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Author(s):

S. Jahandoust and R. Naghipour

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QUINTASYMPTOTIC SEQUENCES OVER AN IDEAL AND QUINTASYMPTOTIC COGRADE

S. JAHANDOUST AND R. NAGHIPOUR*

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Dedicated to Professor Rahim Zaare-Nahandi

Abstract. Let I denote an ideal of a Noetherian ring R. The purpose of this article is to introduce the concepts of quintasymptotic sequences over I and quintasymptotic cograde of I, and to show that they play a role analogous to quintessential sequences over I and quintessential cograde of I. We show that, if R is local, then the quintasymptotic cograde of I is unambiguously defined and behaves well when passing to certain related local rings. Also, we use this cograde to characterize two classes of local rings.
Keywords: Quintasymptotic prime, quintasymptotic sequence, quasi-unmixed ring.
MSC(2010): Primary: 13E05; Secondary: 13B2, 13B22.

1. Introduction

Since the notion of regular sequences was first given, commutative algebraists have been able to enrich their arsenal with powerful tools. In the early 90's several kinds of sequences have been considered by various mathematicians working on different problems which they are generalizations of regular sequences. D. Katz and L. J. Ratliff, Jr., in [4] and [15] introduced interesting concepts of essential prime divisors, essential sequences, and essential grade of an ideal I in a Noetherian ring

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*Corresponding author.

R, and asymptotic prime divisors, asymptotic sequences, and asymptotic grade of an ideal I in a Noetherian ring R; and therein they showed that these concepts are excellent analogues of, associated prime divisors, R-regular sequences, and the standard grade of I, in the classical theory, respectively.

On the other hand, D. Rees introduced the important concept of an asymptotic sequence over an ideal I in a Noetherian local ring R in [17], and in [8], McAdam and Ratliff, showed that asymptotic sequences over an ideal I in a Noetherian ring R and asymptotic cograde of I (when R is local) have some useful properties; and several bounds on this cograde were established in [8]. The main purpose of the present article is to introduce the concepts of quintasymptotic sequences over an ideal I in a Noetherian ring R and quintasymptotic cograde of I. We show that, quintasymptotic sequences over I behave nicely when passing to certain rings related to R and that the quintasymptotic cograde of I is well defined (when R is local) and satisfies certain rather natural inequalities. Also, we show that if R is local, then any two maximal quintasymptotic sequences over I have the same length, and

$$\operatorname{qacogd}(I) = \min\{\dim R^*/IR^* + z \mid z \text{ is a minimal prime of } R^*\}.$$

Finally, we show that, for every ideal I in a complete local ring R, $\operatorname{agd}(I) + \operatorname{qacogd}(I) = \dim R$ if and only if, for every prime ideal $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} = 1$, $\operatorname{agd}(\mathfrak{p}) + \operatorname{qacogd}(\mathfrak{p}) = \dim R$, if and only if R has a unique minimal prime divisor of zero.

Throughout this paper, all rings considered will be commutative and Noetherian and will have non-zero identity elements. Such a ring will be denoted by R. For terminology, we follow [2], [5] and [10].

Let I be an ideal of R. We denote by $\mathscr{R}$ the *Rees ring* $R[u, It] := \oplus_{n\in\mathbb{Z}} I^n t^n$ of R w.r.t. I, where t is an indeterminate and $u = t^{-1}$. Also, the *radical* of I, denoted by $\operatorname{Rad}(I)$, is defined to be the set $\{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$. For each R-module L, we denote by $\operatorname{mAss}_R L$ the set of minimal primes of $\operatorname{Ass}_R L$. If $(R, \mathfrak{m})$ is Noetherian local, then R^* denotes the completion of R with respect to the $\mathfrak{m}$-adic topology. Then R is said to be an *unmixed* (respectively, *quasi-unmixed*) ring if for every $\mathfrak{p} \in \operatorname{Ass}_{R^*} R^*$ (respectively, $\mathfrak{p} \in \operatorname{mAss}_{R^*} R^*$), the condition $\dim R^*/\mathfrak{p} = \dim R$ is satisfied. More generally, if R is not necessarily local, R is a *locally unmixed* (respectively, *locally quasi-unmixed*) ring if for any $\mathfrak{p} \in \operatorname{Spec} R$, $R_\mathfrak{p}$ is an unmixed (respectively, quasi-unmixed) ring. A prime ideal $\mathfrak{p}$ of R is called a *quintessential* (respectively, *quintasymptotic*) prime ideal of I precisely when there exists $\mathfrak{q} \in \operatorname{Ass}_{R^*_\mathfrak{p}} R^*_\mathfrak{p}$

(respectively, $\mathfrak{q} \in \mathrm{mAss}_{R^*_\mathfrak{p}} R^*_\mathfrak{p}$) such that $\mathrm{Rad}(IR^*_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}R^*_\mathfrak{p}$. The set of *quintessential* (respectively, *quintasymptotic*) primes of I is denoted by $Q(I)$ (respectively, $\overline{Q^*}(I)$). Then the *essential* (respectively, *asymptotic*) primes of I, denoted by $E(I)$ (respectively, $\overline{A^*}(I)$), is defined to be the set $\{\mathfrak{q} \cap R \mid \mathfrak{q} \in Q(u\mathscr{R})\}$ (respectively, $\{\mathfrak{q} \cap R \mid \mathfrak{q} \in \overline{Q^*}(u\mathscr{R})\}$). Finally, we shall use $A^*(I)$ to denote the ultimately constant value of the sequence $\mathrm{Ass}_R R/I^n$, which is a well defined finite set of prime ideals, (cf. [1]).

A brief summary of the contents of this paper will now be given. Let R be a commutative Noetherian ring and I an arbitrary ideal of R. In the second section, the notion of the quintasymptotic sequences over I is introduced, and it is shown that most of the basic properties of the quintessential sequences (respectively, essential sequences) over I, given in [16] (respectively, [4]), extend to the quintasymptotic sequences over I. In fact, it is shown in this section that the quintasymptotic sequence over I behaves nicely with respect to passing to certain rings related to R. In the third section, the concept of quintasymptotic cograde of an ideal is developed and it is shown that most of bounds on the essential, asymptotic and quintessential cograde of I given in [4], [8] and [16] have a valid analogous for the quintasymptotic cograde of I. Finally, in section 4 we characterize two classes of local rings by using the quintasymptotic cograde.

2. Quintasymptotic sequences over an ideal

In this section we introduce the notion of quintasymptotic sequences over an ideal I of a Noetherian ring R and we show that they have most of basic properties enjoyed by quintessential sequences, essential sequences and asymptotic sequences over I. We begin with the following definitions.

