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REVERSIBILITY OF A MODULE WITH RESPECT TO THE BIFUNCTORS HOM AND REJ

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Abstract. Let M_R be a non-zero module and $\mathcal{F} : \sigma[M_R]\times\sigma[M_R] \to$ Mod-$\mathbb{Z}$ a bifunctor. The $\mathcal{F}$-reversibility of M is defined by $\mathcal{F}(X, Y) = 0 \Rightarrow \mathcal{F}(Y, X) = 0$ for all non-zero X, Y in $\sigma[M_R]$. Hom (resp. Rej)-reversibility of M is characterized in different ways. Among other things, it is shown that R_R $({}_RR)$ is Hom-reversible if and only if $R = \bigoplus_{i=1}^{n} R_i$ such that each R_i is a perfect ring with a unique simple module (up to isomorphism). In particular, for a duo ring, the concepts of perfectness and Hom-reversibility coincide.
Keywords: Co-retractable, Kasch module, perfect ring, prime module, cogenerator.
MSC(2010): Primary: 16D60, Secondary: 16L30; 16D10.

1. **Introduction and Preliminaries**

Throughout this paper rings will have non-zero identity elements and modules will be unitary. Unless stated otherwise modules will be right modules. Let R be a ring, M a non-zero R-module, $\mathcal{F}$: $\sigma[M_R] \times \sigma[M_R] \to$ Mod-$\mathbb{Z}$ a bifunctor and $\emptyset \neq \mathcal{C}, \mathcal{D} \subseteq \sigma[M_R]$, where $\sigma[M_R]$ is the full subcategory of R-modules whose objects are all R-modules subgenerated by M. We say that M_R is *$\mathcal{CD}$-$\mathcal{F}$-reversible* if for all $C\in\mathcal{C}, D\in\mathcal{D}$, $\mathcal{F}(C, D) = 0$ implies $\mathcal{F}(D, C) = 0$. If for $\mathcal{C} = \mathcal{D} = \sigma[M_R]$ the above condition holds, we say that M_R is *F-reversible*. There are several concepts that may be stated in terms of the notion of reversibility.

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For instance, M_R is *prime* in the sense of Bican [6] (i.e., $\mathrm{Rej}(M, N) = 0$ for all nonzero submodules N of M) if and only if M_R is $\mathcal{CD}$-Rej-reversible for $\mathcal{C} = \{$all non-zero submodules of $M_R\}$, $\mathcal{D} = \{M_R\}$. Here $\mathrm{Rej}(C, D) = \cap_{f:C\to D} \ker f$. Also a module M_R is fully Kasch in the sense of Albu and Wisbauer [2] if and only if M_R is $\mathcal{CD}$-Hom-reversible for $\mathcal{C} = \{$all simple R-modules in $\sigma[M_R]\}$ and $\mathcal{D} = \{$cyclic modules in $\sigma[M_R]\}$. Note that if S is a simple module in $\sigma[M_R]$, then $S \simeq N/K$ for some cyclic submodule $N \leq M^{(\Lambda)}$; see also [5]. The conditions retractable [9], co-retractable [1, 2.13], [7] and weak generator [11] can also be stated by $\mathcal{CD}$-$\mathcal{F}$-reversibility by choosing suitable $\mathcal{C}, \mathcal{D}$ and $\mathcal{F} =$ Hom. These observations motivated us to consider questions, such as: *"what are $\mathcal{F}$-reversible modules when $\mathcal{F}$ is Hom or Rej?"*. In this paper, we answer these questions and show first that Hom-reversible modules with semiprime endomorphism rings are precisely semisimple modules [Theorem 2.4]. Then, Hom (resp. Rej)-reversible modules are characterized in several ways in Theorem 2.8 (resp. Theorem 2.14). In particular, it is proved that a duo ring R is perfect if and only if the module R_R is Hom-reversible [Theorem 2.10]. We use the notations $N \trianglelefteq M$ and $N \leq_e M$ to denote respectively that N is a fully invariant and essential submodule of M. We follow [4] and [12] for the terms not defined here, and for the basic results on module and ring theory that are relevant to this work.

