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# STRONG CONVERGENCE OF MODIFIED NOOR ITERATION IN CAT(0) SPACES

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(Communicated by Abbas Salemi)

Abstract. We prove a strong convergence theorem for the modified Noor iterations in the framework of CAT(0) spaces. Our results extend and improve the corresponding results of X. Qin, Y. Su and M. Shang, T. H. Kim and H. K. Xu and S. Saejung and some others.
**Keywords:** Modified noor iteration, CAT(0) spaces, nonexpansive mapping, strong convergence.
**MSC(2010):** Primary: 47H09; Secondary: 47H10.

## 1. Introduction

Let $(X, d)$ be a metric space and let $C$ be a nonempty subset of $X$. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

A point $x \in C$ is called a fixed point of $T$ if $x = Tx$. The set of all fixed points of $T$ is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

In 1967, Halpern [6] introduced the following iterative scheme in Hilbert spaces which was referred to as *Halpern iteration* for approximating a fixed point of $T$:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where $x_1, u \in C$ are arbitrarily chosen, and $\{\alpha_n\}$ is a sequence in [0,1]. Wittmann [14] studied the iterative scheme (1.1) in a Hilbert space and

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obtained a strong convergence of the iteration. Reich [10] and Shioji and Takahashi [12] extended Wittmann's result to a real Banach space.

The modified version of Halpern iteration was investigated widely by many mathematicians. For instance, Kim and Xu [7] studied the sequence $\{x_n\}$ generated as follows:

$$\begin{cases} y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $x_1, u \in C$ are arbitrarily chosen and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1]. They proved the strong convergence of iterative scheme (1.2) in the framework of a uniformly smooth Banach space.

In 2006, Su and Qin [13] introduced the composite iteration scheme as follows:

$$\begin{cases} w_n &= \delta_n x_n + (1-\delta_n)Tx_n, \\ z_n &= \gamma_n x_n + (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $x_1, u \in C$ are arbitrarily chosen and $\{\alpha_n\}, \{\beta_n\}$ are sequences in (0,1) and $\{\gamma_n\}, \{\delta_n\}$ in [0,1]. They proved the strong convergence of iterative scheme (1.3) in the framework of a uniformly smooth Banach space.

In 2008, Qin, Su and Shang [9] further modified the iterative process (1.2) and obtained a strong convergence theorem in a uniformly smooth Banach space as follows:

$$\begin{cases} z_n &= \gamma_n x_n + (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

Recently, Saejung [11] extended the results of Halpern [6], Reich [10], Shioji and Takahashi [12] and Wittmann [14], to the case of a CAT(0) space.

The purpose of this paper is to extend Su-Qin's result (1.3) to a special kind of metric spaces, namely, CAT(0) spaces.

## 2. **Preliminaries**

Let $(X,d)$ be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from $x$ to $y$) is a map $c$ from a closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $c(0)=x$, $c(l)=y$, and $d(c(t),c(t'))=|t-t'|$ for all $t,t' \in [0,l]$. In particular, $c$ is an isometry and $d(x,y)=l$. The image $\alpha$ of $c$ is called a *geodesic* (or *metric*) *segment* joining $x$ and $y$. The space $(X,d)$ is said to be a *geodesic space* if every two points of $X$ are joined by a geodesic, and $X$ is said to be *uniquely geodesic* if there is exactly one geodesic joining $x$ and $y$ for each $x,y \in X$. A subset $Y \subseteq X$ is said to be *convex* if $Y$ includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic metric space $(X,d)$ consists of three points $x_1,x_2,x_3$ in $X$ and three geodesic segment joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x_1,x_2,x_3)$ is the triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean space $\mathbb{E}^2$ such that $d(x_i,x_j)=d_{\mathbb{E}^2}(\overline{x}_i,\overline{x}_j)$ for all $i,j=1,2,3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): let $\triangle$ be a geodesic triangle in $X$, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$, $d(x,y) \le d_{\mathbb{E}^2}(\overline{x},\overline{y})$.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space.Other examples are pre-Hilbert spaces, $\mathbb{R}$-*trees* (see [1]), the complex Hilbert ball with a hyperbolic metric (see [5]), and many others. A thorough discussion of these spaces and their important role in various branches of mathematics are given in [1, 2].