Definition 2.1. *Let I and $\mathfrak{p}$ be ideals of a Noetherian ring R such that $\mathfrak{p}$ is prime. Then $\mathfrak{p}$ is called a quintasymptotic (respectively, quintessential) prime ideal of I precisely when there exists $z \in \mathrm{mAss}_{R^*_\mathfrak{p}} R^*_\mathfrak{p}$ (respectively, $z \in \mathrm{Ass}_{R^*_\mathfrak{p}} R^*_\mathfrak{p}$) such that $\mathrm{Rad}(IR^*_\mathfrak{p} + z) = \mathfrak{p}R^*_\mathfrak{p}$. The set of quitasymptotic (respectively, quintessential) primes of I is denoted by $\overline{Q^*}(I)$ (respectively, $Q(I)$).*

Definition 2.2. *Let I denote an ideal of a Noetherian ring R. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called a quintasymptotic (respectively, quintessential) sequence over I if, $(I, (\mathbf{x})) \neq R$ and for all*

$1 \leq i \leq n$, *we have* $x_i \notin \bigcup\{\mathfrak{p} \in \overline{Q^*}((I,(x_1,\ldots,x_{i-1})))$ *(respectively,* $x_i \notin \bigcup\{\mathfrak{p} \in Q((I,(x_1,\ldots,x_{i-1})))$*. A quintasymptotic (respectively, quintessential) sequence over* (0) *is simply called a quintasymptotic (respectively, quintessential) sequence in* R

A quintasymptotic (respectively, quintessential) sequence $\mathbf{x} = x_1,\ldots,x_n$ of elements of R over I is *maximal* if $x_1,\ldots,x_n,x_{n+1}$ is not a quintasymptotic (respectively, quintessential) sequence over I for any $x_{n+1} \in R$. If R is local, then it is shown in Theorem 3.2 (respectively, [16, Theorem 3.2]) that all maximal quintasymptotic (respectively, quintessential) sequences over I have the same length. This allows us to introduce the fundamental notion of *quintasymptotic cograde* (respectively, *quintessential cograde*), $\operatorname{qacogd}(I)$ (respectively, $\operatorname{qecogd}(I)$), of I. Also, it is shown in Corollary 2.15 (respectively, [9, Proposition 4.3]) that all maximal quintasymptotic (respectively, quintessential) sequences coming from I have the same length. Therefore, we define the fundamental notion of *quintasymptotic grade* (respectively, *quintessential grade*), $\operatorname{qagd}(I)$ (respectively, $\operatorname{qegd}(I)$) of I.

The following lemma is needed in the proof of the main results of this paper.

Lemma 2.3. *Let* I *and* J *be ideals in a Noetherian ring* R*. Then the following hold:*

(i) *If* $\mathfrak{p}$ *is a minimal prime divisor of* I*, then* $\mathfrak{p} \in \overline{Q^*}(I)$*.*

(ii) *If* $\operatorname{Rad}(I) = \operatorname{Rad}(J)$*, then* $\overline{Q^*}(I) = \overline{Q^*}(J)$*.*

(iii) $\overline{Q^*}(I) \subseteq \overline{A^*}(I) \cap Q(I)$ *and* $\overline{A^*}(I) \cup Q(I) \subseteq E(I) \subseteq A^*(I)$*.*

(iv) *If* $I \subseteq \mathfrak{p} \in \operatorname{Spec} R$ *and* S *is a multiplicatively closed subset of* R *which is disjoint from* $\mathfrak{p}$*, then* $\mathfrak{p} \in \overline{Q^*}(I)$ *if and only if* $\mathfrak{p}R_S \in \overline{Q^*}(IR_S)$*.*

(v) $\mathfrak{p} \in \overline{Q^*}(I)$ *if and only if there is* $z \in \operatorname{mAss}_R R$ *such that* $z \subseteq \mathfrak{p}$ *and* $\mathfrak{p}/z \in \overline{Q^*}(I(R/z))$*.*

(vi) *If* $z \in \operatorname{mAss}_R R$ *and* $\mathfrak{p}$ *is a minimal prime over* $I + z$*, then* $\mathfrak{p} \in \overline{Q^*}(I)$*.*

(vii) *Let the ring* T *be a faithfully flat Noetherian extension of* R*. Let* $\mathfrak{q}$ *be a prime ideal in* T *such that* $\mathfrak{p} = \mathfrak{q} \cap R$*. If* $\mathfrak{q} \in \overline{Q^*}(IT)$*, then* $\mathfrak{p} \in \overline{Q^*}(I)$ *and* $\mathfrak{q} \in \overline{Q^*}(\mathfrak{p}T)$*. Moreover, if* $\mathfrak{p} \in \overline{Q^*}(I)$ *and* $\mathfrak{q}$ *is minimal over* $\mathfrak{p}T$*, then* $\mathfrak{q} \in \overline{Q^*}(IT)$*.*

(viii) *Let the ring* T *be a finite module extension of* R*. If* $\mathfrak{p} \in \overline{Q^*}(I)$*, then there is a* $\mathfrak{q} \in \overline{Q^*}(IT)$ *such that* $\mathfrak{q} \cap R = \mathfrak{p}$*. Moreover, if all minimal primes in* T *lies over a minimal prime in* R*, then the converse holds.*

(ix) $\overline{Q^*}((I,X)R[X]) = \{(\mathfrak{p},X)R[X] \mid \mathfrak{p} \in \overline{Q^*}(I)\}$.

Proof. (i) and (ii) follow readily from definition. (iii)-(viii) are proved in [7, Lemmas 2.1, 3.4 and Propositions 3.6, 3.8]. To prove (ix), let $\mathfrak{p} \in \overline{Q^*}(I)$. Then by (iv),(v) and (vii), we may assume that R is a complete local domain with maximal ideal $\mathfrak{p}$. Then, in view of [11, Propositions 6 and 7], R and $R[X]$ are locally unmixed. Thus $\overline{Q^*}(J) = Q(J)$ for all ideals J in R and in $R[X]$. Therefore $(\mathfrak{p},X)R[X] \in \overline{Q^*}((I,X)R[X])$, by [16, Lemma 2.7]. The other inclusion is similar. □

The next result is a consequence of Lemma 2.3(i) and Definition 2.2.

Corollary 2.4. *Let I be an ideal in a Noetherian ring R and let $\mathbf{x} = x_1,\ldots,x_n$ be a sequence of elements of R.*

(i) *If $\mathbf{x}$ is a quintasymptotic sequence over I, then* $\operatorname{height}(I,(\mathbf{x})) \geq \operatorname{height} I + n$. *Therefore by the* Generalized Principal Ideal Theorem *if $\mathbf{x}$ is a quintasymptotic sequence in R, then* $\operatorname{height}((\mathbf{x})) = n$.

(ii) *The sequence $\mathbf{x}$ is a maximal quintasymptotic sequence over I if and only if $\mathbf{x}$ is a quintasymptotic sequence over I and for each maximal ideal $\mathfrak{m}$ in R containing $(I,(\mathbf{x}))$ it holds that $\mathfrak{m} \in \overline{Q^*}((I,(\mathbf{x})))$.*

The following proposition shows that the quintasymptotic sequences over an ideal are well behaved when passing to localization.