2. **Main results**

We begin with some definitions and lemmas. A non-zero R-module M is called:

- *Kasch* if any simple module in $\sigma[M_R]$ can be embedded in M_R [2].
- *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for any non-zero submodule N of M_R [9].
- *co-retractable* if $\mathrm{Hom}_R(M/N, M) \neq 0$ for any proper submodule N of M_R [1].

In [2], a module M_R is called *fully Kasch* if all modules in $\sigma[M_R]$ are Kasch. We say that M_R is *fully retractable* (resp. *fully co-retractable*) if all modules in $\sigma[M_R]$ are retractable (resp. co-retractable). Also we call M_R, *fully max* if any non-zero module in $\sigma[M_R]$ has a maximal submodule. In [3], ring R for which the module R_R is fully co-retractable was studied where the authors used the term completely co-retractable for such rings. It is easy to verify that all of the above conditions on a module are Morita invariant properties.

Lemma 2.1. *Let R be a semiprime ring. Then the following statements hold.*
(i) $\forall\ 0 \neq I \leq R_R\ ({}_RR),\ \exists\ a \in I$ *with* $a^2 \neq 0$.
(ii) $I \leq_e R_R\ (I \leq_e {}_RR) \Rightarrow\ \text{r.ann}_R I = 0\ (\text{l.ann}_R I = 0)$.

Proof. We only prove (i). Let $I \leq R_R$. If $a^2 = 0$ for all $a \in I$ then $(x+y)^2 = 0$ for all $x, y \in I$. Consequently, $xy = -yx$ for all $x, y \in I$. It follows that $(xI)^2 = 0$ for all $x \in I$. Thus for all $x \in I$, $xI = 0$ by the semiprime condition on R, hence $I^2 = 0$, a contradiction. Therefore, there exists $a \in I$ such that $a^2 \neq 0$. □

Lemma 2.2. *Let M be a non-zero R-module with semiprime endomorphism ring S.*
(i) *If M_R is retractable, $N \trianglelefteq_e M_R$ and $I =$* $\text{Hom}_R(M, N)$ *then $I \leq_e {}_SS$.*
(ii) *If $M = N_1 \oplus N_2$ is a direct sum of submodules then* $\text{Hom}_R(N_1, N_2) = 0$ *if and only if* $\text{Hom}_R(N_2, N_1) = 0$.

Proof. Let $0 \neq g \in S$. Since $N \leq_e M_R$, $g(M) \cap N = W \neq 0$, $\text{Hom}_R(M, W) =: A$ is a non-zero right ideal of S by the retractable condition on M. Thus by Lemma 2.1 there exists $f \in A$ such that $f^2 \neq 0$. Since $f^2(M) \subseteq fg(M) \subseteq N$, we have $0 \neq fg \in I$. It follows that $I \leq_e {}_SS$.
(ii) Let $A_i = \text{End}_R(N_i)$ $(i = 1, 2)$, $B = \text{Hom}_R(N_1, N_2)$ and $C = \text{Hom}_R(N_2, N_1)$. It is easy to verify that $S \simeq \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix}$. Now if $B = 0$, then $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ is an ideal of S with zero square. Hence, $C = 0$ by hypothesis on S. Similarly, $C = 0$ implies $B = 0$. □

Corollary 2.3. *Let M be a non-zero R-module.*
(i) *If for every non-zero $N \leq M_R$, the ring* $\text{End}_R(M \oplus N)$ *is semiprime, then M is a retractable R-module.*
(ii) *If for every $N < M_R$, the ring* $\text{End}_R(M \oplus M/N)$ *is semiprime, then M is a co-retractable R-module.*

Proof. Apply Lemma 2.2(ii) and note that for every non-zero $N < M_R$, $\text{Hom}_R(N, M)$ and $\text{Hom}_R(M, M/N)$ are non-zero. □

Theorem 2.4. *The following statements are equivalent for M_R.*
(i) *For every non-zero $X \in \sigma[M]$, $\mathrm{End}_R(X)$ is a semiprime ring.*
(ii) *M_R is Hom-reversible and $\mathrm{End}_R(M)$ is a semiprime ring.*
(iii) *M_R is semisimple.*

