

δ -DOUBLE DERIVATIONS ON C^* -ALGEBRAS

M. MIRZAVAZIRI* AND E. OMIDVAR TEHRANI

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ABSTRACT. Let $\mathcal{A}$ be an algebra and $\delta, \varepsilon : \mathcal{A} \rightarrow \mathcal{A}$ be linear mappings. We say that a linear mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is a (δ, ε) -double derivation if $d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$ for all $a, b \in \mathcal{A}$. By a δ -double derivation we mean a (δ, δ) -double derivation. Giving some elementary facts concerning double derivations, we prove that if $\mathcal{A}$ is a C^* -algebra, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a $*$ -linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ is a continuous δ -double derivation then δ is continuous. We also show that if $\mathcal{A}$ is a C^* -algebra, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a continuous linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ is a $*$ - δ -double derivation then d is continuous. Similar facts concerning (δ, ε) -double derivations on C^* -algebras are also given.

1. Introduction

Let $\mathcal{A}$ be a subalgebra of an algebra $\mathcal{B}$, $\mathcal{X}$ be a $\mathcal{B}$ -bimodule and $\sigma : \mathcal{A} \rightarrow \mathcal{B}$ be a linear mapping. A linear mapping $d : \mathcal{A} \rightarrow \mathcal{X}$ is called a σ -derivation (see [5] and [6]) if

$$(1.1) \quad d(ab) = d(a)\sigma(b) + \sigma(a)d(b) \quad a, b \in \mathcal{A}.$$

Clearly, if $\sigma = id$, the identity mapping on $\mathcal{A}$, then a σ -derivation is an ordinary derivation. On the other hand, each homomorphism d is a $\frac{d}{2}$ -derivation. Thus, the theory of σ -derivations combines the theory of

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*Corresponding author

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derivations and homomorphisms. If $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is an ordinary derivation and $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ is a homomorphism, then $d = \delta\sigma$ is a σ -derivation. Although, a σ -derivation is not necessarily of the form $\delta\sigma$, but it seems that the generalized Leibniz rule, $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$, comes from this observation. Taking ideas from this fact, we motivate to consider two derivations $\delta, \varepsilon : \mathcal{A} \rightarrow \mathcal{A}$ to find a similar rule, for $d = \delta\varepsilon$. In this case, we see that d satisfies

$$(1.2) \quad d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) \quad a, b \in \mathcal{A}.$$

Fortunately, this can be perceived as a generalization of the notion of a σ -derivation. We say that a linear mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is a (δ, ε) -double derivation if it satisfies (1.2).

The problem of automatic continuity of derivations is an important problem in the theory of derivations. In 1960, H. Sakai [11] proved that every derivation on a C^* -algebra is automatically continuous and later in 1972, J. R. Ringrose [10] showed that every derivation from a C^* -algebra into a Banach $\mathcal{A}$ -bimodule is continuous. The problem of automatic continuity is also considered for σ -derivations. In 2006, M. Mirzavaziri and M. S. Moslehian [5] acquired some results about automatic continuity of σ -derivations. Suppose that $\mathcal{A}$ is a C^* -algebra acting on a Hilbert space $\mathcal{H}$. In [5], it is proved that if $\sigma : \mathcal{A} \rightarrow B(\mathcal{H})$ is a continuous $*$ -linear mapping then every σ -derivation from $\mathcal{A}$ to $B(\mathcal{H})$ is automatically continuous. Moreover, the converse is established in [5] in the sense that if $d : \mathcal{A} \rightarrow B(\mathcal{H})$ is a continuous $*$ - σ -derivation then there exists a continuous mapping $\Sigma : \mathcal{A} \rightarrow B(\mathcal{H})$ such that d is a $*$ - Σ -derivation. Here, we consider the same problem for double derivations. Since the notion of a double derivation is a generalization of derivation, homomorphism and σ -derivation, our results extend the previous facts. Although our proof are similar to the previous arguments in some cases, but they are essentially new.

The reader is referred to [8] and [9] for the definitions and elementary properties of C^* -algebras, to [1],[2],[4],[5],[6] and [7] for various generalized notions of derivations and to [3],[10],[11] and [12] for more information on automatic continuity of derivations, inner derivations, point derivations and higher derivations.

2. Preliminaries

Definition 2.1. Let $\mathcal{A}$ be an algebra and $\delta, \varepsilon : \mathcal{A} \rightarrow \mathcal{A}$ be linear mappings. A linear mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is called a (δ, ε) -double derivation if

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$$

for all $a, b \in \mathcal{A}$. By a δ -double derivation we mean a (δ, δ) -double derivation.