Proposition 2.5. *Let I be an ideal in a Noetherian ring R and let $\mathbf{x} = x_1,\ldots,x_n$ be a sequence of elements of R. Then the following statements hold:*

(i) *If $\mathbf{x}$ is a quintasymptotic sequence over I and S a multiplicatively closed subset of R such that $(I,(\mathbf{x}))R_S \neq R_S$, then the image of $\mathbf{x}$ in R_S is a quintasymptotic sequence over IR_S. The converse holds if for all $\mathfrak{p} \in \bigcup\{\mathfrak{q} \in \overline{Q^*}((I,(x_1,\ldots,x_i))); i = 0,\ldots,n-1\}$, we have $\mathfrak{p}R_S \neq R_S$.*

(ii) *If $\mathbf{x}$ is a maximal quintasymptotic sequence over I, then for each maximal ideal $\mathfrak{m}$ in R containing $(I,(\mathbf{x}))$ it holds that the image of $\mathbf{x}$ in $R_\mathfrak{m}$ is a maximal quintasymptotic sequence over $IR_\mathfrak{m}$. The converse holds if the x_i are contained in the Jacobson radical of R.*

Proof. (i) follows from Lemma 2.3(iv). The first statement in (ii) follows from part (i) and Corollary 2.4(ii). For the last statement in (ii) it will first be shown that $\mathbf{x}$ is a quintasymptotic sequence over I. To this end, suppose the contrary is true. Then there exists i such that $x_i \in \mathfrak{p}$ for some $\mathfrak{p} \in \overline{Q^*}((I,(x_1,\ldots,x_{i-1})))$. Let $\mathfrak{m}$ be a maximal ideal in R containing $\mathfrak{p}$. Then the hypotheses implies that $(I,(\mathbf{x}))R \subseteq \mathfrak{m}$, and so the hypotheses and Lemma 2.3(iv) imply that the image of x_i is in $\mathfrak{p}R_{\mathfrak{m}} \in \overline{Q^*}((I,(x_1,...,x_{i-1}))R_{\mathfrak{m}})$. But this implies that the image of $\mathbf{x}$ in $R_{\mathfrak{m}}$ is not quintasymptotic sequence over $IR_{\mathfrak{m}}$, in contradiction to the hypotheses. Therefore, $\mathbf{x}$ is a quintasymptotic sequence over I. If $\mathfrak{m}$ is a maximal ideal in R containing $(I,(\mathbf{x}))$, then by Corollary 2.4(ii) and Lemma 2.3(iv) we have $\mathfrak{m} \in \overline{Q^*}((I,(\mathbf{x})))$, and so by Corollary 2.4(ii), $\mathbf{x}$ is a maximal quintasymptotic sequence over I. $\square$

The following result shows that the quintasymptotic sequences over an ideal I of a Noetherian ring R are well behaved when passing to the factor rings modulo minimal primes of R.

Proposition 2.6. *Let I be an ideal in a Noetherian ring R and let $\mathbf{x} = x_1,\dots,x_n$ be a sequence of elements of R. Then the following statements hold:*

(i) $\mathbf{x}$ *is a quintasymptotic sequence over I if and only if the image of* $\mathbf{x}$ *in R/z is a quintasymptotic sequence over $I(R/z)$ for all $z \in \mathrm{mAss}_R R$.*

(ii) $\mathbf{x}$ *is a maximal quintasymptotic sequence over I if and only if the image of* $\mathbf{x}$ *in R/z is a quintasymptotic sequence over $I(R/z)$ for all $z \in \mathrm{mAss}_R R$, and for all maximal ideals $\mathfrak{m}$ in R containing $(I,(\mathbf{x}))$, there exists $z \in \mathrm{mAss}_R R$ such that $z \subseteq \mathfrak{m}$ and $\mathfrak{m}/z \in \overline{Q^*}((I,(\mathbf{x}))(R/z))$.*

Proof. It follows readily from Lemma 2.3(v) and Corollary 2.4(ii). $\square$

The next result shows that the quintasymptotic sequences over an ideal are well behaved when passing to faithfully flat Noetherian extension rings of R. Before bringing it, let us recall the following definition.

Definition 2.7. *Let $R \subseteq T$ be Noetherian rings.*

(i) *We say that R is dominated by T if, for every proper ideal I of R, we have $IT \neq T$ and every maximal ideal of T lies over a maximal ideal of R.*

(ii) *We say that the Theorem of Transition holds for rings R and T if, R is dominated by T and if $\mathfrak{q}$ is a primary ideal of R such that* $\mathrm{Rad}(\mathfrak{q})$ *is a maximal ideal, say $\mathfrak{m}$, then* $\mathrm{length}_T T/\mathfrak{q}T$ *is finite and that*

$$\mathrm{length}_T T/\mathfrak{q}T = (\mathrm{length}_T T/\mathfrak{m}T)(\mathrm{length}_R R/\mathfrak{q}).$$

Proposition 2.8. *Let $R \subseteq T$ be a faithfully flat extension of Noetherian rings. Let I be an ideal of R and let $\mathbf{x} = x_1,\dots,x_n$ be a sequence of elements of R. Then the following hold:*

(i) $\mathbf{x}$ *is a quintasymptotic sequence over I if and only if* $\mathbf{x}$ *is a quintasymptotic sequence over IT.*

(ii) *If $R \subseteq T$ satisfy the Theorem of Transition, then $\mathbf{x}$ is a maximal quintasymptotic sequence over I if and only if $\mathbf{x}$ is a maximal quintasymptotic sequence over IT.*

Proof. (i) follows immediately from Lemma 2.3(vii). In order to prove (ii), let $\mathbf{x}$ be a maximal quintasymptotic sequence over I and $\mathfrak{n}$ a maximal ideal of T containing $(I, (\mathbf{x}))T$. Then by (i), $\mathbf{x}$ is a quintasymptotic sequence over IT. Let $\mathfrak{m} := \mathfrak{n} \cap R$. Since R is dominated by T, it follows that $\mathfrak{m}$ is a maximal ideal containing $(I, (\mathbf{x}))$, and so $\mathfrak{m} \in \overline{Q^*}((I, (\mathbf{x})))$, by Corollary 2.4(ii). Therefore, $\mathfrak{n} \in \overline{Q^*}((I, (\mathbf{x}))T)$ by Lemma 2.3(vii) and hence, $\mathbf{x}$ is a maximal quintasymptotic sequence over IT, by Corollary 2.4(ii).

Now, let $\mathbf{x}$ be a maximal quintasymptotic sequence over IT. Then by (i), $\mathbf{x}$ is a quintasymptotic sequence over I. Let $\mathfrak{m}$ be a maximal ideal of R containing $(I, (\mathbf{x}))$ and let $\mathfrak{n}$ be a maximal ideal in T containing $\mathfrak{m}T$. Then $\mathfrak{n} \in \overline{Q^*}((I, (\mathbf{x}))T)$ by Corollary 2.4(ii). On the other hand, $\mathfrak{m} = \mathfrak{n} \cap R$, and so $\mathfrak{m} \in \overline{Q^*}((I, (\mathbf{x})))$ by Lemma 2.3(vii). Therefore, $\mathbf{x}$ is a maximal quintasymptotic sequence over I, by Corollary 2.4(ii). □

The next result shows that the quintasymptotic sequences over an ideal are well behaved when passing to finite extension rings of R.

Proposition 2.9. *Let $R \subseteq T$ be Noetherian rings, with T a finitely generated R-module. Let I be an ideal of R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then the following statements hold:*

(i) *If $\mathbf{x}$ is a quintasymptotic sequence over IT, then $\mathbf{x}$ is a quintasymptotic sequence over I.*

(ii) *If every minimal prime of T lies over a minimal prime in R, then $\mathbf{x}$ is a quintasymptotic sequence over I if and only if $\mathbf{x}$ is a quintasymptotic sequence over IT.*

(iii) *If every minimal prime of T lies over a minimal prime in R, then $\mathbf{x}$ is a maximal quintasymptotic sequence over I if and only if $\mathbf{x}$ is a quintasymptotic sequence over IT and for each maximal ideal $\mathfrak{m}$ in R that contains $(I, (\mathbf{x}))$, there exists a prime ideal $\mathfrak{n}$ in T such that $\mathfrak{m} = \mathfrak{n} \cap R$ and $\mathfrak{n} \in \overline{Q^*}((I, (\mathbf{x}))T)$.*

Proof. It follows readily from Lemma 2.3 (viii) and Corollary 2.4(ii). □

The next proposition is concerned with the quintasymptotic sequences over I and $IR[X]$, where X is an indeterminate over R.