Proof. i)⇒(ii). Apply Lemma 2.2(ii) and the fact that the class $\sigma[M_R]$ is closed under direct sums.
(ii)⇒(iii). Let $S = \mathrm{End}_R(M)$ and $\mathrm{Soc}(M) = N$. Assume that X is a non-zero submodule of M_R. By [12, 14.9] there exists a submodule $Y \leq X$ such that $\mathrm{Soc}(X/Y) \neq 0$. Let U be a simple R-module embedded in X/Y. It follows by hypothesis that $\mathrm{Hom}_R(U, X) \neq 0$. Hence $\mathrm{Soc}(X)$ is non-zero. This shows that $N \leq_e M$. Also the condition (ii) implies that M_R is retractable. Thus $I = \mathrm{Hom}_R(M, N)$ is an ideal of S such that $I \leq_e {}_SS$ by Lemma 2.2(i). Now if M/N is non-zero, then by hypothesis, there exists a non-zero homomorphism $f : M/N \to M$. Let $g = f\pi$ where $\pi : M \to M/N$ is the canonical homomorphism. Then $0 \neq g \in S$ and $g(N) = 0$. It follows that $gI = 0$ which contradicts Lemma 2.1. Therefore, $M/N = 0$ and M is a semisimple R-module.
(iii)⇒(i). This is a known result but we give a proof for completeness; see also [8, Theorem 2.6(b)]. If M_R is semisimple, then so is every $X \in \sigma[M]$. Hence, it is enough to show that $S = \mathrm{End}_R(M)$ is a semiprime ring. Let now $0 \neq g \in S$ with $(gS)^2 = 0$ and set $K = \ker g$, then there exists $N \leq M_R$ such that $K \oplus N = M$. Suppose that $M = g(N) \oplus W$ for some $W \leq M_R$. Since g is one to one on N, the map $h : M \to M$ defined by $h(g(n)) = n$ and $h(W) = 0$, is an R-homomorphism. Now we have $ghg = g$ and so $e = gh$ is a non-zero idempotent in gS, a contradiction. Therefore S is a semiprime ring. □

Corollary 2.5. *A ring R is semisimple Artinian if and only if all non-zero R-modules have semiprime endomorphism rings.*

Proof. This is an immediate corollary of Theorem 2.4. □

We are now going to investigate the Hom-reversible R-modules.

Lemma 2.6. *If every proper factor of M_R has a maximal submodule and M_R is Kasch then M_R is co-retractable. In particular, finitely generated Kasch modules are co-retractable.*

Proof. Let $Y < M_R$. By hypothesis, there exists a maximal submodule K of M such that $Y \leq K$. Since M_R is Kasch, M/K can be embedded in M. It follows that $\mathrm{Hom}_R(M/Y, M) \neq 0$, proving that M_R is co-retractable. The last statement is now clear. □

Proposition 2.7. (i) *Every fully co-retractable module is a fully Kasch module.*
(ii) *Every fully retractable module is a fully max module.*
(iii) *Every fully Kasch fully max module is fully co-retractable and fully retractable.*

Proof. (i) Let M_R be a fully co-retractable module and $X \in \sigma[M]$. We shall show that X_R is Kasch. Let U be a simple R-module in $\sigma[X]$, then there exist a set Λ, $Y \leq X^{(\Lambda)}$ and a surjective homomorphism $\alpha : Y \to U$. Since Y_R is co-retractable, $\mathrm{Hom}_R(Y/K, Y) \neq 0$ where $K = \ker \alpha$. It follows that U can be embedded in X, proving that X_R is Kasch.
(ii) Let M_R be a fully retractable module and $0 \neq X \in \sigma[M]$. As seen in the proof of (ii)⇒(iii) of Theorem 2.4, X_R has a maximal submodule.
(iii) Let M_R be a fully Kasch max module and $Y < X \in \sigma[M]$. By Lemma 2.6, X_R is co-retractable. It remains to show that X_R is retractable. By hypothesis, Y_R is Kasch. Hence Y contains a simple R-module U. Clearly there exists a non-zero homomorphism $\theta : X \to \mathrm{E}(U)$. By hypothesis, $\ker \theta =: K$ is contained in a maximal submodule N of X_R. Now since $\mathrm{Im}\theta \simeq (X/K)_R$ is Kasch, the simple R-module $X/N \in \sigma[\mathrm{Im}\theta]$ must be embedded in $\mathrm{E}(U)$. It follows that $X/N \simeq U$ and so $\mathrm{Hom}_R(X, Y) \neq 0$, as desired. □

