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A STRONG CONVERGENCE THEOREM FOR SOLUTIONS OF ZERO POINT PROBLEMS AND FIXED POINT PROBLEMS

S. Y. CHO, X. QIN AND L. WANG*

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Abstract. Zero point problems of the sum of two monotone mappings and fixed point problems of a strictly pseudocontractive mapping are investigated. A strong convergence theorem for the common solutions of the problems is established in the framework of Hilbert spaces.
Keywords: Fixed point, inverse-strongly monotone mapping, maximal monotone operator, nonexpansive mapping.
MSC(2010): Primary: 47H09; Secondary: 47H10.

1. Introduction

Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two (possibly simpler) nonlinear operators. The central problem is to iteratively find a zero point of the sum of two monotone operators, that is,

$$0 \in (A+B)(x). \tag{1.1}$$

Many problems can be formulated as a problem of the form (1.1). For instance, a stationary solution to the initial value problem of the evolution

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equation

$$\begin{cases} 0 \in Fu + \frac{\partial u}{\partial t}, \\ u_0 = u(0), \end{cases}$$

can be recast as (1.1) when the governing maximal monotone F is of the form $F = A + B$; for more details, see [10] and the references therein. Fixed point theory as an important branch of nonlinear analysis has been applied in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics. The aim of this paper is to investigate zero point problems of the sum of two monotone mappings and fixed point problems of a strictly pseudocontractive mapping in the framework of Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity iterative algorithm with mixed errors is investigated. And, a strong convergence theorem is established. In Section 4, applications of the main results are discussed.

2. **Preliminaries**

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and $Proj_C$ the metric projection from H onto C.

Let $S : C \to H$ be a mapping. We use $Fix(S)$ to stand for the fixed point set of S; that is, $Fix(S) := \{x \in C : x = Sx\}$.

Recall that S is said to be α-contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be κ-strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C.$$

The class of κ-strictly pseudocontractive mappings was introduced by Browder and Petryshyn [2]. Note that the class of κ-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings. That is, S is nonexpansive iff the coefficient $\kappa = 0$.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be ξ-strongly monotone iff there exists a constant $\xi > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \xi\|x - y\|^2, \quad \forall x, y \in C.$$

A is said to be ξ-inverse-strongly monotone iff there exists a constant $\xi > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \xi\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is not hard to see that ξ-inverse-strongly monotone mappings are Lipschitz continuous. Indeed, we have

$$\xi\|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y\rangle \leq \|Ax - Ay\|\|x - y\|.$$

This shows that $\|Ax - Ay\| \leq \frac{1}{\xi}\|x - y\|$. Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{2.1}$$

One can see that the variational inequality (2.1) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (2.1) iff $u \in C$ satisfies the equation

$$u = Proj_C(u - \lambda Au),$$

where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. If A is α-inverse-strongly monotone and $\lambda \in (0, 2\alpha]$, then the mapping $P_C(I - \lambda A)$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\|(I - \lambda A)x - (I - \lambda A)y\|^2\\
&= \|(x - y) - \lambda(Ax - Ay)\|^2\\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2\\
&\leq \|x - y\|^2 - \lambda(2\alpha - \lambda)\|Ax - Ay\|^2.
\end{aligned}$$

This shows that $P_C(I - \lambda A)$ is nonexpansive.

A multivalued operator $B : H \to 2^H$ with the domain $Dom(B) = \{x \in H : Bx \neq \emptyset\}$ and the range $Ran(B) = \{Bx : x \in Dom(B)\}$ is said to be monotone if for $x_1 \in Dom(B)$, $x_2 \in Dom(B)$, $y_1 \in Bx_1$ and $y_2 \in Bx_2$, we have $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. A monotone operator B is said to be maximal if its graph $Graph(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator. Let I

denote the identity operator on H and let $B : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $\lambda > 0$, a nonexpansive single valued mapping $J_\lambda : H \to H$ by $J_\lambda = (I + \lambda B)^{-1}$. It is called the resolven of B. We know that $B^{-1}0 = Fix(J_\lambda)$ for all $\lambda > 0$ and J_λ is firmly nonexpansive; for more details, see [6], [7], [13-15] and [19] and the references therein.