Proposition 2.10. *Let I be an ideal in a Noetherian ring R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then the following statements are equivalent:*

(i) *The sequence $\mathbf{x}$ is a (respectively, maximal) quintasymptotic sequence over I.*

(ii) *The sequence $x_1, \ldots, x_i, X, x_{i+1}, \ldots, x_n$ is a (respectively, maximal) quintasymptotic sequence over $IR[X]$ for some $i = 0, 1, \ldots, n$.*

(iii) *The sequence $x_1, \ldots, x_i, X, x_{i+1}, \ldots, x_n$ is a (respectively, maximal) quintasymptotic sequence over $IR[X]$ for every $i = 0, 1, \ldots, n$.*

Proof. In view of Lemma 2.3(vii), for $j = 0, 1, \ldots, i$, we have

$$\overline{Q^*}((I, (x_1, \ldots, x_j))R[X]) = \{\mathfrak{p}R[X] \mid \mathfrak{p} \in \overline{Q^*}((I, (x_1, \ldots, x_j))\},$$

(note that, for an ideal J in R, the prime divisors of $JR[X]$ are $\mathfrak{p}R[X]$ such that $\mathfrak{p}$ is a prime divisor of J). Also, it is clear that X is not in any prime divisor of $(I, (x_1, \ldots, x_i))R[X]$. Moreover, for $k = 0, 1, \ldots, n - i$, we have

$$\overline{Q^*}((I, (x_1, \ldots, x_i, X, x_{i+1}, \ldots, x_{i+k}))R[X]) = \{(\mathfrak{p}, X)R[X] \mid \mathfrak{p} \in \overline{Q^*}((I, (x_1, \ldots, x_{i+k})))\},$$

by Lemma 2.3(ix). Now, the result follows from this and Corollary 2.4(ii). Note that the maximal ideals of $R[X]$ containing $(I, X)R[X]$ are the ideals $(\mathfrak{m}, X)R[X]$ such that $\mathfrak{m}$ is a maximal ideal of R containing I. □

The following result is concerned with quintasymptotic sequences over ideals with the same radical.

Proposition 2.11. *Let I and J be ideals in a Noetherian ring R such that* $\mathrm{Rad}(I) = \mathrm{Rad}(J)$*, and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then the following statements hold:*

(i) $\mathbf{x}$ *is a quintasymptotic sequence over I if and only if* $\mathbf{x}$ *is a quintasymptotic sequence over J.*

(ii) $\mathbf{x}$ *is a maximal quintasymptotic sequence over I if and only if* $\mathbf{x}$ *is a maximal quintasymptotic sequence over J.*

Proof. As $\mathrm{Rad}(I) = \mathrm{Rad}(J)$, it follows that

$$\mathrm{Rad}((I, (x_1, \ldots, x_i))) = \mathrm{Rad}((J, (x_1, \ldots, x_i))),$$

for all $i = 0, 1, \ldots, n$, and in view of Lemma 2.3(ii), we have

$$\overline{Q^*}((I, (x_1, \ldots, x_i))) = \overline{Q^*}((J, (x_1, \ldots, x_i))),$$

for all $i = 0, 1, \ldots, n$. Therefore (i) is true by Definition 2.2.

Now, let $\mathbf{x}$ be a maximal quintasymptotic sequence over I. Then by (i), $\mathbf{x}$ is a quintasymptotic sequence over J. Let $\mathfrak{m}$ be a maximal ideal of R containing $(J, (\mathbf{x}))$. Then, $(I, (\mathbf{x})) \subseteq \mathfrak{m}$. Since $\mathbf{x}$ is a maximal quintasymptotic sequence over I, it follows from Corollary 2.4(ii) that $\mathfrak{m} \in \overline{Q^*}((I, (\mathbf{x})))$. Therefore $\mathfrak{m} \in \overline{Q^*}((J, (\mathbf{x})))$, and so by Corollary 2.4(ii), $\mathbf{x}$ is a maximal quintasymptotic sequence over J. The converse will be proved similarly. □

The next remark, which gives us some additional basic information concerning quintasymptotic sequences over an ideal, follows immediately from the Definition 2.2 and Lemma 2.3(ii).

Remark 2.12. *Let I be an ideal in a Noetherian ring R and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then the following statements are equivalent:*

(i) *$x_1, \ldots, x_n$ is a quintasymptotic sequence over I.*

(ii) *${x_1}^{m_1}, \ldots, {x_n}^{m_n}$ is a quintasymptotic sequence over I for some positive integers m_i.*

(iii) *${x_1}^{m_1}, \ldots, {x_n}^{m_n}$ is a quintasymptotic sequence over I for all positive integers m_i.*

(iv) *There exists an integer $i \in \{0, \ldots, n-1\}$ such that $x_1, \ldots, x_i$ is a quintasymptotic sequence over I and $x_{i+1}, \ldots, x_n$ is quintasymptotic sequence over $(I, (x_1, \ldots, x_i))$.*

(v) *For all $i \in \{0, \ldots, n-1\}$, $x_1, \ldots, x_i$ is a quintasymptotic sequence over I and $x_{i+1}, \ldots, x_n$ is a quintasymptotic sequence over $(I, (x_1, \ldots, x_i))$.*

In the remainder of this section, we examine the quintasymptotic sequences over the zero ideal. Before bringing the next results we recall that a sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an *asymptotic* (respectively, *essential*) *sequence over I* if, $(I, (\mathbf{x})) \neq R$ and for all $1 \leq i \leq n$, we have $x_i \notin \bigcup\{\mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{i-1})))$ (respectively, $x_i \notin \bigcup\{\mathfrak{p} \in E((I, (x_1, \ldots, x_{i-1})))$. An asymptotic (respectively, essential) sequence over (0) is simply called an *asymptotic* (respectively, *essential*) *sequence in R*. An asymptotic (respectively, essential) sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R over I is *maximal* if $x_1, \ldots, x_n, x_{n+1}$ is not an asymptotic (respectively, essential) sequence over I for any $x_{n+1} \in R$. If R is local, then it is shown in [3, Theorem 1.9] (respectively, [4, Theorem 4.1]) that all maximal asymptotic (respectively, essential) sequences over I have the same length. This allows us to introduce the fundamental notion of *asymptotic cograde* (respectively, *essential cograde*), $\operatorname{acogd}(I)$ (respectively, $\operatorname{ecogd}(I)$), of I. Also, it is shown in

[15, Theorem 3.1] (respectively, [4, Proposition 3.10]) that all maximal asymptotic (respectively, essential) sequences coming from I have the same length. Therefore, we define the fundamental notion of *asymptotic grade* (respectively, *essential grade*), $\operatorname{agd}(I)$ (respectively, $\operatorname{egd}(I)$) of I.

The next proposition is a consequence of [4, Theorem 3.1] and [16, Proposition 4.1].