Theorem 2.8. *The following statements are equivalent for M_R.*
(i) *M_R is Hom-reversible.*
(ii) *M_R is fully retractable and fully co-retractable.*
(iii) *M_R is fully Kasch and fully max.*

Proof. (i)⇒(ii). This follows by a routine argument.
(ii)⇒(i). Let $X, Y \in \sigma[M]$ and $0 \neq f \in \mathrm{Hom}_R(X, Y)$. We shall show that $\mathrm{Hom}_R(Y, X) \neq 0$. Let $\ker f := K$ and consider the exact sequence $0 \to X/K \xrightarrow{f} Y$. Since X_R is co-retractable, $\mathrm{Hom}_R(X/K, X)$ is non-zero. It follows that there exists non-zero element $h \in \mathrm{Hom}_R(Y, \mathrm{E}(X))$. Now since $h(Y)$ is a retractable R-module, $\mathrm{Hom}_R(h(Y), h(Y) \cap X)$ is non-zero. Thus $\mathrm{Hom}_R(Y, X)$ is non-zero.
(ii)⇔(iii). By Proposition 2.7. □

As an application of Theorem 2.8, we will give a new characterization of certain perfect rings in terms of Hom-reversibility of R. A ring R is called *right fully Kasch* if R/I is a right Kasch ring for any proper ideal I of R. A ring R is right fully Kasch if and only if R_R is fully Kasch [2, Proposition 3.15].

Proposition 2.9. *Let R be a ring. The following statements are equivalent.*
(i) *Every cyclic R-module is co-retractable.*
(ii) *R is a right fully Kasch ring.*
(iii) *$R = \bigoplus_{i=1}^{n} R_i$ such that each R_i is a left perfect ring with a unique simple module (up to isomorphism).*

Proof. (i)⇒(ii). Let M be a non-zero R-module. It is easy to see that every simple module in $\sigma[M_R]$ is isomorphic to a factor of a cyclic submodule of $M^{(\Lambda)}$ for some set Λ. Thus M is a Kasch module by (i).
(ii)⇒(iii). By [5, Proposition 2.8], R is left perfect. Thus $R = \bigoplus_{i=1}^{t} e_iR$ where $e_1, ..., e_t$ are orthogonal idempotents and $e_iR/\mathrm{J}(e_iR) \simeq S_i^{(n_i)}$ such that S_i, S_j are non-isomorphic simple R-modules for $i \neq j$. It is enough to show that $\mathrm{Hom}_R(e_iR, e_jR) = 0$ for $1 \leq i \neq j \leq t$. For $i \neq j$ let $f : e_iR \to e_jR$ be a non-zero homomorphism. By hypothesis, S_i can be embedded in $f(e_iR) \leq e_jR$. It follows that S_i can be essentially embedded in e_jR/K for some $K \leq e_jR$. On the other hand, S_j embeds in e_jR/K by hypothesis. Thus $S_j \simeq S_i$, a contradiction.
(iii)⇒(i). This is by Lemma 2.6 and the fact that a left perfect ring with a unique simple module is a fully Kasch ring. □

Theorem 2.10. *The following are equivalent for a ring R.*
(i) *R_R $({}_RR)$ is Hom-reversible.*
(ii) *R_R $({}_RR)$ is fully co-retractable.*
(iii) *$R = \bigoplus_{i=1}^{n} R_i$ such that each R_i is a perfect ring with a unique simple module.*

Proof. (i)⇒(ii). By Theorem 2.8.
(ii)⇒(iii). By Proposition 2.9 and [3, Theorem 3.10].
(iii)⇒(i). By Theorem 2.8(iii) and the fact that if R is a perfect ring with a unique simple module then R_R is fully Kasch. □

A ring R is said to be duo if every right (left) ideal of R is a two sided ideal.