In [9], Kamimura and Takahashi investigated the problem of finding zero points of a maximal monotone operator by considering the following iterative algorithm

$$x_1 \in H, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 1, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a positive sequence, $B : H \to 2^H$ is a maximal monotone and $J_{\lambda_n} = (I+\lambda_n B)^{-1}$. They showed that the sequence $\{x_n\}$ generated in (2.2) converges strongly to some $z \in B^{-1}(0)$ provided that the control sequence satisfies some restrictions. Further, using this result, they also investigated the case that $B = \partial f$, where $f : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function. Convergence theorems are established in the framework of real Hilbert spaces; for more details, see [9].

In [8], Iiduka and Takahashi investigated the problem of finding a common solution of the variational inequality (2.1) and a fixed point problem involving nonexpansive mappings by considering the following iterative algorithm

$$x_1 \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) S Proj_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping. They showed that the sequence $\{x_n\}$ generated in (2.3) converges strongly to some $z \in VI(C, A) \cap Fix(S)$ provided that the control sequence satisfies some restrictions; for more details, see [8].

Recently, Takahahsi, Takahahsi and Toyoda studied zero point problems of the sum of two monotone mappings and fixed point problems of a nonexpansive mapping by considering the following iterative algorithm: $x_1 \in C$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)), \quad \forall n \geq 1, \tag{2.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping. They showed that the sequence $\{x_n\}$ generated in (2.4) converges strongly to some $z \in (A+B)^{-1}(0) \cap Fix(S)$ provided that the control sequence satisfies some restrictions; for more details, see [20].

Motivated by the above results, we investigate zero point problems of the sum of two monotone mappings and fixed point problems of a strictly pseudocontractive mapping. To obtain our main results, we need the following tools.

Recall that a space is said to satisfy Opial's condition [12] if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, where $\rightharpoonup$ denotes the weak convergence, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Indeed, the above inequality is equivalent to the following

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Lemma 2.1. [1] *Let C be a nonempty, closed, and convex subset of H, $A : C \to H$ a mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $F(J_\lambda(I - \lambda A)) = (A+B)^{-1}(0)$.*

Lemma 2.2. [18] *Suppose that H is a real Hilbert space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. [3] *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $x_n - Sx_n \to 0$, then $\bar{x} \in F(S)$.*

Lemma 2.4 [11] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. **Main results**

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*,* $A : C \to H$ *a* ξ*-inverse-strongly monotone mapping,* $S : C \to H$ *a* κ*-strictly pseudocontractive mapping,* $T : C \to H$ *an* α*-contractive mapping and* B *a maximal monotone operator on* H*. Assume that* $\mathcal{F} = Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real number sequences in* $(0,1)$*. Let* $\{\lambda_n\}$ *be a positive real number sequence. Let* $\{x_n\}$ *be a sequence in* C *generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = Proj_C(\alpha_n T x_n + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) Proj_C(\gamma_n y_n + (1-\gamma_n) S y_n), \quad \forall n \geq 1, \end{cases}$$

where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$*. Assume that the sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *satisfy the following restrictions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < a \leq \beta_n \leq b < 1$;

(c) $\kappa \leq \gamma_n \leq c < 1$, $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(d) $0 < d \leq \lambda_n \leq e < 2\xi$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

where a, b, c, d *and* e *are some real numbers. Then the sequence* $\{x_n\}$ *converges strongly to* $x = Proj_{\mathcal{F}} Tx$*.*

