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EPI-RETRACTABLE MODULES AND SOME APPLICATIONS

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ABSTRACT. Generalizing concepts "*right Bezout* " and "*principal right ideal* " of a ring R to modules, an R-module M is called *n-epi-retractable* (resp. *epi-retractable*) if every n-generated submodule (resp. submodule) of M_R is a homomorphic image of M. It is shown that if M_R is finitely generated quasi-projective 1-epi-retractable, then $\mathrm{End}_R(M)$ is a right Bezout (resp. principal right ideal) ring if and only if M_R is n-epi-retractable for all $n \geq 1$ (resp. epi-retractable). For a ring R and an infinite ordinal $\beta \geq |R|$, the R-module $M = F \oplus N$ is epi-retractable where F is a free R-module with a basis set of cardinality β and N is a γ-generated R-module with $\gamma \leq \beta$. A ring R is *quasi Frobenius* if every injective R-module is epi-retractable. Injective modules in $\sigma[N_R]$ are epi-retractable for every $N \in \sigma[M_R]$ if and only if every non-zero factor ring of S is a quasi Frobenius ring where S is an endomorphism ring of a progenerator in $\sigma[M_R]$

1. **Introduction**

All rings are associative with unit elements and all modules are unitary right modules. Let R be a ring. The ring R is said to be a principal right ideal ring (*pri*) if every right ideal of R is principal. Also, R is said to be a right Bezout ring if every finitely generated right ideal of

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R is principal. Generalizing these concepts to modules, an R-module M is called epi-retractable (resp. n-epi-retractable) if every submodule (resp. n-generated submodule) of M is a homomorphic image of M_R. Therefore, R is a pri (resp. right Bezout) ring if and only if R_R is epi-retractable (resp. n-retractable $\forall n \geq 1$). In [5], morphic modules are introduced and shown to have internal cancellation property by investigating the epi-retractable condition for such modules ([5], Theorem 15). If M_R is n-epi-retractable for some $n \geq 1$, then it is retractable (i.e., $\mathrm{Hom}_R(M, N) \neq 0$ for any non-zero submodule $N \leq M_R$). Retractable modules have been investigated by several authors; see for example, [2], [3], [6], [8]. Here, we reveal some applications of projective, nonsingular, injective epi-retractable modules regarding the characterization of Bezout, pri, quasi Frobenius, rings. Examples are also given where (n)-epi-retractable modules appear. A brief description of the content of the paper will now follow.

In Section 2, it is proved in Theorem 2.2 that if M is a non-zero finitely generated quasi-projective 1-epi-retractable R-module, then:

(i) $\mathrm{End}_R(M)$ is a right Bezout ring if and only if M_R is n-epi-retractable, for all $n \geq 1$.

(ii) $\mathrm{End}_R(M)$ is a pri ring if and only if M_R is epi-retractable.

In particular, the matrix ring $\mathrm{Mat}_{m\times m}(R)$ is pri (resp. right Bezout) if and only if $R_R^{(m)}$ is epi-retractable (resp. $R_R^{(m)}$ is n-epi-retractable $\forall n \geq m$).

Projective epi-retractable modules are investigated and it is shown that for a ring R and an infinite ordinal $\beta \geq |R|$, the R-module $M = F \oplus N$ is epi-retractable where F is a free R-module with a basic set of cardinality β and N is a γ-generated R-module with $\gamma \leq \beta$ (Theorem 2.8). Over a ring in which principal right ideals are projective, finite dimensional torsionfree epi-retractable modules are characterized and shown to be projective. It also shown that a ring R is a pri domain if and only if R_R is uniform and there exists a uniform nonsingular epi-retractable R-module (Proposition 2.16). If every injective R-module is epi-retractable, then R is a quasi Frobenius ring (Proposition 3.2). We end up with Theorem 3.5 that states: For any module M_R with a progenerator $P \in \sigma[M_R]$, any non-zero factor ring of $\mathrm{End}_R(P)$ is a quasi Frobenius ring if and only if for any $N \in \sigma[M_R]$, every injective module in $\sigma[N_R]$ is epi-retractable. Recall that a ring R is said to be a quasi Frobenius ring if it is a (left) right self injective Noetherian ring. Any unexplained terminology, and all the basic results on rings and modules that are used in the sequel can be found in [4] and [7].