It is clear that each σ -derivation $d : \mathcal{A} \rightarrow \mathcal{A}$ is a $(\sigma - id, d)$ -double derivation. Moreover, every homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ is a $(\frac{\varphi}{2} - id, \varphi)$ -double derivation.

Lemma 2.2. *If δ is a derivation on $\mathcal{A}$, then each δ -double derivation $d : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $d = \delta^2 + \varepsilon$, where $\varepsilon : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation.*

Proof. Let $\varepsilon = d - \delta^2$. Then,

$$\begin{aligned} \varepsilon(ab) &= (d - \delta^2)(ab) \\ &= d(a)b + ad(b) + 2\delta(a)\delta(b) - \delta(\delta(a)b + a\delta(b)) \\ &= d(a)b + ad(b) + 2\delta(a)\delta(b) - \delta^2(a)b - 2\delta(a)\delta(b) - a\delta^2(b) \\ &= (d - \delta^2)(a)b + a(d - \delta^2)(b) \\ &= \varepsilon(a)b + a\varepsilon(b). \end{aligned}$$

Hence ε is a derivation. $\square$

Remark 2.3. Lemma 2.2 shows that if $\mathcal{A}$ is an algebra such that each derivation defined on $\mathcal{A}$ is automatically continuous and δ is a derivation then each δ -double derivation is also automatically continuous.

3. δ -double derivations

Recall that if $\mathcal{Y}$ and $\mathcal{Z}$ are normed spaces and $T : \mathcal{Y} \rightarrow \mathcal{Z}$ is a linear mapping, then the set of all z such that there is a sequence $\{y_n\}$ in $\mathcal{Y}$ with $y_n \rightarrow 0$ and $Ty_n \rightarrow z$ is called the separating space $\mathcal{S}(T)$ of T . Clearly, $\mathcal{S}(T) = \overline{\cap_{n=1}^{\infty} \{T(y) : \|y\| < 1/n\}}$ is a closed linear space. If $\mathcal{Y}$

and $\mathcal{Z}$ are Banach spaces, by the closed graph theorem, T is continuous if and only if $\mathcal{S}(T) = \{0\}$. Also, recall that if E is a subset of an algebra $\mathcal{B}$, the right annihilator $\text{ran}(E)$ of E (resp., the left annihilator $\text{lan}(E)$ of E) is defined to be $\{b \in \mathcal{B} : Eb = \{0\}\}$ (resp., $\{b \in \mathcal{B} : bE = \{0\}\}$). The set $\text{ann}(E) := \text{ran}(E) \cap \text{lan}(E)$ is called the annihilator of E .

Lemma 3.1. *Let $\mathcal{A}$ be a C^* -algebra acting on a Hilbert space $\mathcal{H}$. If $d : \mathcal{A} \rightarrow \mathcal{A}$ is a continuous δ -double derivation then $\mathcal{S}(\delta) \subseteq \text{ann}(\delta(\mathcal{A}))$.*

Proof. Let $A \in \mathcal{S}(\delta)$. Thus, there is a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $A_n \rightarrow 0$ and $\delta(A_n) \rightarrow A$. For each $B \in \mathcal{A}$ we have,

$$\lim_{n \rightarrow \infty} d(A_n B) = \lim_{n \rightarrow \infty} A_n d(B) + \lim_{n \rightarrow \infty} d(A_n) B + \lim_{n \rightarrow \infty} 2\delta(A_n)\delta(B).$$

Since d is continuous, we obtain $0 = A\delta(B)$. Similarly, $0 = \delta(B)A$ and so $A \in \text{ann}(\delta(\mathcal{A}))$. $\square$

Theorem 3.2. *Let $\mathcal{A}$ be a C^* -algebra acting on a Hilbert space $\mathcal{H}$, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a $*$ -linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous δ -double derivation. Then, δ is continuous.*

Proof. Let d be continuous, $\mathcal{L}_0 = \bigcup_{A \in \mathcal{A}} \delta(A)(\mathcal{H})$ and $\mathcal{L}$ be the closed linear span of $\mathcal{L}_0$. Then, $\mathcal{H} = \mathcal{L} \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{L}^\perp$. By Lemma 3.2. in [4], we have,

$$\mathcal{K} = \bigcap_{A \in \mathcal{A}} \ker \delta(A).$$

Assume that $\{A_n\} \subseteq \mathcal{A}$, $A_n \rightarrow 0$ and $\delta(A_n) \rightarrow A$. Then, for each $\ell \in \mathcal{L}_0$ there is a $B \in \mathcal{A}$ and there is an $h \in \mathcal{H}$ such that $\ell = \delta(B)(h)$. Now, since $\mathcal{S}(\delta) \subseteq \text{ann}(\delta(\mathcal{A}))$, $A(\ell) = A(\delta(B)(h)) = (A\delta(B))(h) = 0$, then $A = 0$ on $\mathcal{L}_0$ and so $A = 0$ on $\mathcal{L}$. On the other hand,

$$A(k) = \lim_{n \rightarrow \infty} (\delta(A_n))(k) = 0.$$

Thus, $A = 0$ on $\mathcal{K}$ and so $A = 0$ on $\mathcal{H}$. Therefore, $\mathcal{S}(\delta) = \{0\}$ and δ is continuous. $\square$