Proof. First, we show that $\{x_n\}$ is bounded. Notice that $I - \lambda_n A$ is nonexpansive. Indeed, we have

$$\begin{aligned} &\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 \\ &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda_n \langle x-y, Ax-Ay\rangle + {\lambda_n}^2 \|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - \lambda_n(2\xi - \lambda_n)\|Ax-Ay\|^2. \end{aligned}$$

In view of the restriction (d), we find that $I - \lambda_n A$ is nonexpansive. Fixing $p \in \mathcal{F}$, we find from Lemma 2.1 that

$$p = Sp = J_{\lambda_n}(p - \lambda_n Ap).$$

Put $z_n = J_{\lambda_n}(x_n - \lambda_n Ax_n)$. Since J_{λ_n} and $I - \lambda_n A$ are nonexpansive, we have

$$\begin{aligned} \|z_n - p\| &\leq \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.1}$$

Define $G_n x = Proj_C(\gamma_n x + (1-\gamma_n)Sx)$, $\forall x \in C$. It follows from the restriction (c) that

$$\begin{aligned} &\|G_n y_n - p\|^2 \\ &\leq \|(\gamma_n y_n + (1-\gamma_n)Sy_n) - (\gamma_n p + (1-\gamma_n)Sp)\|^2 \\ &\leq \|\gamma_n(y_n - p) + (1-\gamma_n)(Sy_n - Sp)\|^2 \\ &= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|Sy_n - Sp\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|(y_n - p) - (Sy_n - Sp)\|^2 \\ &\leq \gamma_n\|y_n - p\|^2 + (1-\gamma_n)(\|y_n - p\|^2 \\ &\quad + \kappa\|(y_n - p) - (Sy_n - Sp)\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|(y_n - p) - (Sy_n - Sp)\|^2 \\ &= \|y_n - p\|^2 - (1-\gamma_n)(\gamma_n - \kappa)\|(y_n - p) - (Sy_n - Sp)\|^2 \\ &\leq \|y_n - p\|^2. \end{aligned} \tag{3.2}$$

Notice that

$$\begin{aligned} \|y_n - p\| &= \|Proj_C(\alpha_n Tx_n + (1-\alpha_n)z_n) - p\| \\ &\leq \alpha_n\|Tx_n - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq (1 - \alpha_n(1-\alpha))\|x_n - p\| + \alpha_n\|Tp - p\|. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain that

$$\|G_n y_n - p\| \leq (1 - \alpha_n(1-\alpha))\|x_n - p\| + \alpha_n\|Tp - p\|. \tag{3.4}$$

This in turn implies that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|G_n y_n - p\| \\ &\leq \big(1 - \alpha_n(1-\alpha)(1-\beta_n)\big)\|x_n - p\| \\ &\quad + \alpha_n(1-\beta_n)\|Tp - p\|. \end{aligned}$$

Putting $M = \max\{\|x_1 - p\|, \frac{\|Tp-p\|}{1-\alpha}\}$, we find that $\|x_n - p\| \le M$ for all $n \ge 1$. Indeed, it is clear that $\|x_1 - p\| \le M$. Suppose that $\|x_m - p\| \le M$ for some positive integer m. It follows that

$$\begin{aligned}\|x_{m+1} - p\| &\le \big(1 - \alpha_m(1-\alpha)(1-\beta_m)\big)\|x_m - p\| \\ &\quad + \alpha_m(1-\beta_n)\|Tp - p\| \\ &\le \big(1 - \alpha_m(1-\alpha)(1-\beta_m)\big)M \\ &\quad + \alpha_m(1-\beta_n)(1-\alpha)M \\ &= M.\end{aligned}$$