Corollary 2.11. *Let R be a ring Morita equivalent to a duo ring. Then the following statements are equivalent for R.*

(i) *Every cyclic R-module is co-retractable.*
(ii) *R is a perfect ring.*
(iii) R_R *is Hom-reversible.*

Proof. Suppose that R is Morita equivalent to a duo ring T.
(i)⇒(ii). By Proposition 2.9, R and hence T is a left perfect ring. Now since T is a duo ring, it must be a (right) perfect ring.
(ii)⇒(iii). Since T is a duo ring, all idempotent elements in T are central. Now the condition (ii) implies that T satisfies the condition (iii) of Theorem 2.10, proving that R_R is Hom-reversible.
(iii)⇒(i). By Theorem 2.10. □

In the following, we show that the converse of parts (i) and (ii) of Proposition 2.7 do not necessarily hold.

Examples 2.12. (i) *Let R be a local left perfect ring that is not right perfect; see* [10, Example 23.22]. *Hence R_R is fully Kasch but it is not fully co-retractable by Proposition 2.9 and Theorem 2.10.*
(ii) *Let* $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $I = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. *Then since* $I^2 = 0$ *and* $R/I \simeq \mathbb{Q} \oplus \mathbb{Q}$, *it is easily seen that R_R is fully max. But eR is not a retractable R-module, because* $Hom_R(eR, I) = 0$.
(iii) *Let* $M = \mathbb{Q} \oplus A$ *and* $A = \oplus_p \mathbb{Z}_p$ *where p runs over the set of all prime numbers. It is easy to verify that M is a Kasch $\mathbb{Z}$-module but it is not co-retractable, because* $Hom_R(M/N, M) = 0$, *where* $N = \mathbb{Z} \oplus A$.

We are now going to investigate the Rej-reversible modules. We say that M_R is *left* (resp. *right*) *Rej-reversible* whenever for every non-zero $X \in \sigma[M_R]$, $\mathrm{Rej}(X, M) = 0$ (resp. $\mathrm{Rej}(M, X) = 0$) implies $\mathrm{Rej}(M, X) = 0$ (resp. $\mathrm{Rej}(X, M) = 0$). Recall from [12, Proposition 15.1] that $U_e = \oplus\{U \mid U_R \text{ is a finitely generated submodule of } M^{(\mathbb{N})}\}$ is a generator in $\sigma[M_R]$ and $\mathrm{Tr}(U_e, \prod_\Lambda N_\lambda)$ (dented by $\prod_\Lambda^M N_\lambda$) is a product for any family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M_R]$.

Proposition 2.13. *Let M be a non-zero R-module.*
(i) *M is right Rej-reversible if and only if M_R is a co-generator in $\sigma[M_R]$.*
(ii) *M is left Rej-reversible if and only if $\prod_\Lambda^M M$ is a prime module for any set Λ.*

Proof. (i) Just note that for all $X \in \sigma[M_R]$, $\mathrm{Rej}(M, M \oplus X) = 0$.
(ii) ($\Rightarrow$). Let $N \leq \prod_{\Lambda}^{M} M$ for some set Λ. Then $\mathrm{Rej}(N, M) = 0$ and so $\mathrm{Rej}(M, N) = 0$ by hypothesis. Thus $\mathrm{Rej}(\prod_{\Lambda}^{M} M, N) = 0$, proving that $\prod_{\Lambda}^{M} M$ is a prime module.
($\Leftarrow$). Let $X \in \sigma[M_R]$ and $\mathrm{Rej}(X, M) = 0$, then X embeds in $\prod_{\Lambda} M$ for some Λ and so $X = \mathrm{Tr}(U_e, X)$ embeds in $\mathrm{Tr}(U_e, \prod_{\Lambda} M) = \prod_{\Lambda}^{M} M$. Since $\prod_{\Lambda}^{M} M$ is prime, $\mathrm{Rej}(M, X) = 0$, proving that M is left Rej-reversible. □

An R-module M is called *co-semisimple* if any proper submodule of M is an intersection of maximal submodules [12, 23.1].