Theorem 3.3. *Let $\mathcal{A}$ be a C^* -algebra, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a $*$ - δ -double derivation. Then, d is continuous.*

Proof. We may assume that $\mathcal{A}$ is unital. In fact, if $\mathcal{A}$ has no identity, we shall consider the unitization $\mathcal{A}_1$ of $\mathcal{A}$ with unit 1 and define $d(1) = \delta(1) = 0$. Then, d and δ can be uniquely extended to linear mappings d_1 and δ_1 on $\mathcal{A}_1$. Moreover, d_1 is a δ_1 -double derivation since

$$\begin{aligned} d_1[(a + \alpha)(b + \beta)] &= d_1[ab + \alpha b + a\beta + \alpha\beta] \\ &= d_1(ab) + \alpha d_1(b) + d_1(a)\beta + 0 \\ &= ad_1(b) + d_1(a)b + 2\delta_1(a)\delta_1(b) + \alpha d_1(b) + d_1(a)\beta \\ &= (a + \alpha)d_1(b) + d_1(a)(b + \beta) + 2\delta_1(a)\delta_1(b) \\ &= (a + \alpha)d_1(b + \beta) + d_1(a + \alpha)(b + \beta) + 2\delta_1(a + \alpha)\delta_1(b + \beta). \end{aligned}$$

Thus, it is sufficient to prove that a $*$ - δ -double derivation d on a unital C^* -algebra $\mathcal{A}$ is continuous if so is δ . For this, we show that the $*$ -linear mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ defined by $D(a) = d(a) - ad(1)$ for each $a \in \mathcal{A}$ is continuous. Suppose that δ is continuous and a is a self-adjoint element in $\mathcal{A}$. Also, let φ be a state on $\mathcal{A}$ such that $\varphi(a) = \|a\|$. Put $\|a\|1 - a = h^2$ ($h \geq 0, h \in \mathcal{A}$). Then, $\varphi(h^2) = 0$ and

$$\begin{aligned} &| -\varphi(D(a)) - \varphi(2(\delta(h))^2) | \\ &= | \varphi(D(h^2 - \|a\|1)) - \varphi(2(\delta(h))^2) | \\ &= | \varphi(D(h^2)) - \varphi(2(\delta(h))^2) | \\ &= | \varphi(d(h^2) - h^2d(1)) - \varphi(2(\delta(h))^2) | \\ &= | \varphi(hd(h)) + \varphi(d(h)h) - \varphi(h^2d(1)) | \\ &\leq \varphi(h^2)^{1/2}\varphi(d(h)^2)^{1/2} + \varphi(d(h)^2)^{1/2}\varphi(h^2)^{1/2} + \varphi(h^4)^{1/2}\varphi(d(1)^2)^{1/2} \\ &= 0. \end{aligned}$$

Hence, $\varphi(D(a)) = -\varphi(2(\delta(h))^2)$. Suppose that $\{a_n\}$ is a sequence of self-adjoint elements in $\mathcal{A}$ such that $a_n \rightarrow 0$ and $D(a_n) \rightarrow b (\neq 0)$. Let φ_n be a state on $\mathcal{A}$ such that $\varphi_n(b + a_n) = \|b + a_n\|$, and let φ_0 be an accumulation point of $\{\varphi_n\}$ in the state space of $\mathcal{A}$. Then, we have,

$$\begin{aligned} |\varphi_{n_j}(b + a_{n_j}) - \varphi_0(b)| &\leq |\varphi_{n_j}(b + a_{n_j}) - \varphi_{n_j}(b)| + |\varphi_{n_j}(b) - \varphi_0(b)| \\ &\leq \|b + a_{n_j} - b\| + |\varphi_{n_j}(b) - \varphi_0(b)| \rightarrow 0 \end{aligned}$$

for some subsequence $\{n_j\}$ of $\{n\}$. Hence, $\varphi_0(b) = \|b\|$ and so

$$\varphi_0(D(b)) = -\varphi_0(2(\delta(h_b))^2),$$

where $\|b\|1 - b = h_b^2$. Also, if $\|b + a_{n_j}\|1 - (b + a_{n_j}) = h_{b+a_{n_j}}^2$ then

$$-\varphi_{n_j}(2(\delta(h_{b+a_{n_j}}))^2) = \varphi_{n_j}(D(b + a_{n_j})) = \varphi_{n_j}(D(b) + D(a_{n_j})) \rightarrow \varphi_0(D(b) + b).$$