This competes the proof that $\{x_n\}$ is bounded. Notice that

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\ &\quad + \|J_{\lambda_{n+1}}(x_n - \lambda_n Ax_n) - J_{\lambda_n}(x_n - \lambda_n Ax_n)\| \\ &\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + \|J_{\lambda_{n+1}}(x_n - \lambda_n Ax_n) - J_{\lambda_n}(x_n - \lambda_n Ax_n)\|,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|(\alpha_{n+1}Tx_{n+1} + (1-\alpha_{n+1})z_{n+1}) \\ &\quad - (\alpha_n Tx_n + (1-\alpha_n)z_n)\| \\ &\le \alpha_{n+1}\|Tx_{n+1} - Tx_n\| + (1-\alpha_{n+1})\|z_{n+1} - z_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|z_n - Tx_n\| \\ &\le \alpha_{n+1}\alpha\|x_{n+1} - x_n\| + (1-\alpha_{n+1})\|z_{n+1} - z_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|z_n - Tx_n\|.\end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6), we arrive at

$$\begin{aligned}&\|y_{n+1} - y_n\| \\ &\le (1 - \alpha_{n+1}(1-\alpha))\|x_{n+1} - x_n\| \\ &\quad + (1-\alpha_{n+1})|\lambda_{n+1} - \lambda_n|\|Ax_n\| \\ &\quad + (1-\alpha_{n+1})\|J_{\lambda_{n+1}}(x_n - \lambda_n Ax_n) - J_{\lambda_n}(x_n - \lambda_n Ax_n)\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|z_n - Tx_n\|.\end{aligned} \tag{3.7}$$

Put $u_n = x_n - \lambda_n Ax_n$. Since B is monotone, we see that

$$\langle J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n, \frac{u_n - J_{\lambda_{n+1}}u_n}{\lambda_{n+1}} - \frac{u_n - J_{\lambda_n}u_n}{\lambda_n}\rangle \ge 0.$$

It follows that

$$\langle J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n, (1-\frac{\lambda_{n+1}}{\lambda_n})(u_n - J_{\lambda_n}u_n)\rangle \geq \|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\|^2.$$

This in turn implies that

$$\|J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n\| \leq \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_n}\|u_n - J_{\lambda_n}u_n\|. \tag{3.8}$$

Substituting (3.8) into (3.7), we find that

$$\begin{aligned}\|y_{n+1}-y_n\| \leq\ & (1-\alpha_{n+1}(1-\alpha))\|x_{n+1}-x_n\| \\ & + (1-\alpha_{n+1})|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ & + (1-\alpha_{n+1})\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_n}\|u_n - J_{\lambda_n}u_n\| \\ & + |\alpha_{n+1}-\alpha_n|\|z_n - Tx_n\|.\end{aligned} \tag{3.9}$$

On the other hand, we see that the mappign $G_n : C \to C$ is nonexpansive. Indeed, we see from the restriction (c) that

$$\begin{aligned}\|G_nx - G_ny\|^2 &\leq \|\gamma_n(x-y) + (1-\gamma_n)(Sx-Sy)\|^2 \\ &= \gamma_n\|x-y\|^2 + (1-\gamma_n)\|Sx-Sy\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|(x-y)-(Sx-Sy)\|^2 \\ &\leq \gamma_n\|x-y\|^2 + (1-\gamma_n)(\|x-y\|^2 \\ &\quad + \kappa\|(x-y)-(Sx-Sy)\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|(x-y)-(Sx-Sy)\|^2 \\ &= \|x-y\|^2 - (1-\gamma_n)(\gamma_n-\kappa)\|(x-y)-(Sx-Sy)\|^2 \\ &\leq \|x-y\|^2, \quad \forall x,y \in C.\end{aligned}$$

This shows that G_n is nonexpansive. Therefore, we have

$$\begin{aligned}&\|G_{n+1}y_{n+1} - G_ny_n\| \\ &\leq \|G_{n+1}y_{n+1} - G_{n+1}y_n + G_{n+1}y_n - G_ny_n\| \\ &\leq \|y_{n+1}-y_n\| + \|Proj_C(\gamma_{n+1}y_n + (1-\gamma_{n+1})Sy_n) \\ &\quad - Proj_C(\gamma_ny_n + (1-\gamma_n)Sy_n)\| \\ &\leq \|y_{n+1}-y_n\| + |\gamma_{n+1}-\gamma_n|\|y_n - Sy_n\|.\end{aligned} \tag{3.10}$$