Theorem 2.14. *The following statements are equivalent for a non-zero module M_R.*
(i) *M is left and right Rej-reversible.*
(ii) *$\mathrm{Rej}(X, Y) = 0$ for all non-zero $X, Y \in \sigma[M_R]$.*
(iii) *M_R is semi-Artinian co-semisimple and $\sigma[M_R]$ has a unique simple module.*
(iv) *Any non-zero $X \in \sigma[M_R]$ is a co-generator in $\sigma[M_R]$.*
(v) *Any non-zero $X \in \sigma[M_R]$ is prime.*
(vi) *M_R is Rej-reversible.*

Proof. (i)$\Rightarrow$(ii). Let $X, Y \in \sigma[M_R]$ be non-zero R-modules. Then by (i) and Proposition 2.13 (i), $\mathrm{Rej}(X, M) = 0 = \mathrm{Rej}(Y, M)$ and so $\mathrm{Rej}(M, Y) = 0$, because M is left Rej-reversible. Thus $\mathrm{Rej}(X, Y) = 0$.
(ii)$\Rightarrow$(iii). Let S be a non-zero simple R-module in $\sigma[M_R]$ and $0 \neq X \in \sigma[M_R]$. By (ii), $\mathrm{Rej}(X, S) = 0$, therefore X is co-semisimple, $\sigma[M_R]$ has a unique simple module and X is semi-Artinian because by hypothesis $\mathrm{Rej}(S, X) = 0$.
(iii)$\Rightarrow$(iv). Let $0 \neq X \in \sigma[M_R]$. Since M_R is co-semisimple, X is co-semisimple by [12, 23.1] and since $\sigma[M_R]$ has a unique simple module, $\mathrm{Rej}(X, S) = 0$ for any simple R-module S in $\sigma[M_R]$. Hence $\mathrm{Rej}(X, Y) = 0$ for all non-zero $Y \in \sigma[M_R]$, because Y has a copy of S.
(iv)$\Rightarrow$(v). This is clear.
(v)$\Rightarrow$(i). By (v), $\prod_{\Lambda}^{M} M$ is prime hence by Proposition 2.13 (ii), M is left Rej-reversible and since $M \bigoplus X$ is prime for all non-zero X in $\sigma[M_R]$, it is easy to see that M is a co-generator in $\sigma[M_R]$. Thus M_R is right Rej-reversible.
(ii)$\Rightarrow$(vi)$\Rightarrow$(i). These are clear by definitions. □

Corollary 2.15. *If M_R is Rej-reversible then M_R is Hom-reversible.*

Proof. This follows from Theorems 2.8 and 2.14. □

An R-module M is called *semi-projective* if $\mathrm{Hom}_R(M, fM) = f\mathrm{End}_R(M)$ for any $f \in \mathrm{End}_R(M)$ [12, p. 260].

Proposition 2.16. *Let M be a semi-projective R-module. Then M is Rej-reversible if and only if M is homogeneous semisimple.*

Proof. The sufficiency is clear. Conversely, by Corollary 2.15, M is Hom-reversible. Hence by Theorem 2.4 it is enough to show that $\mathrm{End}_R(M)$ is semiprime. Let I be a non-zero principal right ideal in $\mathrm{End}_R(M)$. Since M is Rej-reversible, it can be embedded in $\prod_\Lambda IM$ for some set Λ. Hence there exists $f \in \mathrm{Hom}_R(M, IM)$ such that $f|_{IM} \neq 0$. Thus $0 \neq fI \in I^2 = \mathrm{Hom}_R(M, IM)I$. Therefore $\mathrm{End}_R(M)$ is semisimple and by Theorem 2.14(iii), M is homogeneous semisimple. □

We remark that if M_R is Ext-reversible, then a module in $\sigma[M_R]$ is M-projective if and only if it is M-injective. The diligent readers may then be interested in the following project.

Project: Let R be a ring and M_R be a non-zero module. Characterize the Ext-reversibility of M_R.

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