The following definitions and lemmas are also required in the main results. In this paper, we write $(1-t)x \oplus ty$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that

$$d(x,z)=td(x,y),\ d(y,z)=(1-t)d(x,y).$$

We also denote by $[x,y]$ the geodesic segment joining $x$ to $y$, that is, $[x,y]=\{(1-t)x \oplus ty : t \in [0,1]\}$.

Let $\mu$ be a continuous linear functional on $\ell^\infty$, the Banach space of bounded real sequences, satisfying $\|\mu\|=1=\mu(1)$. Then we know that $\mu$ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n; n \in \mathbb{N}\} \le \mu(a) \le \sup\{a_n; n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. According to time and circumstances, we used $\mu_n(a_n)$ instead of $\mu(a)$. A mean $\mu$ on $\mathbb{N}$ is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$.

**Lemma 2.1.** ( [12], Proposition 2 ). *Let $a$ be a real number and let $(a_1, a_2, \ldots) \in \ell^\infty$ be such that $\mu_n(a_n) \leq a$ for all Banach limits $\mu$ and $\limsup_n(a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq a$.*

The following lemmas are observed by S. Saejung.

**Lemma 2.2.** ( [11], Lemma 2.1). *Let $C$ be a closed convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0, 1)$, the mapping $S_t : C \to C$ defined by*

$$S_t x = tu \oplus (1 - t)Tx \text{ for } x \in C$$

*has a unique fixed point $z_t \in C$, that is,*

$$z_t = S_t z_t = tu \oplus (1 - t)Tz_t. \tag{2.1}$$

**Lemma 2.3.** ( [11], Lemma 2.2). *Let $C$, $T$ be as the preceding lemma. Then $F(T) \neq \emptyset$ if and only if $\{z_t\}$ given by the formula (2.1) remains bounded as $t \to 0$. In this case, the following statements hold:*

- (i) *$\{z_t\}$ converges to the unique fixed point $z$ of $T$ which is nearest to $u$;*
- (ii) *$d(u, z)^2 \leq \mu_n d(u, x_n)^2$ for all Banach limits $\mu$ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

The important property of the CAT(0) space that we use is the following lemma.

**Lemma 2.4.** ([3, 4, 8]). *Let $X$ be a CAT(0) space. Then, for each $x, y, z \in X$ and $t, s \in [0, 1]$, we have*

- (i) $d\big((1 - t)x \oplus ty, z\big) \leq (1 - t)d(x, z) + td(y, z)$.
- (ii) $d\big((1 - t)z \oplus tx, (1 - t)z \oplus ty\big) \leq td(x, y)$.
- (iii) $d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$.
- (iv) $d\big((1 - t)x \oplus ty, (1 - s)x \oplus sy\big) = |t - s|d(x, y)$.

Consequently, we also need the following lemma that is extracted from H.K. Xu.

**Lemma 2.5.** ( [15], Lemma 2.1). *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \ n \geq 0,$$

*where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that*

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

*Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.*

## 3. Main results

From now on, we let $C$ be a nonempty closed and convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$.