Substituting (3.9) into (3.10), we find that

$$\begin{aligned}
&\|G_{n+1}y_{n+1} - G_n y_n\| \\
&\le (1-\alpha_{n+1}(1-\alpha))\|x_{n+1}-x_n\| \\
&\quad + (1-\alpha_{n+1})|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad + (1-\alpha_{n+1})\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_n}\|u_n - J_{\lambda_n}u_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|z_n - Tx_n\| + |\gamma_{n+1}-\gamma_n|\|y_n - Sy_n\| \\
&\le \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_n}\|u_n - J_{\lambda_n}u_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|z_n - Tx_n\| + |\gamma_{n+1}-\gamma_n|\|y_n - Sy_n\|.
\end{aligned}$$

This yields from the restrictons (a), (c), and (d) that

$$\limsup_{n\to\infty}(\|G_{n+1}y_{n+1} - G_n y_n\| - \|x_{n+1}-x_n\|) \le 0.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty}\|G_n y_n - x_n\| = 0. \tag{3.11}$$

In view of

$$x_{n+1} - x_n = (1-\beta_n)(G_n y_n - x_n),$$

we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Since $\|\cdot\|^2$ is convex, we see from (3.2) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|G_n y_n - p\|^2 \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|\alpha_n(Tx_n - p) \\
&\quad + (1-\alpha_n)(J_{\lambda_n}(x_n - \lambda_n Ax_n) - p)\|^2 \\
&\le \beta_n\|x_n-p\|^2 + \alpha_n(1-\beta_n)\|Tx_n - p\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\|J_{\lambda_n}(x_n - \lambda_n Ax_n) - p\|^2 \\
&\le \|x_n-p\|^2 + \alpha_n\|Tx_n-p\|^2 \\
&\quad - \lambda_n(2\xi - \lambda_n)(1-\beta_n)\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.13}$$

Notice that

$$\begin{aligned}
&\lambda_n(2\xi-\lambda_n)(1-\beta_n)\|Ax_n-Ap\|^2\\
&\le \|x_n-p\|^2+\alpha_n\|Tx_n-p\|^2-\|x_{n+1}-p\|^2\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\alpha_n\|Tx_n-p\|^2.
\end{aligned}$$

In view of the restrictions (a), (c), and (d), we obtain from (3.12) that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.14}$$

Notice that

$$\begin{aligned}
&\|z_n-p\|^2\\
&=\|J_{\lambda_n}(x_n-\lambda_nAx_n)-J_{\lambda_n}(p-\lambda_nAp)\|^2\\
&\le \langle (x_n-\lambda_nAx_n)-(p-\lambda_nAp), z_n-p\rangle\\
&=\frac{1}{2}\big(\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\|^2+\|z_n-p\|^2\\
&\quad -\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)-(z_n-p)\|^2\\
&\le \frac{1}{2}\big(\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n-\lambda_n(Ax_n-Ap)\|^2\big)\\
&\le \frac{1}{2}\big(\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2\\
&\quad +2\lambda_n\|x_n-z_n\|\|Ax_n-Ap\|\big)\\
&\le \frac{1}{2}\big(\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2\\
&\quad +2\lambda_n\|x_n-y_n\|\|Ax_n-Ap\|\big).
\end{aligned}$$

It follows that

$$\|z_n-p\|^2\le\|x_n-p\|^2-\|x_n-z_n\|^2+2\lambda_n\|x_n-y_n\|\|Ax_n-Ap\|. \tag{3.15}$$