2. **Epi-Retractable condition for projective modules**

We begin with the following observation.

Proposition 2.1. *Let M be a non-zero quasi-projective 1-epi-retractable R-module with $End_R(M) = S$. If S is an n-epi-retractable right S-module (resp. a pri ring), then M_R is n-epi-retractable (resp. epi-retractable).*

Proof. Let $X = x_1R + \cdots + x_nR$ be an n-generated submodule of M. Then, by hypothesis, M is $(x_1R \oplus \cdots \oplus x_nR)$-projective. Thus, for every $h \in \mathrm{Hom}_R(M, X)$, there exists $\bar{h} : M \to (x_1R \oplus \cdots \oplus x_nR)$ such that $h = \mu\bar{h}$ where $\mu : (x_1R \oplus \cdots \oplus x_nR) \to X$ is the natural surjective homomorphism. It follows that $\mathrm{Hom}_R(M, X) = \sum_{i=1}^{n} \mathrm{Hom}_R(M, x_iR)$. Now, since M_R is 1-epi-retractable, then each x_iR is equal to $f_i(M)$ for some $f_i \in S$ and we have $\mathrm{Hom}_R(M, x_iR) = f_iS$ $(1 \leq i \leq n)$, by the quasi-projective condition on M_R. Hence, by the hypothesis on S, $\mathrm{Hom}_R(M, X) = \sum_{i=1}^{n} f_iS = gS$ for some $g \in S$. We now show that $X = g(M)$. Clearly, $g(M) \subseteq X$. Let x be an element in X. Then, there exists $\theta \in S$ such that $\theta(M) = xR$ and we have $\theta S = \mathrm{Hom}_R(M, \theta(M)) \subseteq \mathrm{Hom}_R(M, X) = gS$. Hence, $\theta = gt$ for some $t \in S$ and so $\theta(M) \subseteq g(M)$. It follows that $x \in g(M)$. Consequently, $X = g(M)$ is a homomorphic image of M_R, proving that M_R is n-epi-retractable. The other case of the result is proved similarly. □

Theorem 2.2. *If M is a non-zero quasi-projective 1-epi-retractable finitely generated R-module, then:*
(i) $End_R(M)$ is a right Bezout ring if and only if M_R is n-epi-retractable, $\forall n \geq 1$.
(ii) $End_R(M)$ is a pri ring if and only if M_R is epi-retractable.

Proof. We prove *(i)*. One direction follows from Proposition 2.1. Conversely, set $\mathrm{End}_R(M) = S$ and let I be a finitely generated right ideal of S. Because M_R is finitely generated, then IM is a finitely generated R-submodule of M. Since M_R is n-epi-retractable for all $n \geq 1$, then we must have $IM = f(M)$ for some $f \in S$. Hence, by hypothesis, $fS = \mathrm{Hom}_R(M, f(M)) = I$. □

Corollary 2.3. *Let m be a positive integer. Then, the following statements are equivalent on a ring R.*
(i) $R_R^{(m)}$ is epi-retractable (resp. $R_R^{(m)}$ is n-epi-retractable $\forall n \geq m$).
(ii) The matrix ring $Mat_{m\times m}(R)$ is pri (resp. right Bezout).

Proof. Apply Theorem 2.2 for $M = R^{(m)}$, and note that M_R is a k-epi-retractable R-module for any $k < m$. □

Examples 2.4. (1) Let I be a right ideal in a regular ring R. Then I_R is n-epi-retractable for every $n \geq 1$. In fact, if I contains a finitely generated right ideal J of R, then J is a direct summand of R and hence a direct summand of I_R. It follows that J is a homomorphic image of I_R.
(2) For any non-zero R-module X, the R-module $M = R/\mathrm{ann}_R(X) \oplus X$ is 1-epi-retractable. Note that for any $m \in M$, we have $\mathrm{ann}_R(X) \subseteq \mathrm{ann}_R(m)$. Hence, $mR \simeq R/\mathrm{ann}_R(m)$ is a homomorphic image of $R/\mathrm{ann}_R(X)$ and so of M_R.
(3) Over a PID, every finitely generated module is epi-retractable. To see this, let R be a PID and let M be a finitely generated R-module, and $N \leq M$. Then, by a well known result, $M \simeq (\bigoplus_{i=1}^{n} R_i) \oplus (\bigoplus_{i=1}^{m} R/(p_i^{k_i}))$ and $N \simeq (\bigoplus_{i=1}^{t} R_i) \oplus (\bigoplus_{i=1}^{s} R/(p_i^{s_i}))$, where $R_i = R, 0 \leq t \leq n$, $0 \leq s \leq m$, and $0 \leq s_i \leq k_i$. From this, we see that N is a homomorphic image of M_R.