Proposition 2.13. *Let R be a Noetherian ring such that $\operatorname{Ass}_{R_{\mathfrak{p}}^*} R_{\mathfrak{p}}^*$ has no embedded prime ideals for all $\mathfrak{p} \in \operatorname{Spec} R$. Let I be an ideal of R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then $\overline{Q^*}(I) = Q(I) \subseteq \overline{A^*}(I) = E(I)$. Moreover, the following statements are equivalent:*

(i) $\mathbf{x}$ *is a quintasymptotic sequence in R;*

(ii) $\mathbf{x}$ *is an asymptotic sequence in R;*

(iii) $\mathbf{x}$ *is a quintessential sequence in R;*

(iv) $\mathbf{x}$ *is an essential sequence in R.*

Proof. It is easy to see that $\overline{Q^*}(I) = Q(I)$. Also, by [4, Theorem 3.1] we have $Q(I) \subseteq \overline{A^*}(I) = E(I)$. Now, in view of [16, Proposition 4.1] and [4, Proposition 3.10], we have (iii) $\Longleftrightarrow$ (iv) and (ii) $\Longleftrightarrow$ (iii), and the first statement shows that (i) $\Longleftrightarrow$ (iii). $\square$

Now we show that the quintasymptotic grade of I is well defined and equals with asymptotic grade of I.

Proposition 2.14. *Let I be an ideal of a Noetherian ring R. If I is generated by a quintasymptotic sequence of elements of R, then $\overline{Q^*}(I) = \overline{A^*}(I)$.*

Proof. In view of Lemma 2.3(iii) it suffices to show that $\overline{A^*}(I) \subseteq \overline{Q^*}(I)$. To do this, let $\mathbf{x} = x_1, \ldots, x_n$ be a quintasymptotic sequence of elements of R and $I = (\mathbf{x})$. Let $\mathfrak{p} \in \overline{A^*}(I)$. Then $\mathfrak{p}R_{\mathfrak{p}} \in \overline{A^*}(IR_{\mathfrak{p}})$. Further, if $\mathfrak{p}R_{\mathfrak{p}} \in \overline{Q^*}(IR_{\mathfrak{p}})$, then by Lemma 2.3(iv), $\mathfrak{p} \in \overline{Q^*}(I)$. Also by Proposition 2.5, $IR_{\mathfrak{p}}$ is generated by a quintasymptotic sequence in $R_{\mathfrak{p}}$. Thus, we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. Moreover, $\mathfrak{p}R^* \in \overline{A^*}(IR^*)$ by [6, Proposition 3.18], and if $\mathfrak{p}R^* \in \overline{Q^*}(IR^*)$, then $\mathfrak{p} \in \overline{Q^*}(I)$, by Lemma 2.3(vii). Also, IR^* is generated by a quintasymptotic sequence in R^* by Proposition 2.8. Therefore, we may assume that $(R, \mathfrak{p})$ is a complete local ring. Finally, by [6, Proposition 3.18] there exists $z \in \operatorname{mAss}_R R$ such that $\mathfrak{p}/z \in \overline{A^*}(I(R/z))$, and if $\mathfrak{p}/z \in \overline{Q^*}(I(R/z))$, then $\mathfrak{p} \in \overline{Q^*}(I)$, by Lemma 2.3(v). Also, $I(R/z)$ is generated by a quintasymptotic sequence in R/z by Proposition 2.6. Consequently, we may assume that $(R, \mathfrak{p})$ is a complete local domain. Now, since by [11, Proposition 6] R is locally unmixed, it follows from Proposition 2.13 that I is generated by a quintessential sequence in R and $\overline{Q^*}(I) = Q(I) \subseteq \overline{A^*}(I) = E(I)$. Whence by [4, Theorem 2.5], we have $\overline{Q^*}(I) = \overline{A^*}(I)$, as required. □

Corollary 2.15. *Let R be a Noetherian ring and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then $\mathbf{x}$ is an asymptotic sequence in R if and only if $\mathbf{x}$ is a quintasymptotic sequence in R. In particular* $\operatorname{agd}(I) = \operatorname{qagd}(I)$ *for all ideals I in R.*

Proof. It follows from Lemma 2.3(iii) and Proposition 2.14. □

Corollary 2.16. *Let R be a Noetherian ring such that $\operatorname{Ass}_{R^*_{\mathfrak{m}}} R^*_{\mathfrak{m}}$ has no embedded prime ideals for all maximal ideals $\mathfrak{m}$ in R. Let I be an ideal of R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Then the following statements are equivalent:*

(i) $\mathbf{x}$ *is a quintasymptotic sequence in R.*
(ii) $\mathbf{x}$ *is an asymptotic sequence in R.*
(iii) $\mathbf{x}$ *is a quintessential sequence in R*
(iv) $\mathbf{x}$ *is an essential sequence in R.*

In particular, $\operatorname{qagd}(I) = \operatorname{agd}(I) = \operatorname{qegd}(I) = \operatorname{egd}(I)$.

Proof. It follows from Corollary 2.15, [16, Proposition 4.1] and [4, Proposition 3.10]. □

Remark 2.17. *It has been shown in* [6, Lemma 6.13] *that, if $\mathfrak{p} \in \overline{A^*}(I)$, $\mathbf{x} = x_1, \ldots, x_n$ is an asymptotic sequence over I and $\mathfrak{q}$ a minimal prime divisor of $(\mathfrak{p}, (\mathbf{x}))$, then $\mathfrak{q} \in \overline{A^*}((I, (\mathbf{x})))$. Also, this statement is proved in* [4, Theorem 5.1] *for essential primes and essential sequences. But the statement is not true for quintasymptotic primes and quintasymptotic sequences. Because otherwise, it is easy to show that it holds for quintessential primes and quintessential sequences, and this contradicts* [4, Example 7.3].

The next corollary is a weaker result compared to the above remark for quintasymptotic primes and quintasymptotic sequences.

Corollary 2.18. *Let $\mathbf{x} = x_1, \ldots, x_n$ be a quintasymptotic sequence in a Noetherian ring R. Let $1 \leq i < n$ and $\mathfrak{p} \in \overline{Q^*}((x_1, \ldots, x_i))$. If $\mathfrak{q}$ is a minimal prime divisor of $(\mathfrak{p}, (x_{i+1}, \ldots, x_n))$, then $\mathfrak{q} \in \overline{Q^*}((\mathbf{x}))$.*

Proof. By Proposition 2.14, we have $\overline{Q^*}((x_1, \ldots, x_j)) = \overline{A^*}((x_1, \ldots, x_j))$, for all $1 \leq j \leq n$. Therefore, $x_{i+1}, \ldots, x_n$ is an asymptotic sequence over $x_1, \ldots, x_i$, and so the conclusion follows from [6, Lemma 6.13]. □

3. **Quintasymptotic cograde**

In this section we show that the quintasymptotic cograde of an ideal in a Noetherian local ring is unambiguously defined and behaves well when passing to certain related local rings. We begin with the following useful lemma which is proved by L. J. Ratliff, Jr.