This implies that

$$\begin{aligned}
\|y_n-p\|^2&\le \alpha_n\|Tx_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2\\
&\le \alpha_n\|Tx_n-p\|^2+\|x_n-p\|^2-(1-\alpha_n)\|x_n-z_n\|^2\\
&\quad +2\lambda_n\|x_n-y_n\|\|Ax_n-Ap\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|G_n y_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n\|Tx_n - p\|^2 \\
&\quad - (1-\alpha_n)(1-\beta_n)\|x_n - z_n\|^2 \\
&\quad + 2\lambda_n(1-\beta_n)\|x_n - y_n\|\|Ax_n - Ap\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\|x_n - z_n\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n\|Tx_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\lambda_n(1-\beta_n)\|x_n - y_n\|\|Ax_n - Ap\| \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|Tx_n - p\|^2 \\
&\quad + 2\lambda_n(1-\beta_n)\|x_n - y_n\|\|Ax_n - Ap\|.
\end{aligned}$$

By virtue of the restrictions (a), (b), and (d), we find from (3.12) and (3.14) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.16}$$

Notice that $y_n - z_n = \alpha_n(Tx_n - z_n)$. It follows that

$$\lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.17}$$

Put

$$Gx = Proj_C(\gamma x + (1-\gamma)Sx), \quad \forall x \in C,$$

where $\gamma = \lim_{n\to\infty}\gamma_n$. It is no hard to see that $G : C \to C$ is nonexpansive with $Fix(G) = Fix(S)$. Notice that

$$\begin{aligned}
&\|Gy_n - y_n\| \\
&\le \|Gy_n - G_n y_n\| + \|G_n y_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\| \\
&\le \|(\gamma y_n + (1-\gamma)Sy_n) - (\gamma_n y_n + (1-\gamma_n)Sy_n)\| \\
&\quad + \|G_n y_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\| \\
&\le |\gamma - \gamma_n|\|y_n - Sy_n\| + \|G_n y_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\|.
\end{aligned}$$

From (3.11), (3.16), and (3.17), we find that

$$\lim_{n\to\infty}\|Gy_n - y_n\| = 0. \tag{3.18}$$

Notice that

$$\begin{aligned}\|Gz_n - z_n\| &\le \|Gz_n - Gy_n\| + \|Gy_n - y_n\| + \|y_n - z_n\| \\ &\le \|Gy_n - y_n\| + 2\|y_n - z_n\|.\end{aligned}$$

It follows from (3.17) and (3.18) that

$$\lim_{n\to\infty} \|Gz_n - z_n\| = 0. \tag{3.19}$$

Since $Proj_{\mathcal{F}}T$ is contractive, we see that there exists a unique fixed point, say x. Next, we show that $\limsup_{n\to\infty}\langle Tx - x, z_n - x\rangle \le 0$. To show this, we can choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle Tx - x, z_n - x\rangle = \lim_{i\to\infty}\langle Tx - x, z_{n_i} - x\rangle.$$

Since z_{n_i} is bounded, we can choose a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to some point h. We may assume, without loss of generality, that z_{n_i} converges weakly to h. Now, we are in a position to show $h \in Fix(G)$. Assume that $h \notin Fix(G)$. In view of Opial's condition, we find from (3.19) that

$$\begin{aligned}\liminf_{i\to\infty} \|z_{n_i} - h\| &< \liminf_{i\to\infty} \|z_{n_i} - Gh\| \\ &= \liminf_{i\to\infty} \|z_{n_i} - Gz_{n_i} + Gz_{n_i} - Gh\| \\ &\le \liminf_{i\to\infty} \|z_{n_i} - h\|.\end{aligned}$$