We shall now investigate when a projective module is epi-retractable and vice versa.

Proposition 2.5. *Let R be a right hereditary ring. Then, R is a pri ring if and only if every free right R-module is epi-retractable.*

Proof. ($\Leftarrow$). This is obtained from the definitions.
($\Rightarrow$). By Kaplansky's Theorem ([4], Theorem 2.24), any submodule N of a free right R-module $F = \bigoplus_{\alpha\in\Omega} e_\alpha R$, is isomorphic to $\bigoplus_{\alpha\in\Omega} J_\alpha$, where the J_α are right ideals of R. Thus, the result is proved by the fact that if $f_\alpha : e_\alpha R \to J_\alpha$ is surjective R-homomorphism for all $\alpha \in \Omega$, then the homomorphism $\bigoplus_{\alpha\in\Omega} f_\alpha$ is also surjective. □

The following Lemmas are needed.

Lemma 2.6. *Let R be a ring of cardinality $|R| = \alpha$. Then any free R-module F with an infinite basis set X of cardinality $\beta \geq \alpha$ is an epi-retractable R-module.*

Proof. We first show that $|F| = \beta$. Let $X_n = \{\sum_{i=1}^{n} x_i r_i \mid x_i \in X, r_i \in R\}$ where $n \geq 1$. Then, there naturally exist a surjective map $Y_n := (X \times R)^{(n)} \to X_n$ and an injective map $X \to X_n$. Consequently, $|X| \leq |X_n| \leq |Y_n| = \beta$ for any $n \geq 1$. It follows from $F = \bigcup_{n \in \mathbb{N}} X_n$ that $|F| = \beta$, as desired. Now, any submodule N of F_R is a homomorphic image of a free R-module G with a basic set of cardinality $\gamma \leq \beta$. Thus, G and hence N is a homomorphic image of F_R, proving that F_R is epi-retractable. □

Lemma 2.7. *The following statements are equivalent for a module M.*
(i) M is epi-retractable.
(ii) There exist surjective homomorphisms $M \to N$ and $N \to M$ for some epi-retractable module N.
(iii) There exists a surjective homomorphism $M/K \to M$ for some epi-retractable factor module M/K.

Proof. (i)$\Rightarrow$ (ii). This is clear.
(ii)$\Rightarrow$ (iii). Suppose that there exist an epi-retractable module N and surjective homomorphisms $\alpha : M \to N$, $\beta : N \to M$. Let $K = \ker \alpha$. Then, α induces an isomorphism $\bar{\alpha} : M/K \to N$. Thus, M/K is an epi-retractable module.
(iii)$\Rightarrow$(i). Let L be any submodule of M. By our assumption, there exists an isomorphism $\bar{\varphi} : M/K' \to M$ for some submodule K' of M with $K \subseteq K'$. Let $\bar{\varphi}(N/K') = L$ for some submodule N of M. Since M/K is assumed epi-retractable, then there exists a surjective homomorphism $\theta : M/K \to N/K$. Consider $\alpha : N/K \to N/K'$ with $\alpha(n+K) = n+K'$, and the canonical epimorphism $\pi : M \to M/K$. Then, $\bar{\varphi}\alpha\theta\pi : M \to L$ is a surjective homomorphism, proving that M is epi-retractable. □

Theorem 2.8. *Let R be a ring and β be an infinite ordinal $\geq |R|$. Suppose that $M = F \oplus N$ where F is a free R-module with a basic set of* *cardinality β and N is a γ-generated R-module with $\gamma \leq \beta$. Then, M_R is epi-retractable.*

Proof. By Lemma 2.6, F_R is epi-retractable. By hypothesis, N is a homomorphic image of F_R. Since $F \oplus F \simeq F$, then there exist surjective homomorphisms $M \to F$ and $F \to M$. Thus, the result holds by Lemma 2.7. □