**Lemma 3.1.** *Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$, $\{\delta_n\}$ in $[0,1]$, the sequence $\{x_n\}$ is defined iteratively by*

$$\begin{cases} w_n &= \delta_n x_n \oplus (1-\delta_n)Tx_n, \\ z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{3.1}$$

*Then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded sequences.*

*Proof.* If we take a fixed point $p$ of $T$, then from (3.1) we estimate as follows:

$$\begin{aligned} d(w_n,p) &= d(\delta_n x_n \oplus (1-\delta_n)Tx_n, p) \\ &\leq \delta_n d(x_n,p) + (1-\delta_n)d(Tx_n,p) \\ &\leq \delta_n d(x_n,p) + (1-\delta_n)d(x_n,p) \end{aligned}$$

$$d(w_n,p) \leq d(x_n,p). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} d(z_n,p) &= d(\gamma_n x_n \oplus (1-\gamma_n)Tw_n, p) \\ &\leq \gamma_n d(x_n,p) + (1-\gamma_n)d(Tw_n,p) \\ &\leq \gamma_n d(x_n,p) + (1-\gamma_n)d(x_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

$$\begin{aligned} d(y_n, p) &= d(\beta_n x_n \oplus (1-\beta_n)Tz_n, p) \\ &\le \beta_n d(x_n, p) + (1-\beta_n) d(Tz_n, p) \\ &\le \beta_n d(x_n, p) + (1-\beta_n) d(z_n, p) \\ &\le \beta_n d(x_n, p) + (1-\beta_n) d(x_n, p) \\ &\le d(x_n, p). \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n u \oplus (1-\alpha_n) y_n, p) \\ &\le \alpha_n d(u, p) + (1-\alpha_n) d(y_n, p) \\ &\le \alpha_n d(u, p) + (1-\alpha_n) d(x_n, p) \\ &\le \max\{d(u, p), d(x_n, p)\}. \end{aligned}$$

By induction we have

$$d(x_n, p) \le \max\{d(u, p), d(x_0, p)\}, \quad n \ge 0.$$

This proves the boundedness of the sequence $\{x_n\}$, which implies that the sequences $\{y_n\}, \{z_n\}$ and $\{w_n\}$ are also bounded. □

**Lemma 3.2.** *If $C$, $T$, $\{x_n\}$, $\{y_n\}, \{z_n\}$ ,$\{w_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$ *and* $\sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$

*then* $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

*Proof.* By (3.1), we have

$$\begin{aligned} x_{n+1} &= \alpha_n u \oplus (1-\alpha_n) y_n, & x_n &= \alpha_{n-1} u \oplus (1-\alpha_{n-1}) y_{n-1} \\ y_n &= \beta_n x_n \oplus (1-\beta_n) T z_n, & y_{n-1} &= \beta_{n-1} x_{n-1} \oplus (1-\beta_{n-1}) T z_{n-1}. \end{aligned}$$