This is a contradiction. That is, $h = Gh$. This shows that $h \in Fix(S)$. Since $z_n = J_{\lambda_n}(x_n - \lambda_n A x_n)$, we find that

$$x_n - \lambda_n A x_n \in (I + \lambda_n B)z_n.$$

That is,

$$\frac{x_n - z_n}{\lambda_n} - Ax_n \in Bz_n.$$

Since B is monotone, we get, for any $(\mu, \nu) \in B$, that

$$\langle z_n - \mu, \frac{x_n - z_n}{\lambda_n} - Ax_n - \nu\rangle \ge 0.$$

Replacing n by n_i and letting $i \to \infty$, we obtain from (3.16) that

$$\langle h - \mu, -Ah - \nu\rangle \ge 0.$$

This means $-Ah \in Bh$, that is, $0 \in (A + B)(h)$. Hence we get $h \in (A + B)^{-1}(0)$. This completes the proof that $h \in \mathcal{F}$. It follows that

$$\limsup_{n\to\infty}\langle Tx - x, z_n - x\rangle = \langle Tx - x, h - x\rangle \le 0.$$

Notice that

$$
\begin{aligned}
&\|x_{n+1} - x\|^2 \\
&\le \beta_n\|x_n - x\|^2 + (1-\beta_n)\|G_n y_n - x\|^2 \\
&\le \beta_n\|x_n - x\|^2 + (1-\beta_n)\|y_n - x\|^2 \\
&\le \beta_n\|x_n - x\|^2 + (1-\beta_n)\|\alpha_n(Tx_n - x) + (1-\alpha_n)(z_n - x)\|^2 \\
&\le \beta_n\|x_n - x\|^2 + (1-\beta_n)((1-\alpha_n)^2\|z_n - x\|^2 \\
&\quad + 2\alpha_n\langle Tx_n - x, z_n - x\rangle) \\
&\le (1-\alpha_n(1-\beta_n))\|x_n - x\|^2 + 2\alpha_n(1-\beta_n)\langle Tx_n - x, z_n - x\rangle.
\end{aligned}
$$

In view of Lemma 2.4, we find that $\lim_{n\to\infty}\|x_n - x\| = 0$. This completes the proof. □

If both the mapping T and S are self mappings, then we have from Theorem 3.1 the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a ξ-inverse-strongly monotone mapping, $S : C \to C$ a κ-strictly pseudocontractive mapping, $T : C \to C$ an α-contractive mapping and B a maximal monotone operator on H such that the domain of B is included in C. Assume that $\mathcal{F} = Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*

$$
\begin{cases}
x_1 \in C, \\
y_n = \alpha_n T x_n + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n), \\
x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n) S y_n), \quad \forall n \ge 1,
\end{cases}
$$

where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(b) $0 < a \le \beta_n \le b < 1$;
(c) $\kappa \le \gamma_n \le c < 1$, $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$;
(d) $0 < d \le \lambda_n \le e < 2\xi$, $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

where a, b, c, d and e are some real numbers. Then the sequence $\{x_n\}$ converges strongly to $x = Proj_{\mathcal{F}}Tx$.

If $Ty = x$, for all $y \in C$, where x is a fixed element in C and $\gamma_n = \kappa = 0$, then we find from Corollary 3.2 the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a ξ-inverse-strongly monotone mapping, $S : C \to H$ a nonexpansive mapping, and B a maximal monotone operator on H such that the domain of B is included in C. Assume that $\mathcal{F} = Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n x_1 + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S y_n, \quad \forall n \geq 1, \end{cases}$$

where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < a \leq \beta_n \leq b < 1$;
(c) $0 < d \leq \lambda_n \leq e < 2\xi$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

where a, b, d and e are some real numbers. Then the sequence $\{x_n\}$ converges strongly to $x = Proj_{\mathcal{F}} x_1$.