Corollary 2.9. *Let R be an infinite ring. Then, every free R-module with a basic set of cardinality $\geq |R|$ is epi-retractable.*

Proof. It follows from Theorem 2.8. □

Proposition 2.10. *Let R be a countable semiprime pri ring. Then, any free R-module is epi-retractable.*

Proof. In view of Theorem 2.8, we need to show that any finitely generated free R-module is epi-retractable. Because R is a pri semiprime ring, then by a well known result $R = \oplus_{i=1}^{t} S_i$ is a finite product of prime pri rings. Hence, $\mathrm{Mat}_{n\times n}(R) \simeq \bigoplus_{i=1}^{t} \mathrm{Mat}_{n\times n}(S_i)$ and each $\mathrm{Mat}_{n\times n}(S_i)$ is a pri ring; see [1]. The result now follows from Corollary 2.3. □

We now investigate when a direct summand of an epi-retractable module is epi-retractable.

Proposition 2.11. *Let M be an epi-retractable R-module. Then:*
(i) M/N is epi-retractable for any fully invariant submodule N of M_R.
(ii) If $M = L \oplus N$ such that $Hom_R(L, N) = 0$, then N_R is epi-retractable.

Proof. *(i)* Let N be a fully invariant submodule of M, and let K/N be any submodule of M/N. There is a surjective homomorphism $\varphi : M \to K$. Now, $\varphi(N) \subseteq N$ by our assumption, and so $\bar{\varphi} : M/N \to K/N$, with $\bar{\varphi}(m + N) = \varphi(m) + N$ is a surjective homomorphism.
(ii) Note that $\mathrm{End}_R(M) = \begin{bmatrix} \mathrm{End}_R(L) & \mathrm{Hom}_R(N,L) \\ 0 & \mathrm{End}_R(N) \end{bmatrix}$. Hence, $\mathrm{End}_R(M) \begin{bmatrix} L \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} L \\ 0 \end{bmatrix}$. It follows that $(L \oplus 0)$ is a fully invariant submodule of M_R. Now, apply *(i)*. □

Remark 2.12. Let G be a free $\mathbb{Z}$-module with an infinite countable basic set and X be any countable $\mathbb{Z}$-module which is not epi-retractable (e.g., $X = \mathbb{Q}$). Then, the $\mathbb{Z}$-module $M = X \oplus G$ is epi-retractable by Theorem 2.8. This shows that a direct summand (and hence a submodule or a factor module) of an epi-retractable module need not be epi-retractable.

We are now going to investigate when an epi-retractable module is projective. A ring R is called *right Rickart* or *right principally projective* if every principal right ideal in R is projective (as a right R-module). Semi-hereditary rings and domains are clearly Rickart. A module M has a *finite uniform dimension* (or *finite rank*) if M contains no infinite direct sum of non-zero submodules or equivalently there exist independent uniform submodules $U_1, \cdots, U_n$ in M such that $\oplus_{i=1}^{n} U_i$ is an essential submodule of M. In this case, it is written $\mathrm{u.dim}(M) = n$.

Proposition 2.13. *Let R be a right Rickart ring. Then, every finite dimensional nonsingular epi-retractable R-module M is isomorphic to a direct sum of principal right ideals of R. In particular, M_R is projective.*

Proof. Let m be any non-zero element of M. Since M is nonsingular, then the right annihilator m in R is not an essential right ideal. Let $B = \mathrm{r.ann}_R(m)$ and $B \cap A = 0$ for some non-zero principal right ideal A of R. Thus, $mA \simeq A$. It follows that every non-zero submodule of M_R isomorphically contains a non-zero principal right ideal of R. Now, let $\mathcal{P}$ be a maximal independent family of elements in the set $\{N \leq M_R \mid N$ is isomorphic to a principal right ideal of $R\}$ and let $L = \bigoplus \mathcal{P}$. Then, L is an essential submodule of M_R. By the epi-retractable condition on M_R, there exists a surjective homomorphism from M to L. Also, by hypothesis on R, L_R is projective. Hence, L is isomorphic to a direct summand K of M_R. Now, $\mathrm{u.dim}(K) = \mathrm{u.dim}(L) = \mathrm{u.dim}(M)$. It follows that $L \simeq K = M$, as desired. □

Corollary 2.14. *Let R be a right and left Ore domain. Then, every finitely generated torsion free epi-retractable R-module is a free R-module.*

Proof. This is obtained by Proposition 2.13 and the well known result from Gentile and Levy which states that over a semiprime right and left Goldie ring, finitely generated torsion free modules have finite ranks. □

We end this section with an application of nonsingular epi-retractable modules.