It follows from Lemma 2.4 that

$$
\begin{aligned}
& d(x_{n+1}, x_n) \\
& = d(\alpha_n u \oplus (1-\alpha_n)y_n, \alpha_{n-1}u \oplus (1-\alpha_{n-1})y_{n-1}) \\
& \leq d(\alpha_n u \oplus (1-\alpha_n)y_n, \alpha_n u \oplus (1-\alpha_n)y_{n-1}) \\
& \ + d(\alpha_n u \oplus (1-\alpha_n)y_{n-1}, \alpha_{n-1}u \oplus (1-\alpha_{n-1})y_{n-1}) \\
& \\
& \leq (1-\alpha_n)d(y_n, y_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) \\
& = (1-\alpha_n)d(\beta_n x_n \oplus (1-\beta_n)Tz_n, \beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})Tz_{n-1}) \\
& \ + |\alpha_n - \alpha_{n-1}|d(u, \beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})Tz_{n-1}) \\
& \\
& \leq (1-\alpha_n)\Big[d(\beta_n x_n \oplus (1-\beta_n)Tz_n, \beta_n x_{n-1} \oplus (1-\beta_n)Tz_n) \\
& \ + d(\beta_n x_{n-1} \oplus (1-\beta_n)Tz_n, \beta_n x_{n-1} \oplus (1-\beta_n)Tz_{n-1}) \\
& \ + d(\beta_n x_{n-1} \oplus (1-\beta_n)Tz_{n-1}, \beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})Tz_{n-1})\Big] \\
& \ + |\alpha_n - \alpha_{n-1}|\Big[\beta_{n-1}d(u, x_{n-1}) + (1-\beta_{n-1})d(u, Tz_{n-1})\Big] \\
& \\
& \leq (1-\alpha_n)\Big[\beta_n d(x_n, x_{n-1}) + (1-\beta_n)d(Tz_n, Tz_{n-1}) \\
& \ + |\beta_n - \beta_{n-1}|d(x_{n-1}, Tz_{n-1})\Big] \\
& \ + |\alpha_n - \alpha_{n-1}|\Big[\beta_{n-1}d(u, x_{n-1}) + (1-\beta_{n-1})d(u, Tz_{n-1})\Big] \\
& \\
& = (1-\alpha_n)\beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n)d(Tz_n, Tz_{n-1}) \\
& \ + (1-\alpha_n)|\beta_n - \beta_{n-1}|d(x_{n-1}, Tz_{n-1}) \\
& \ + |\alpha_n - \alpha_{n-1}|\beta_{n-1}d(u, x_{n-1}) + |\alpha_n - \alpha_{n-1}|(1-\beta_{n-1})d(u, Tz_{n-1}) \\
& \\
& \leq (1-\alpha_n)\beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n)d(Tz_n, Tz_{n-1}) \\
& \ + (1-\alpha_n)|\beta_n - \beta_{n-1}|d(x_{n-1}, Tz_{n-1}) \\
& \ + |\alpha_n - \alpha_{n-1}|\beta_{n-1}d(u, x_{n-1}) - |\alpha_n - \alpha_{n-1}|\beta_{n-1}d(u, Tz_{n-1}) \\
& \ + |\alpha_n - \alpha_{n-1}|d(u, Tz_{n-1}) \\
& \\
& \leq (1-\alpha_n)\beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n)d(Tz_n, Tz_{n-1}) \\
& \ + (1-\alpha_n)|\beta_n - \beta_{n-1}|d(x_{n-1}, Tz_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, Tz_{n-1}) \\
& \ + |\alpha_n - \alpha_{n-1}|\beta_{n-1}\Big[d(u, Tz_{n-1}) + d(Tz_{n-1}, x_{n-1})\Big] \\
& \ - |\alpha_n - \alpha_{n-1}|\beta_{n-1}d(u, Tz_{n-1})
\end{aligned}
$$

$$\begin{aligned}
&d(x_{n+1}, x_n)\\
&\leq (1-\alpha_n)\beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n)d(Tz_n, Tz_{n-1})\\
&\quad + (1-\alpha_n)|\beta_n - \beta_{n-1}|d(x_{n-1}, Tz_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, Tz_{n-1})\\
&\quad + |\alpha_n - \alpha_{n-1}|\beta_{n-1}d(x_{n-1}, Tz_{n-1})\\
&\leq (1-\alpha_n)(1-\beta_n)d(Tz_n, Tz_{n-1}) + (1-\alpha_n)\beta_n d(x_n, x_{n-1})\\
&\quad + |(\beta_n - \beta_{n-1})(1-\alpha_n) + (\alpha_n - \alpha_{n-1})\beta_{n-1}|d(x_{n-1}, Tz_{n-1})\\
&\quad + |\alpha_n - \alpha_{n-1}|d(u, Tz_{n-1}).
\end{aligned}$$

This implies that

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq (1-\alpha_n)(1-\beta_n)d(z_n, z_{n-1}) + (1-\alpha_n)\beta_n d(x_n, x_{n-1})\\
&\quad + |(\beta_n - \beta_{n-1})(1-\alpha_n) + (\alpha_n - \alpha_{n-1})\beta_{n-1}|d(x_{n-1}, Tz_{n-1})\\
&\quad + |\alpha_n - \alpha_{n-1}|d(u, Tz_{n-1}). \qquad (3.3)
\end{aligned}$$