4. **Applications**

Let H be a Hilbert space and $f : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y\rangle, \quad z \in H\}, \quad \forall x \in H.$$

From Rockafellar [16,17], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y\in H} f(y)$. Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{4.1}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 4.1 [20] *Let C be a nonempty closed convex subset of a real Hilbert space H, $Proj_C$ the metric projection from H onto C, ∂I_C the subdifferential of I_C, where I_C is as defined in (4.1) and $J_\lambda = (I + \lambda \partial I_C)^{-1}$. Then*

$$y = J_\lambda x \Longleftrightarrow y = Proj_C x, \quad x \in H, y \in C.$$

Now, we consider a variation inequality problem.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a ξ-inverse-strongly monotone mapping, $S : C \to H$ a κ-strictly pseudocontractive mapping, and $T : C \to H$ an α-contractive mapping. Assume that $\mathcal{F} = Fix(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = Proj_C(\alpha_n T x_n + (1-\alpha_n) Proj_C(x_n - \lambda_n A x_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) Proj_C(\gamma_n y_n + (1-\gamma_n) S y_n), \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < a \leq \beta_n \leq b < 1$;
(c) $\kappa \leq \gamma_n \leq c < 1$, $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(d) $0 < d \leq \lambda_n \leq e < 2\xi$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

where a, b, c, d and e are some real numbers. Then the sequence $\{x_n\}$ converges strongly to $x = Proj_{\mathcal{F}} T x$.

Proof. Put $Bx = \partial I_C$. Next, we show that $VI(C, A) = (A + \partial I_C)^{-1}(0)$. Notice that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}(0) &\Longleftrightarrow 0 \in Ax + \partial I_C x \\ &\Longleftrightarrow -Ax \in \partial I_C x \\ &\Longleftrightarrow \langle Ax, y - x \rangle \geq 0 \\ &\Longleftrightarrow x \in VI(C, A). \end{aligned}$$

From Lemma 4.1, we can conclude the desired conclusion immediately. □

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

In this work, we use $EP(F)$ to denote the solution set of the equilibrium problem (4.2).

To study the equilibrium problems (4.2), we may assume that F satisfies the following conditions:

(A1) $F(x,x)=0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y)+F(y,x) \le 0$ for all $x,y \in C$;
(A3) for each $x,y,z \in C$,

$$\limsup_{t \downarrow 0} F(tz+(1-t)x,y) \le F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semicontinuous.

Putting $F(x,y)=\langle Ax, y-x\rangle$ for every $x,y \in C$, we see that the equilibrium problem (4.2) is reduced to the variational inequality (2.1).

The following lemma can be found in [4] and [5].

Lemma 4.2. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r>0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \ge 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \ge 0, \quad \forall y \in C\} \tag{4.3}$$

for all $r>0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*
(b) *T_r is firmly nonexpansive, i.e., for any $x,y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x-y\rangle;$$

(c) *$F(T_r)=EP(F)$;*
(d) *$EP(F)$ is closed and convex.*

Lemma 4.3 [20] *Let C be a nonempty closed convex subset of a real Hilbert space H, F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4) and A_F a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \ge \langle y-x, z\rangle, \quad \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.4}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F)=A_F^{-1}(0)$ and

$$T_r x = (I+rA_F)^{-1}x, \quad \forall x \in H, r>0,$$

where T_r is defined as in (4.3).

The following result is now derived based on Theorem 3.1 and Lemma 4.3.

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to H$ a κ-strictly pseudocontractive mapping, $T : C \to H$ an α-contractive mapping and F_B a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Assume that $\mathcal{F} = Fix(S) \cap EP(F_B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = Proj_C(\alpha_n T x_n + (1-\alpha_n) z_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) Proj_C(\gamma_n y_n + (1-\gamma_n) S y_n), \quad \forall n \geq 1, \end{cases}$$

where $z_n \in C$ such that

$$F_B(z_n, u) + \frac{1}{\lambda_n}\langle u - z_n, z_n - x_n \rangle \geq 0, \quad \forall u \in C.$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < a \leq \beta_n \leq b < 1$;
(c) $\kappa \leq \gamma_n \leq c < 1$, $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(d) $0 < d \leq \lambda_n \leq e < \infty$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

where a, b, c, d and e are some real numbers. Then the sequence $\{x_n\}$ converges strongly to $x = Proj_{\mathcal{F}} Tx$.

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