Lemma 2.15. *A non-zero module M_R is uniform nonsingular epi-retractable if and only if $Z(M) \neq M$ and $M \simeq N$, for all non-zero submodules N of M.*

Proof. ($\Rightarrow$). We have $\mathrm{Z}(M) \neq M$ and for each non-zero submodule N of M, there exists a surjective homomorphism $f : M \to N$, and so $M/\ker f \hookrightarrow M$. Since M_R is nonsingular, then $M/\ker f$ is nonsingular and hence $\ker f = 0$, because M is uniform. Thus, $M \simeq N$.
($\Leftarrow$) By hypothesis, M_R is epi-retractable and if $\mathrm{Z}(M) \neq 0$, then we must have $M \simeq \mathrm{Z}(M)$. Consequently, $M = \mathrm{Z}(M)$, which is a contradiction. Therefore $\mathrm{Z}(M) = 0$. Also, for all $0 \neq x \in M$, $M \simeq xR$, and so M_R is Noetherian. Therefore, M_R has a uniform submodule U ([4], Proposition 6.4). It follows that $M \simeq U$ is uniform. □

Proposition 2.16. *The following statement are equivalent for a ring R.*
(i) R is a principal right ideal domain.
(ii) R_R is uniform and there exists a uniform nonsingular epi-retractable R-module.

Proof. *(i)⇒(ii).* Apply Lemma 2.15 for $M = R$.
(ii)⇒(i). Let M be a uniform nonsingular epi-retractable R-module. Since R_R is uniform and $\mathrm{Z}(M) \neq M$, then there exists $x \in M$ such that $\mathrm{ann}_R(x) = 0$. Thus, R can be embedded in M_R and hence $M \simeq R$, by Lemma 2.15. It follows that R is a right nonsingular principal right ideal ring with uniform dimension 1. Now, if $ab = 0$ and $0 \neq b$ for some $a, b \in R$, then $0 \neq \mathrm{r.ann}_R(a)$ is an essential right ideal of R, which implies that $a \in \mathrm{Z}(R_R) = 0$. The proof is now complete. □

3. Epi-Retractable condition for injective objects

By a *class* $\mathcal{C}$ of R-modules we mean a collection $\mathcal{C}$ of R-modules which contains a non-zero module and which is closed under taking isomorphisms. Let $\mathcal{C}$ be a class of modules. A module $X \in \mathcal{C}$ is called an *injective* module in $\mathcal{C}$, if every exact sequence $0 \to X \to A \to B \to 0$ with A, $B \in \mathcal{C}$ splits. We denote by $\sigma[M_R]$, the full subcategory of mod-R whose objects are all R-submodules of M-generated modules. In this

section, we observe that every non-zero factor ring of a ring R is a quasi Frobenius ring if and only if for any R-module N, in the class $\sigma[N_R]$, injective R-modules are epi-retractable. It is well known that injective modules are *continuous*. We first record that continuous nonsingular epi-retractable modules are semisimple. Recall that an R-module M is said to be a continuous module if it satisfies the following conditions:
(C_1) Every submodule of M is essential inside a direct summand of M_R.
(C_2) Every submodule of M that is isomorphic to a summand of M is itself a summand of M_R.

Proposition 3.1. *The following are equivalent for a nonsingular R-module M.*
(i) M_R is semisimple.
(ii) M_R is continuous epi-retractable.