Again, from (3.1), we have

$$\begin{aligned}
w_n &= \delta_n x_n \oplus (1-\delta_n)Tx_n,\\
w_{n-1} &= \delta_{n-1}x_{n-1} \oplus (1-\delta_{n-1})Tx_{n-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(w_n, w_{n-1}) &= d\left(\delta_n x_n \oplus (1-\delta_n)Tx_n, \delta_{n-1}x_{n-1} \oplus (1-\delta_{n-1})Tx_{n-1}\right)\\
&\leq d\left(\delta_n x_n \oplus (1-\delta_n)Tx_n, \delta_n x_n \oplus (1-\delta_n)Tx_{n-1}\right)\\
&\quad + d\left(\delta_n x_n \oplus (1-\delta_n)Tx_{n-1}, \delta_n x_{n-1} \oplus (1-\delta_n)Tx_{n-1}\right)\\
&\quad + d\left(\delta_n x_{n-1} \oplus (1-\delta_n)Tx_{n-1}, \delta_{n-1}x_{n-1} \oplus (1-\delta_{n-1})Tx_{n-1}\right)\\
&\leq (1-\delta_n)d(Tx_n, Tx_{n-1}) + |\delta_n - \delta_{n-1}|d(x_{n-1}, Tx_{n-1})\\
&\quad + \delta_n d(x_n, x_{n-1})\\
&\leq (1-\delta_n)d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}|d(x_{n-1}, Tx_{n-1})\\
&\quad + \delta_n d(x_n, x_{n-1}).
\end{aligned}$$

That is,

$$d(w_n, w_{n-1}) \leq d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}|d(x_{n-1}, Tx_{n-1}). \qquad (3.4)$$

Similarly, we have

$$\begin{aligned}
z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tw_n,\\
z_{n-1} &= \gamma_{n-1}x_{n-1} \oplus (1-\gamma_{n-1})Tw_{n-1}.
\end{aligned}$$

Thus,

$$d(z_n, z_{n-1}) \leq (1-\gamma_n)d(Tw_n, Tw_{n-1}) + \gamma_n d(x_n, x_{n-1}) \\ + |\gamma_n - \gamma_{n-1}|d(x_{n-1}, Tw_{n-1})$$

$$d(z_n, z_{n-1}) \leq (1-\gamma_n)d(w_n, w_{n-1}) + \gamma_n d(x_n, x_{n-1}) \\ + |\gamma_n - \gamma_{n-1}|d(x_{n-1}, Tw_{n-1}). \tag{3.5}$$

By substituting (3.4) into (3.5), we have

$$\begin{aligned} d(z_n, z_{n-1}) &\leq (1-\gamma_n)\left[d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}|d(x_{n-1}, Tx_{n-1})\right] \\ &\quad + \gamma_n d(x_n, x_{n-1}) + |\gamma_n - \gamma_{n-1}|d(x_{n-1}, Tw_{n-1}) \\ &\leq d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}|d(x_{n-1}, Tx_{n-1}) \\ &\quad + |\gamma_n - \gamma_{n-1}|d(x_{n-1}, Tw_{n-1}). \end{aligned}$$

It follows that

$$d(z_n, z_{n-1}) \leq d(x_n, x_{n-1}) + \left(|\delta_n - \delta_{n-1}| + |\gamma_n - \gamma_{n-1}|\right) M_1 \tag{3.6}$$

where $M_1$ is an appropriate constant such that

$$M_1 \geq \max\left\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tw_{n-1})\right\}.$$

