&= \beta^*_Q(q \cdot q'),
\end{aligned}$$

$$k^\bullet(\beta^*_Q(q)) = (\beta^*_Q(q), \omega_{\iota^\bullet C}).$$

We then define an isomorphism of cat^1-groups $(\lambda, Id) : \mathcal{B} \to \iota^{\bullet\bullet}\mathcal{A}$,

$$\begin{tikzcd}
Q/\beta^*_Q \ltimes \iota^\bullet C/\beta^*_{\iota^\bullet C} \arrow[r, "\lambda"] \arrow[d, shift right, "t^\bullet"'] \arrow[d, shift left, "h^\bullet"] & \iota^{\bullet\bullet}(P/\beta^*_P \ltimes C/\beta^*_C) \arrow[d, shift right, "t^{\bullet\bullet}"'] \arrow[d, shift left, "h^{\bullet\bullet}"] \\
Q/\beta^*_Q \arrow[r, "Id"'] \arrow[u, bend left=60, "k^\bullet"] & Q/\beta^*_Q \arrow[u, bend right=60, "k^{\bullet\bullet}"']
\end{tikzcd}$$

where

$$\lambda\Big(\beta^*_Q(q'), \beta^*_{\iota^\bullet C}(q,c)\Big) = \Big(\beta^*_Q(q'), (\beta^*_P(\iota(q')), \beta^*_C(c)), \beta^*_Q(q \cdot q')\Big)$$

First note that $\lambda(\beta^*_Q(q'), \beta^*_{\iota^\bullet C}) \in \iota^{\bullet\bullet}(P/\beta^*_P \ltimes C/\beta^*_C)$ because

$$t(\beta^*_P(\iota q'), \beta^*_C(c)) = \beta^*_P(\iota(q')) = \iota^*(\beta^*_Q(q'))$$

and

$$\begin{aligned} h(\beta^*_P(\iota q'), \beta^*_C(c)) &= \mathcal{D}(\beta^*_C(c)) \odot \iota^*(\beta^*_Q(q')) \\ &= \iota^*(\beta^*_Q(q)) \odot \iota^*(\beta^*_Q(q')) \\ &= \iota^*(\beta^*_Q(q) \oslash \beta^*_Q(q'))) \\ &= \iota^*(\beta^*_Q(q \cdot q')). \end{aligned}$$

We verify that λ is a homomorphism as follows:

$$\begin{aligned} &\lambda\Big((\beta^*_Q(q'_1), \beta^*_{\iota\bullet C}(q_1, c_1))(\beta^*_Q(q'_2), \beta^*_{\iota\bullet C}(q_2, c_2))\Big) \\ &= \Big((\beta^*_Q(q'_1 \cdot q'_2), \Big(\iota^*(\beta^*_Q(q'_1 \cdot q'_2)), {}^{[\iota^*(\beta^*_Q(q'_1))]}[\beta^*_C(c_1)]\Big), \beta^*_Q(q'_1 \cdot q \cdot \cdot q'_2 \cdot q_2)\Big) \end{aligned}$$

and

$$\begin{aligned} &\lambda\Big(\beta^*_Q(q'_1), \beta^*_{\iota\bullet C}(q_1, c_1)\Big)\lambda\Big(\beta^*_Q(q'_2), \beta^*_{\iota\bullet C}(q_2, c_2)\Big) \\ &= \Big(\beta^*_Q(q'_1), (\beta^*_P(\iota(q'_1)), \beta^*_C(c_1)), \beta^*_Q(q_1 \cdot q'_1)\Big)\Big(\beta^*_Q(q'_2), (\beta^*_P(\iota(q'_2)), \beta^*_C(c_2)), \\ &\qquad \beta^*_Q(q_2 \cdot q'_2)\Big) \\ &= \Big(\beta^*_Q(q_1) \oslash \beta^*_Q(q_2),\ (\beta^*_P(\iota(q'_1)), \beta^*_C(c_1)) \cdot (\beta^*_P(\iota(q'_2)), \beta^*_C(c_2))), \beta^*_Q(q_1 \cdot q'_1) \\ &\qquad \oslash \beta^*_Q(q_2 \cdot q'_2)\Big) \\ &= \Big(\beta^*_Q(q_1 \cdot q_2),\ (\iota^*(\beta^*_Q(q'_1)), \beta^*_C(c_1)) \cdot (\iota^*(\beta^*_Q(q'_2)), \beta^*_C(c_2))), \\ &\qquad \beta^*_Q(q_1 \cdot q'_1 \cdot q_2 \cdot q'_2)\Big) \\ &= \Big(\beta^*_Q(q_1 \cdot q_2),\ \Big(\iota^*(\beta^*_Q(q'_1)) \odot \iota^*(\beta^*_Q(q'_2)),\ {}^{[\iota^*(\beta^*_Q(q'_1))]}[\beta^*_C(c_1)] \otimes \beta^*_C(c_2)\Big), \\ &\qquad \beta^*_Q(q_1 \cdot q'_1 \cdot q_2 \cdot q'_2)\Big) \end{aligned}$$

The inverse of λ is given by

$$\lambda^{-1}\Big(\beta^*_Q(q_1), \Big(\beta^*_P(p), \beta^*_C(c)\Big), \beta^*_Q(q_2)\Big) = \Big(\beta^*_Q(q_1), \beta^*_Q(q_1^{-1} \cdot q_2), \beta^*_C(c)\Big).$$

Then,

$$
\begin{aligned}
t^{\bullet\bullet}\lambda\Big(\beta^*_Q(q'),\beta^*_{\iota^\bullet C}(q,c)\Big) &= t^{\bullet\bullet}\Big(\beta^*_Q(q'),(\beta^*_P(\iota(q')),\beta^*_C(c)),\beta^*_Q(q\cdot q')\Big)\\
&= \beta^*_Q(q')\\
&= t^{\bullet}\Big(\beta^*_Q(q'),\beta^*_{\iota^\bullet C}(q,c)\Big),
\end{aligned}
$$

$$
\begin{aligned}
h^{\bullet\bullet}\lambda\Big(\beta^*_Q(q'),\beta^*_{\iota^\bullet C}(q,c)\Big) &= h^{\bullet\bullet}\Big(\beta^*_Q(q'),(\beta^*_P(\iota(q')),\beta^*_C(c)),\beta^*_Q(q\cdot q')\Big)\\
&= \beta^*_Q(q\cdot q')\\
&= h^{\bullet}\Big(\beta^*_Q(q'),\beta^*_{\iota^\bullet C}(q,c)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\lambda k^{\bullet}(\beta^*_Q(q)) &= \lambda\Big(\beta^*_Q(q,\ (\omega_Q,\omega_C)\Big)\\
&= \Big(\beta^*_Q(q),\ (\iota^*(\beta^*_Q(q)),\omega_C),\ \beta^*_Q(q)\Big)\\
&= k^{\bullet\bullet}(\beta^*_Q(q)).
\end{aligned}
$$

Therefore, the diagram commutes. □

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