Proof. *(i)⇒(ii).* This is clear.
(ii)⇒(i). Let N be a submodule of M_R. By assumption, there exists an R-epimorphism f from M to N. Because $N \simeq M/\ker f$ is nonsingular, then $\ker f$ is an essentially closed submodule of M_R. Then, by the C_1-condition on M_R, $\ker f$ is a direct summand of M_R. It follows that N is isomorphic to a direct summand of M_R. Now, the C_2-condition implies that N is a direct summand of M_R, proving that M_R is semisimple. □

Proposition 3.2. *If R is a ring such that every injective R-module is epi-retractable, then R is a quasi Frobenius ring.*

Proof. By Remark 15.10 in [4], we need to show that every projective R-module is injective. Now, let X be a projective R-module and let E be the injective hull of X_R. Then, by our assumption, there exists a surjective R-homomorphism f from E to X. Because X_R is projective, then the $\ker f$ is a direct summand of E. It follows that $X = E$, as desired. □

It is known that if R is a ring such that every non-zero factor ring of R is a quasi Frobenius ring, then every R-module is a direct sum of *homo-uniserial* modules. An R-module M is called homo-uniserial if for any non-zero finitely generated submodules $K, L \subseteq M$, the factor modules $K/\mathrm{J}(K)$ and $L/\mathrm{J}(L)$ are simple and isomorphic. In particular, if $M \neq$

J(M) and Soc(M) $\neq 0$, then M_R is finitely generated and $M/\mathrm{J}(M) \simeq$ Soc(M); see [7, 56].

Theorem 3.3. *The following are equivalent on a ring R.*
(i) Every non-zero factor ring of R is a quasi Frobenius ring.
(ii) For any R-module N, in the class $\sigma[N_R]$, injective R-modules are epi-retractable.

Proof. *(i) ⇒ (ii).* Suppose that N is a non-zero R-module and set $B = \mathrm{ann}_R(N)$. Because R is an Artinian ring by *(i)*, then it is well known that there are $x_1, \cdots, x_t \in N$ such that $B = \bigcap_{i=1}^{t} \mathrm{ann}_R(x_i)$. Consequently, R/B embeds in $N^{(t)}$. It follows that $\sigma[N] =$ Mod-R/B. Hence, it is enough to show that every injective R/B-module is epi-retractable. By hypothesis, if W is an R/B-module, then $W = \oplus U_{\lambda \in \Lambda}$ is a direct sum of homo-uniserial modules. Because R/B is an Artinian ring, then it is easy to verify that $\mathrm{J}(U_\lambda) \neq U_\lambda$ for each $\lambda \in \Lambda$. Thus, by the above remarks $W/\mathrm{J}(W) \simeq \mathrm{Soc}(W)$. Now, let E be any injective R/B-module and Y be a submodule of E. Because Soc(Y) is a direct summand of Soc(E), then there exists a surjective homomorphism f from E to $Y/\mathrm{J}(Y)$. On the other hand, the injective R/B-module E is also projective as an R/B-module. Hence, there exists a homomorphism $g : E \to Y$ such that $\pi g = f$ where $\pi : Y \to Y/\mathrm{J}(Y)$ is the canonical projection. Because πg is a surjective homomorphism, then we can conclude that $g(E)+\mathrm{J}(Y) = Y$. But, by hypothesis, J(Y) is a small submodule of Y, and hence $g(E) = Y$. This shows that E is an epi-retractable R/B-module, as desired.
(ii) ⇒ (i). Let A be a proper ideal of R and set $N = R/A$. Then, $\sigma[N_R]$ = Mod-R/A. The result follows now Proposition 3.2. □

Lemma 3.4. *Being epi-retractable is a Morita invariant property.*

Proof. In fact, a module M_R is epi-retractable if and only if for any $X \in$ Mod-R with an injective homomorphism $X_R \to M_R$ there exists a surjective homomorphism $M_R \to X_R$. Thus, the result follows from the fact that any category equivalence preserves injective and surjective homomorphisms. □

Theorem 3.5. *Let M be an R-module and let S be the endomorphism ring of a progenerator in $\sigma[M_R]$. Then, the following statements are equivalent.*

(i) For any $N \in \sigma[M_R]$, *in the class* $\sigma[N_R]$, *injective* R*-modules are epi-retractable.*
(ii) Every non-zero factor ring of S *is a quasi Frobenius ring.*

Proof. By Theorem 46.2 in [7], there exists a category equivalence between $\sigma[M_R]$ and Mod-S. Hence, for any $X \in$ Mod-S, the class $\sigma[X_S]$ corresponds to the class $\sigma[N_R]$ for some suitable $N \in \sigma[M_R]$ and vice versa. The result is then obtained by Lemma 3.4 and Theorem 3.3. □

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