Note that $h_{b+a_{n_j}}^2 \rightarrow h_b^2$ and $h_{b+a_{n_j}}$ and h_b are positive so that $h_{b+a_{n_j}} \rightarrow h_b$. Now, since δ is continuous, one can show that the left hand side of the above equality tends to $-\varphi_0(2(\delta(h_b))^2)$. Therefore,

$$-\varphi_0(2(\delta(h_b))^2) = \varphi_0(D(b) + b) = -\varphi_0(2(\delta(h_b))^2) + \varphi_0(b),$$

that is, $\varphi_0(b) = 0$, which is a contradiction. So the closed graph theorem shows that D is continuous and therefore d is continuous. $\square$

Recall that if T is a linear mapping and we define T^* by $T^*(a) = T(a^*)^*$ for all $a \in \mathcal{A}$, then T^* is also linear.

Lemma 3.4. *Let $\mathcal{A} \subseteq \mathcal{A}$, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a δ -double derivation. Then, d^* is a δ^* -double derivation.*

Proof. For each $a, b \in \mathcal{A}$,

$$\begin{aligned} d^*(ab) &= d(b^*a^*)^* \\ &= [b^*d(a^*) + d(b^*)a^* + 2\delta(b^*)\delta(a^*)]^* \\ &= d^*(a)b + ad^*(b) + 2\delta^*(a)\delta^*(b). \end{aligned}$$

Hence, d^* is a δ^* -double derivation.

Proposition 3.5. *Let $\mathcal{A}$ be a C^* -algebra, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a $*$ -linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a δ -double derivation. Then, d is of the form $d_1 + d_2$, where d_1 is a $*$ - δ -double derivation and d_2 is a derivation.*

Proof. We can write,

$$d = \frac{d + d^*}{2} + \frac{d - d^*}{2}.$$

Put $d_1 = \frac{d+d^*}{2}$ and $d_2 = \frac{d-d^*}{2}$. Then, d_1 is a $*$ - δ -double derivation and d_2 is a derivation, since for each $a, b \in \mathcal{A}$,

$$\begin{aligned} d_1(ab) &= \frac{d + d^*}{2}(ab) \\ &= \frac{1}{2}(ad(b) + d(a)b + 2\delta(a)\delta(b) + ad^*(b) + d^*(a)b + 2\delta^*(a)\delta^*(b)) \\ &= a\frac{d + d^*}{2}(b) + \frac{d + d^*}{2}(a)b + \frac{1}{2}(4\delta(a)\delta(b)) \\ &= ad_1(b) + d_1(a)b + 2\delta(a)\delta(b) \end{aligned}$$

and

$$\begin{aligned}
 d_2(ab) &= \frac{d - d^*}{2}(ab) \\
 &= \frac{1}{2}(ad(b) + d(a)b + 2\delta(a)\delta(b) - ad^*(b) - d^*(a)b - 2\delta^*(a)\delta^*(b)) \\
 &= a\frac{d - d^*}{2}(b) + \frac{d - d^*}{2}(a)b.
 \end{aligned}$$

□

Corollary 3.6. *Let $\mathcal{A}$ be a C^* -algebra, $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous $*$ -linear mapping and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a δ -double derivation. Then, d is continuous.*

We also have the following two results.

Theorem 3.7. *Let $\mathcal{A}$ be a C^* -algebra, $\delta, \varepsilon : \mathcal{A} \rightarrow \mathcal{A}$ be two continuous $*$ -linear mappings and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a (δ, ε) -double derivation. Then, d is continuous.*

Theorem 3.8. *Let $\mathcal{A}$ be a C^* -algebra, $\delta, \varepsilon : \mathcal{A} \rightarrow \mathcal{A}$ be two continuous $*$ -linear mappings and $d : \mathcal{A} \rightarrow \mathcal{A}$ be a (δ, ε) -double derivation. Then, d is continuous.*

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Madjid Mirzavaziri

Elaheh Omidvar Tehrani

Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran.

Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Iran.

Email: `mirzavaziri@math.um.ac.ir`

Email: `elahe.tehrani@gmail.com`