Lemma 3.1. *Let* $(R, \mathfrak{m})$ *be a complete Noetherian local ring that has only one prime divisor of zero and let* I *be an ideal of* R*. Then* $\mathrm{ecogd}(I) = \dim R/I$ *and* $\mathrm{egd}(I) + \mathrm{ecogd}(I) = \dim R$*.*

Proof. See [16, Lemma 3.1]. □

Theorem 3.2. *Let* I *denote an ideal in a Noetherian local ring* $(R, \mathfrak{m})$*. Then any two maximal quintasymptotic sequences over* I *have the same length. In fact,*

$$\begin{aligned}\mathrm{qacogd}(I) =&\min\{\dim R^*/IR^* + z \mid z \in \mathrm{mAss}_{R^*} R^*\}\\ =&\min\{\dim R^*/z - \mathrm{height}(IR^* + z/z) \mid z \in \mathrm{mAss}_{R^*} R^*\}.\end{aligned}$$

Proof. Let $\mathbf{x} = x_1, \dots, x_n$ be a maximal quintasymptotic sequence over I. Then by Proposition 2.8, $\mathbf{x}$ is a maximal quintasymptotic sequence over IR^*. Also by Proposition 2.6, the image of $\mathbf{x}$ in R^*/z is a quintasymptotic sequence over $IR^* + z/z$, for all $z \in \mathrm{mAss}_{R^*} R^*$ and this image is a maximal quintasymptotic sequence over $IR^* + z/z$ for such z. Now, as R^*/z is a complete local domain, it follows that the image of $\mathbf{x}$ in R^*/z is a quintessential sequence over $IR^* + z/z$, and so for all $z \in \mathrm{mAss}_{R^*} R^*$, we have $n \leq \mathrm{qecogd}(IR^* + z/z)$; and the equality holds for some such z. Therefore,

$$n = \min\{\mathrm{qecogd}(IR^* + z/z) \mid z \in \mathrm{mAss}_{R^*} R^*\}.$$

On the other hand, in view of Lemma 3.1,

$$\mathrm{qecogd}(IR^* + z/z) = \dim R^*/IR^* + z.$$

Consequently, it follows that $\mathrm{qacogd}(I)$ is unambiguously defined and

$$\mathrm{qacogd}(I) = \min\{\dim R^*/IR^* + z \mid z \in \mathrm{mAss}_{R^*} R^*\}.$$

Finally, the last equality follows from the fact that:

$$\dim R^*/IR^* + z = \dim(R^*/z)/(IR^* + z/z)$$

and

$$\mathrm{height}(IR^* + z/z) + \dim(R^*/z)/(IR^* + z/z) = \dim R^*/z.$$

□

The following theorem shows that qacogd(I) behaves nicely when passing to certain related rings and ideals.

Theorem 3.3. *Let I and J be ideals in a Noetherian local ring $(R, \mathfrak{m})$. Then*

(i) *If $I \subseteq J$, then* $\operatorname{qacogd}(J) \leq \operatorname{qacogd}(I)$.

(ii) *If* $\operatorname{Rad}(I) = \operatorname{Rad}(J)$, *then* $\operatorname{qacogd}(I) = \operatorname{qacogd}(J)$.

(iii) $\operatorname{qacogd}(I) = \min\{\operatorname{qacogd}(I + z/z) \mid z \in \operatorname{mAss}_R R\}$.

(iv) *If T is a faithfully flat Noetherian extension of R, then* $\operatorname{qacogd}(I) \leq \operatorname{qacogd}(IT_{\mathfrak{n}})$ *for all prime ideals $\mathfrak{n}$ in T lying over $\mathfrak{m}$ and equality holds if* $\operatorname{height}\mathfrak{m} = \operatorname{height}\mathfrak{n}$. *In particular* $\operatorname{qacogd}(I) = \operatorname{qacogd}(IR^*)$

(v) *If T is a finite extension ring of R such that every minimal prime of T lies over a minimal prime of R, then* $\operatorname{qacogd}(I) \leq \operatorname{qacogd}(IT_{\mathfrak{n}})$ *for all prime ideals $\mathfrak{n}$ in T lying over $\mathfrak{m}$, and equality holds for some such $\mathfrak{n}$.*

(vi) $\operatorname{qacogd}(I) = \operatorname{qacogd}((I, X)R[X]_{(\mathfrak{m},X)})$.

Proof. (i) and (ii) follow from Theorem 3.2 and (iii) follows from Theorem 3.2 and Proposition 2.6. In order to prove (iv), let $\mathbf{x} = x_1, \ldots, x_n$ be a maximal quintasymptotic sequence over I and let $\mathfrak{n}$ be a prime ideal in T lying over $\mathfrak{m}$. As $(I, (\mathbf{x}))T \subseteq \mathfrak{n}$, it follows that $\mathbf{x}$ is a quintasymptotic sequence over $IT_{\mathfrak{n}}$, by Propositions 2.5 and 2.8. Whence $\operatorname{qacogd}(I) \leq \operatorname{qacogd}(IT_{\mathfrak{n}})$, by Theorem 3.2. Now, if $\operatorname{height}\mathfrak{m} = \operatorname{height}\mathfrak{n}$, then $\mathfrak{n}$ is a minimal prime over $\mathfrak{m}T$. Thus $\mathfrak{n}T_{\mathfrak{n}} \in \overline{Q^*}((I, (\mathbf{x}))T_{\mathfrak{n}})$ by Lemma 2.3(iv),(vii). Therefore, $\operatorname{qacogd}(I) = \operatorname{qacogd}(IT_{\mathfrak{n}})$, by Theorem 3.2.

For (v), let $\mathbf{x} = x_1, \ldots, x_n$ be a maximal quintasymptotic sequence over I and let $\mathfrak{n}$ be a prime ideal in T lying over $\mathfrak{m}$. Since $\mathfrak{n}$ contains $(I, (\mathbf{x}))T$, it follows that $\mathbf{x}$ is a quintasymptotic sequence over $IT_{\mathfrak{n}}$, by Propositions 2.5 and 2.9. Whence $\operatorname{qacogd}(I) \leq \operatorname{qacogd}(IT_{\mathfrak{n}})$ for all prime ideals $\mathfrak{n}$ in T lying over $\mathfrak{m}$, by Theorem 3.2. Also, by Proposition 2.9, there exists a maximal ideal $\mathfrak{n}'$ in T such that $\mathfrak{n}' \in \overline{Q^*}((I, (\mathbf{x}))T)$. Thus $\mathfrak{n}'T_{\mathfrak{n}'} \in \overline{Q^*}((I, (\mathbf{x}))T_{\mathfrak{n}'})$ by Lemma 2.3(iv), and hence $\mathbf{x}$ is a maximal quintasymptotic sequence over $IT_{\mathfrak{n}'}$. Therefore, $\operatorname{qacogd}(I) = \operatorname{qacogd}(IT_{\mathfrak{n}'})$ by Theorem 3.2. Finally, (v) follows immediately from Proposition 2.10 and Theorem 3.2. □

In remainder of this section we give several bounds on quintasymptotic cograde of an ideal.

Theorem 3.4. *Let I be an ideal in a Noetherian local ring R. Then the following hold:*

(i) $\operatorname{qacogd}(I) \leq \dim R/I$.

(ii) $\operatorname{qacogd}(I) \geq \min\{\dim R^*/\mathfrak{q} \mid \mathfrak{q} \in \overline{Q^*}(IR^*)\}$.

Proof. For (i) we have,

$$\dim R/I = \dim R^*/IR^* \geq \min\{\dim R^*/IR^* + z \mid z \in \operatorname{mAss}_{R^*} R^*\},$$

and so, by Theorem 3.2, we have $\operatorname{qacogd}(I) \leq \dim R/I$.