By substituting (3.6) into (3.3), we get that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (1-\alpha_n)d(x_n, x_{n-1}) \\ &\quad + (1-\alpha_n)(1-\beta_n)\left(|\delta_{n-1} - \delta_n| + |\gamma_{n-1} - \gamma_n|\right) M_1 \\ &\quad + \left|(\beta_n - \beta_{n-1})(1-\alpha_n) + (\alpha_n - \alpha_{n-1})\beta_{n-1}\right| d(x_{n-1}, Tz_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}|d(u, Tz_{n-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (1-\alpha_n)d(x_n, x_{n-1}) \\ &\quad + M\big(|\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|\big), \end{aligned} \tag{3.7}$$

where $M$ is an appropriate constat such that

$$M \geq \max\left\{d(u, Tz_{n-1}), d(x_{n-1}, Tz_{n-1}), M_1\right\}$$

for all $n$. By assumptions (C1) - (C2), we have

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and}$$

$$\sum_{n=1}^{\infty}(|\gamma_n-\gamma_{n-1}|+|\beta_n-\beta_{n-1}|+2|\alpha_n-\alpha_{n-1}|+|\delta_n-\delta_{n-1}|)<\infty.$$

Now, applying Lemma 2.5 to (3.7), we obtain

$$\lim_{n\to\infty} d(x_{n+1},x_n)=0.$$

□

**Lemma 3.3.** *If $C$, $T$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ ,$\{w_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy conditions (C1), (C2) and*

$$C(3):\ \beta_n+(1+\beta_n)(1-\gamma_n)(2-\delta_n)\in[0,a)$$

*for some $a\in(0,1)$, then $\lim_{n\to\infty} d(x_n,Tx_n)=0$.*

*Proof.* First, by using condition (C1), we get that

$$d(x_{n+1},y_n)=d(\alpha_n u\oplus(1-\alpha_n)y_n,y_n)$$

$$d(x_{n+1},y_n)\le\alpha_n d(u,y_n)\to 0,\quad \text{as}\quad n\to\infty. \tag{3.8}$$

We estimate as follows:

$$\begin{aligned}
d(x_n,Tx_n)&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+d(y_n,Tz_n)+d(Tz_n,Tx_n)\\
&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+d(y_n,Tz_n)+d(z_n,x_n)\\
&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+\beta_n d(x_n,Tz_n)+d(z_n,x_n)\\
&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+\beta_n d(x_n,Tx_n)\\
&\quad+\beta_n d(Tx_n,Tz_n)+d(z_n,x_n)\\
&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+\beta_n d(x_n,Tx_n)\\
&\quad+\beta_n d(x_n,z_n)+d(z_n,x_n)\\
&= d(x_n,x_{n+1})+d(x_{n+1},y_n)+\beta_n d(x_n,Tx_n)\\
&\quad+(1+\beta_n)d(x_n,z_n)\\
&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+\beta_n d(x_n,Tx_n)\\
&\quad+(1+\beta_n)(1-\gamma_n)d(x_n,Tw_n)\\
&\le d(x_n,x_{n+1})+d(x_{n+1},y_n)+\beta_n d(x_n,Tx_n)\\
&\quad+(1+\beta_n)(1-\gamma_n)\big[d(x_n,Tx_n)+d(Tx_n,Tw_n)\big]
\end{aligned}$$

$$\begin{aligned} d(x_n, Tx_n) \le\ & d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ & + (1+\beta_n)(1-\gamma_n) d(x_n, Tx_n) + (1+\beta_n)(1-\gamma_n) d(x_n, w_n) \\ \le\ & d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ & + (1+\beta_n)(1-\gamma_n) d(x_n, Tx_n) \\ & + (1+\beta_n)(1-\gamma_n)(1-\delta_n) d(x_n, Tx_n). \end{aligned}$$

It follows that

$$\left\{1 - \left[\beta_n + (1+\beta_n)(1-\gamma_n) + (1+\beta_n)(1-\gamma_n)(1-\delta_n)\right]\right\} d(x_n, Tx_n)$$
$$\le d(x_n, x_{n+1}) + d(x_{n+1}, y_n).$$

That is,

$$\left\{1 - \left[\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n)\right]\right\} d(x_n, Tx_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, y_n).$$

Hence, from condition (C3), (3.8) and Lemma 3.2 we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

□

By combining the ideas and the techniques in [11] and [13], we obtain the strong convergence theorem of the modified Noor iterative scheme (3.1) in the CAT(0) space.