In order to prove (ii), in view of Theorem 3.2, there exists a minimal prime z of R^* such that $\operatorname{qacogd}(I) = \dim R^*/IR^* + z$. Thus there exists a minimal prime divisor $\mathfrak{q}$ of $IR^* + z$ such that $\operatorname{qacogd}(I) = \dim R^*/\mathfrak{q}$. Hence, by Lemma 2.3(vi), $\mathfrak{q} \in \overline{Q^*}(IR^*)$, and so the result follows. □

Lemma 3.5. *Let I and J be ideals in a Noetherian ring R and let $\mathfrak{p} \in \operatorname{Spec} R$ such that $I \subseteq J \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \overline{Q^*}(I)$. Then $\mathfrak{p} \in \overline{Q^*}(J)$.*

Proof. It follows from Definition 2.1. □

Proposition 3.6. *Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$ and let $y_1, \ldots, y_k$ be an asymptotic sequence of elements of R such that $y_j \in I$ for all $j(1 \leq j \leq k)$. Then there is a maximal quintasymptotic sequence over I, say $x_1, \ldots, x_n$, such that $y_1, \ldots, y_k, x_1, \ldots, x_n$ is an asymptotic sequence in R. In particular qacogd$(I) + \operatorname{agd}(I)) \leq \operatorname{agd}(\mathfrak{m})$.*

Proof. Let n be the length of a maximal quintasymptotic sequence over I. If $n = 0$ we are done. If $n > 0$, then $\mathfrak{m} \notin \overline{Q^*}(I)$, and so by Lemma 3.5, $\mathfrak{m} \notin \overline{Q^*}((y_1, \ldots, y_k))$. Pick $x_1 \in \mathfrak{m}$ with $x_1 \notin \cup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{Q^*}(I)\}$ and $x_1 \notin \cup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{Q^*}((y_1, \ldots, y_k))\}$. Now, x_1 is a quintasymptotic sequence over I and the length of a maximal quintasymptotic sequence over $(I, (x_1))$ is $n - 1$. Since the choice of x_1 assures that $y_1, \ldots, y_k, x_1$ is an asymptotic sequence in R, we now may use induction. □

The next result determines when the inequality in Proposition 3.6 is equality.

Theorem 3.7. *Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$. Then the following statements are equivalent:*

(i) $\operatorname{qacogd}(I) + \operatorname{agd}(I) = \operatorname{agd}(\mathfrak{m})$.

(ii) *For all* $z \in \operatorname{mAss}_{R^*} R^*$ *with* $\operatorname{agd}(\mathfrak{m}) = \dim R^*/z$*, we have* $\operatorname{agd}(I) = \operatorname{height}(IR^* + z/z)$ *and* $\operatorname{qacogd}(I) = \dim R^*/z - \operatorname{height}(IR^* + z/z)$.

(iii) *There exists* $z \in \operatorname{mAss}_{R^*} R^*$ *such that* $\operatorname{agd}(I) = \operatorname{height}(IR^* + z/z)$ *and* $\operatorname{qacogd}(I) = \dim R^*/z - \operatorname{height}(IR^* + z/z)$.

Proof. To prove (i) $\Longrightarrow$ (ii), let $\operatorname{qacogd}(I) + \operatorname{agd}(I) = \operatorname{agd}(\mathfrak{m})$ and let $z \in \operatorname{mAss}_{R^*} R^*$ such that $\operatorname{agd}(\mathfrak{m}) = \dim R^*/z$. Then by Theorem 3.2 and [6, Proposition 6.10], we have

$$\operatorname{qacogd}(I) \leq \dim R^*/z - \operatorname{height}(IR^* + z/z) \leq \operatorname{agd}(\mathfrak{m}) - \operatorname{agd}(I) = \operatorname{qacogd}(I).$$

Therefore, $\operatorname{agd}(I) = \operatorname{height}(IR^* + z/z)$ and $\operatorname{qacogd}(I) = \dim R^*/z - \operatorname{height}(IR^* + z/z)$.

In order to prove the implication (ii) $\Longrightarrow$ (iii), it suffices to show that, there exists $z \in \operatorname{mAss}_{R^*} R^*$ such that $\operatorname{agd}(\mathfrak{m}) = \dim R^*/z$. To this end we use [6, Proposition 6.10]. Finally, the implication (iii) $\Longrightarrow$ (i) is obvious, by [6, Proposition 6.10] and Proposition 3.6. $\square$

Now we give some lower bounds on $\operatorname{qacogd}(I)$.

Remark 3.8. *Let I be an ideal in a Noetherian ring R. Then the following hold:*

(i) *It follows from the Lemma* 2.3(iii)*, Corollary* 2.15 *and* [4, Proposition 3.10]*, that*

$$\operatorname{egd}(I) = \operatorname{qeg}(I) \leq \operatorname{qagd}(I) = \operatorname{agd}(I).$$

(ii) *If R is local, then by the Lemma* 2.3(iii)*, we have,*

$$\operatorname{ecogd}(I) \leq \operatorname{qecogd}(I) \leq \operatorname{qacogd}(I) \text{ and}$$
$$\operatorname{ecogd}(I) \leq \operatorname{acogd}(I) \leq \operatorname{qacogd}(I).$$

Corollary 3.9. *Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$. Then, for all large n,* $\operatorname{qacogd}(I) \geq \operatorname{egd}(\mathfrak{m}/I^n)$.

Proof. This is clear by [16, Theorem 5.5] and Remark 3.8. $\square$

Corollary 3.10. *Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$. Then, for all large n,* $\operatorname{qacogd}(I) \geq \operatorname{grade}(\mathfrak{m}/I^n)$.

Proof. This is clear by Corollary 3.9 and Lemma 2.3(iii). $\square$

Corollary 3.11. *Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$ such that for all large n, $(R/I^n)^*$ has no imbedded prime divisors of zero. Then* $\operatorname{qacogd}(I) \geq \operatorname{agd}(\mathfrak{m}/I^n)$*, for all large n.*

Proof. This is clear by Corollaries 3.9 and 2.16. $\square$

This section will be closed with the another lower bound for $\operatorname{qacogd}(I)$ in connection with analytic spread. Let us, firstly, recall the important notion of the *analytic spread* of I in a local ring $(R, \mathfrak{m})$, which was introduced by Northcott and Rees in [12] and is defined as $l(I) := \dim \mathscr{R}/(\mathfrak{m}, u)\mathscr{R}$.

Theorem 3.12. *Let I be an ideal in a Noetherian local ring $(R, \mathfrak{m})$. Then,*

$$\operatorname{qacogd}(I) \geq \operatorname{agd}(\mathfrak{m}) - l(I).$$

Proof. By Theorem 3.2 there exists $z \in \operatorname{mAss}_{R^*} R^*$ such that

$$\operatorname{qacogd}(I) = \operatorname{height}(\mathfrak{m}R^*/z) - \operatorname{height} IR^* + z/z.$$

Also, by [6, Proposition 6.10], we have $\operatorname{agd}(\mathfrak{m}) \leq \operatorname{height}(\mathfrak{m}R^*/z)$. On the other hand, by [4, Remark 5.5.4] and [6, Lemma 6.5],

$$\operatorname{height}(IR^* + z/z) \leq l(IR^* + z/z) \leq l(I).$$

Now the desired result follows. □

4. Quintasymptotic cograde and unmixedness

In this section we use quintasymptotic cograde to obtain some characterizations of quasi-unmixed rings and another related class of local rings. We begin with the quasi-unmixed case.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then the following are equivalent:*

(i) *R is quasi-unmixed.*
(ii) $\operatorname{agd}(I) = \operatorname{height} I$ *for all ideals I of R.*
(iii) $\operatorname{agd}(\mathfrak{m}) = \operatorname{height} \mathfrak{m}$.
(iv) $\operatorname{qacogd}(0) = \dim R$.
(v) $\operatorname{qacogd}(I) = \dim R/I$ *for every ideal I of R generated by an asymptotic sequence of elements of R.*

Proof. Since by [13, Lemma 2.5] every quasi-unmixed local ring is locally quasi-unmixed ring, it follows from [6, Corollary 5.8] that (i)-(iii) are equivalent. Assume that (ii) holds and let $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic sequence in R and let $I := (\mathbf{x})$. Then there are elements $x_{n+1}, \ldots, x_r$ in R such that $x_1, \ldots, x_n, x_{n+1}, \ldots, x_r$ is a maximal asymptotic sequence in R, and so $\operatorname{qacogd}(I) = r - n$. By assumption and Corollary 2.4(i), we have $\operatorname{height} I = n$ and $\operatorname{height} \mathfrak{m} = r$. Hence $n + \dim R/I \leq \dim R = \operatorname{height} \mathfrak{m} = r$, and so $\dim R/I \leq r - n = \operatorname{qacogd}(I)$. Now, the implication (ii) $\Longrightarrow$ (v) follows from Theorem 3.4. The implications (v) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (iii) obviously are true. □

Theorem 4.2. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring. Consider the following conditions:*

(i) $\operatorname{qacogd}(I) = \dim R/I$ *for every ideal* I *of* R.