**Theorem 3.4.** *Let $C$ be a nonempty closed and convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u, x_0 \in C$ is arbitrarily chosen, and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$, $\{\delta_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, *and* $\sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in [0, a)$ *for some* $a \in (0,1)$.

*The sequence $\{x_n\}$ is defined iteratively by (3.1). Then $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to $u$.*

*Proof.* By Lemma 3.1, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded sequences, and it follows from Lemma 3.3 that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Next, we prove $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to $u$. From Lemma 2.2, let $z = \lim_{t\to 0} z_t$ where $z_t$ is given by (2.1). Then, $z$ is the point of $F(T)$ which is nearest to $u$. We observe that

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\alpha_n u \oplus (1-\alpha_n)y_n, z) \\ &\le \alpha_n d^2(u,z) + (1-\alpha_n)d^2(y_n,z) - \alpha_n(1-\alpha_n)d^2(u,y_n) \\ &\le \alpha_n d^2(u,z) + (1-\alpha_n)d^2(x_n,z) - \alpha_n(1-\alpha_n)d^2(u,y_n) \\ &= (1-\alpha_n)d^2(x_n,z) + \alpha_n\Big(d^2(u,z) - (1-\alpha_n)d^2(u,y_n)\Big). \end{aligned}$$

That is,

$$d^2(x_{n+1}, z) \le (1-\alpha_n)d^2(x_n,z) + \alpha_n\Big(d^2(u,z) - (1-\alpha_n)d^2(u,y_n)\Big). \tag{3.9}$$

By Lemma 2.3, we have $\mu_n\big(d^2(u,z) - d^2(u,x_n)\big) \le 0$ for all Banach limit $\mu$. Moreover, since $d(x_{n+1}, x_n) \to 0$, we have

$$\limsup_{n\to\infty}\Big(d^2(u,z) + d^2(u,x_{n+1}) - d^2(u,z) - d^2(u,x_n)\Big) = 0.$$

It follows from $d(x_n, y_n) \to 0$ and Lemma 2.1 that

$$\limsup_{n\to\infty}\Big(d^2(u,z) - (1-\alpha_n)d^2(u,y_n)\Big) = \limsup_{n\to\infty}\Big(d^2(u,z) - d^2(u,x_n)\Big) \le 0.$$

Applying Lemma 2.5 to (3.9) we get that $\lim_{n\to\infty} d(x_n, z) = 0$. □

The following results are immediate consequences of Theorem 3.4 by taking $\delta_n = 1$ in Theorem 3.4.

**Corollary 3.5.** *Let $C$ be a nonempty closed and convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u$, $x_0 \in C$ is arbitrarily chosen, and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$;

(C3) $\beta_n + (1+\beta_n)(1-\gamma_n) \in [0,a)$ *for some* $a \in (0,1)$.

*The sequence* $\{x_n\}$ *is defined iteratively by*

$$\begin{cases} z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tx_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases}$$

*Then* $\{x_n\}$ *converges to a fixed point* $z \in F(T)$ *which is nearest to* $u$.

As a consequence of Corollary 3.5, we obtain the following result.

**Corollary 3.6.** *Let* $C$ *be a nonempty closed and convex subset of a complete CAT(0) space* $X$ *and let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Given a point* $u$, $x_0 \in C$ *are arbitrarily chosen and given sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0,1)$, *the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(C3) $\beta_n \in [0, a)$ *for some* $a \in (0,1)$.

*The sequence* $\{x_n\}$ *is defined iteratively by*

$$\begin{cases} y_n &= \beta_n x_n \oplus (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases}$$

*Then* $\{x_n\}$ *converges to a fixed point* $z \in F(T)$ *which is nearest to* $u$.

*Proof.* By taking $\gamma_n = 1$, in Corollary 3.5, $\{x_n\}$ converges strongly to a fixed point $z \in F(T)$ which is nearest to $u$. □

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