(ii) $\operatorname{agd}(I) + \operatorname{qacogd}(I) = \dim R$ *for every ideal* I *of* R.

(iii) $\operatorname{agd}(\mathfrak{p}) + \operatorname{qacogd}(\mathfrak{p}) = \dim R$ *for every prime ideal* $\mathfrak{p}$ *of* R *with* $\dim R/\mathfrak{p} = 1$.

(iv) R *is quasi-unmixed and there exists a unique minimal prime divisor in* R.

Then (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv).

Proof. (i) $\Longrightarrow$ (iv): If (i) holds and $I = (0)$, then by Proposition 4.1, R is a quasi-unmixed ring. Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct minimal prime divisors in R. Then, by [10, 34.5], we have

$$\dim R/\mathfrak{p} = \dim R/\mathfrak{q} = \dim R.$$

Let z be a minimal prime divisor of $\mathfrak{q}R^*$ such that $\dim R^*/z = \dim R^*/\mathfrak{q}R^*$, and so $\dim R^*/z = \dim R$. Thus $z \in \operatorname{mAss}_{R^*} R^*$. Therefore, by Theorem 3.2, we have

$$\operatorname{qacogd}(\mathfrak{p}) \leq \dim R^*/\mathfrak{p}R^* + z.$$

As $\mathfrak{p}$ and $\mathfrak{q}$ are distinct, it yields that

$$\dim R^*/\mathfrak{p}R^* + z < \dim R^*/\mathfrak{q}R^* = \dim R/\mathfrak{q} = \dim R/\mathfrak{p}.$$

Hence $\operatorname{qacogd}(\mathfrak{p}) < \dim R/\mathfrak{p}$, which is a contradiction.

(i) $\Longrightarrow$ (ii): If (i) holds, then R is a quasi-unmixed ring, and so $\operatorname{agd}(I) = \operatorname{height} I$, by Proposition 4.1. Thus (ii) is true by [10, 34.5]. It is clear that (ii) $\Longrightarrow$ (iii).

In order to prove the implication (iii) $\Longrightarrow$ (i), suppose, the contrary, that (i) is not true. Then, there is an ideal I of R such that $\operatorname{qacogd}(I) \neq \dim R/I$ and $\dim R/I \leq \dim R/J$ for every ideal J of R with $\operatorname{qacogd}(J) \neq \dim R/J$. Now, suppose $\dim R/I = d$. Then by Theorem 3.4(i), $d > 0$. If $d = 1$, then $\operatorname{qacogd}(I) = 0$ by Theorem 3.4(i), and so by Theorem 3.3(i), $\operatorname{qacogd}(\mathfrak{p}) = 0$, for all prime ideals $\mathfrak{p}$ in R containing I, which is a contradiction. Therefore $d > 1$ and by [14, Proposition 2.2] there exists infinitely many prime ideals $\mathfrak{p}$ in R such that $I \subseteq \mathfrak{p}$ and $\dim R/\mathfrak{p} = d - 1$. Let $\mathscr{P}$ be the set of these prime ideals. Then, by Theorem 3.2 there exist $z \in \operatorname{mAss}_{R^*} R^*$ such that $\operatorname{qacogd}(I) = \dim R^*/IR^* + z$. Let

$$\mathscr{Q} = \{\mathfrak{q} \in \operatorname{Spec} R^* \mid \text{there is } \mathfrak{p} \in \mathscr{P} \text{ with } \mathfrak{p}R^* + z \subseteq \mathfrak{q} \text{ and } \dim R^*/\mathfrak{p}R^* + z = \dim R^*/\mathfrak{q}\}.$$

We now show that $\mathscr{Q}$ is infinite and $\dim R^*/\mathfrak{q} = d-1$ for all $\mathfrak{q} \in \mathscr{Q}$. To do this, let $\mathfrak{q} \in \mathscr{Q}$. Then there exists $\mathfrak{p} \in \mathscr{P}$ such that $\mathfrak{p}R^* + z \subseteq \mathfrak{q}$ and $\dim R^*/\mathfrak{p}R^* + z = \dim R^*/\mathfrak{q}$. By the choice of d we have $\mathrm{qacogd}(\mathfrak{p}) = \dim R/\mathfrak{p} = d-1$. Therefore,

$$d-1 = \mathrm{qacogd}(\mathfrak{p}) \leq \dim R^*/\mathfrak{p}R^* + z = \dim R^*/\mathfrak{q}.$$

In other hand, we have

$$\dim R^*/\mathfrak{q} \leq \dim R^*/\mathfrak{p}R^* = \dim R/\mathfrak{p} = d-1.$$

Therefore, $\dim R^*/\mathfrak{q} = d-1$ and $\mathfrak{q}$ is a minimal prime divisor of $\mathfrak{p}R^*$. Consequently, there are infinitely many $\mathfrak{q}$, as there are infinitely many $\mathfrak{p}$. Since

$$d-1 = \dim R^*/\mathfrak{q} \leq \dim R^*/\mathfrak{p}R^* + z \leq$$

$$\dim R^*/IR^* + z = \mathrm{qacogd}(I) < \dim R/I = d,$$

it follows that $\dim R^*/IR^* + z = d-1$. Hence $\mathfrak{q}$ is a minimal prime divisor of $IR^* + z$, and therefore, $IR^* + z$ has infinitely many minimal prime divisors, which is a contradiction. That is, (i) holds. □

Theorem 4.3. *Let R be a complete Noetherian local ring. Then the following conditions are equivalent:*

(i) $\mathrm{qacogd}(I) = \dim R/I$, *for every ideal I of R.*

(i) $\mathrm{agd}(I) + \mathrm{qacogd}(I) = \dim R$, *for every ideal I of R.*

(iii) $\mathrm{agd}(\mathfrak{p}) + \mathrm{qacogd}(\mathfrak{p}) = \dim R$, *for every prime ideal $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} = 1$.*

(iv) *R has a unique minimal prime divisor of zero.*

Proof. In view of Theorem 4.2, it suffices to show that (iv) $\Longrightarrow$ (i); and this follows from Theorem 3.2. □

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(Saeed Jahandoust) Department of Mathematics, University of Tabriz, Tabriz, Iran

E-mail address: saeed.e.jahan@gmail.com

(Reza Naghipour) Department of Mathematics, University of Tabriz, Tabriz, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM) P.O. Box 19395-5746, Tehran, Iran

E-mail address: naghipour@ipm.ir; naghipour@tabrizu.ac.ir