ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 40 (2014), No. 4, pp. 1047–1055

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 4, pp. 1047–1055
Online ISSN: 1735-8515

CONTINUOUS FRAMES AND G-FRAMES

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(Communicated by Behzad Djafari-Rouhani)

Abstract. In this note, we aim to show that several known generalizations of frames are equivalent to the continuous frame defined by Ali et al. in 1993. Indeed, it is shown that these generalizations can be considered as an operator between two Hilbert spaces.
Keywords: G-frame, continuous frame, Sun's g-frame.
MSC(2010): Primary: 42C15; Secondary: 41A58, 46C05.

1. **Introduction**

Frames have a myriad of applications in mathematics and engineering including sampling theory, wavelet theory, signal and image processing, operator theory, harmonic analysis, filter banks, geophysics, quantum computing and more. We refer to [6, 10, 13, 17, 18, 19, 24] for an introduction to the frame theory and its applications.

Various kinds of frames have been proposed recently. For example, bounded quasi-projector [15, 14], frames of subspaces [4, 5], pseudo frames [21], oblique frames [7, 12] and outer frames [2]. In [23] Sun introduced a g-frame which generalized the mentioned frames above but not the continuous frame defined by Ali et al. in [3] (for continuous frame see also [20]). In recent years, some researchers have generalized Sun's g-frame and continuous frame [9, 11]. Our aim is to show that all of these g-frames are equivalent to the continuous frame defined by Ali et al. in [3]. In fact, the main idea in this note is that these g-frames

Article electronically published on August 23, 2014.
Received: 27 December 2012, Accepted: 21 July 2013.
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can be considered as operators of some Hilbert space H to some Hilbert space K.

At this point, we review some notations and definitions. Let H be a Hilbert space. The collection $(f_i)_{i\in I} \subset H$ (I is not necessarily countable) is called a (discrete) frame for H if there exist $A, B > 0$ such that $A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2$, for all $f \in H$. The constants A, B are called lower and upper bounds, respectively. If $A = B$, it is called a tight frame and it is said to be a normalized tight or Parseval frame, if $A = B = 1$. The collection $(f_i)_{i\in I} \subset H$ is called Bessel if the above second inequality holds. In this case, B is called the Bessel bound.

This paper is organized as follows. In Section (2) we give some basic definitions of several kinds of frames from [1, 9, 16, 22, 23]. In Section (3) we introduce our g-frame and show that it is equivalent to the continuous frames. Then, we show that the g-frames mentioned in Section (2) are equivalent to the continuous frame defined by Ali et al. in [3].

2. Some basic definitions and preliminaries

In this section we recall definitions of some g-frames from [23, 1, 9, 22, 16] which will be required in the sequel. Note that for a collection $\{K_i\}_{i\in I}$ of Hilbert spaces, there exists a Hilbert space K such that for all $i \in I, K_i \subset K$. Indeed, consider $K = \bigoplus_{i\in I} K_i$, see [8].

The following definition is [23, Definition 1.1.]

Definition 2.1. *([23]) Let H be a Hilbert space, $(K_j)_{j\in J}$, (J is at most countable) a family of Hilbert spaces and $\Lambda_j : H \to K_j$, $j \in J$ bounded linear operators. The set $\{\Lambda_j : j \in J\}$ is called a Sun g-frame if there exist $A, B > 0$ such that $A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2$ for all $x \in H$.*

The following definition can be found in [16].

Definition 2.2. *Let (Ω, μ) be a measure space and H a Hilbert space. The mapping $F : (\Omega, \mu) \to H$ is called a continuous frame with bounds A, B, if $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω for every $f \in H$ and*

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \qquad (f \in H).$$

The mapping F is called a Bessel map if the right hand inequality holds.

Let $F : \Omega \to H$ be a continuous frame. Then the operator $T_F : H \to L^2(\Omega, \mu)$ given by $T_F(x)(\omega) = \langle x, F(\omega)\rangle$ is a well-defined bounded linear operator. This operator is called the frame transform or analysis operator. It is 1-1 and bounded below if and only if F is a continuous frame.

The following definition is [1, Definition 2.7.].

Definition 2.3. *Let (Ω,μ) be a measure space, $(K_\omega)_{\omega\in\Omega}$ a collection of Hilbert spaces. Define $((K_\omega)_{\omega\in\Omega},(\Omega,\mu))$ as the set of all $F:\Omega\to\bigcup_{\omega\in\Omega}K_\omega$ such that*

- *$F(\omega)\in K_\omega$ for all $\omega\in\Omega$,*
- *$\omega\to\|F(\omega)\|$ is a measurable map in which $\|F(\omega)\|$ is in K_ω,*
- *$\int_\Omega\|F(\omega)\|^2d\mu(\omega)<\infty$.*

Put $\|F\|_2^2=\int_\Omega\|F(\omega)\|^2d\mu(\omega)$ and

$$\langle F,G\rangle=\int_\Omega\langle F(\omega),G(\omega)\rangle d\mu(\omega)$$

where $F,G\in((K_\omega)_{\omega\in\Omega},(\Omega,\mu))$.

It is easy to see that $((K_\omega)_{\omega\in\Omega},(\Omega,\mu))$ is a Hilbert space with the above inner product.

Example 2.4. *If $K_\omega=\mathbb{C}$ for any $\omega\in\Omega$, then $((K_\omega)_{\omega\in\Omega},(\Omega,\mu))=L^2(\Omega,\mu)$*

Definition 2.5. *([1]) Let H be a Hilbert space, (Ω,μ) a measure space and $(K_\omega)_{\omega\in\Omega}$ a family of Hilbert spaces. A family $\{\Lambda_\omega\in B(H,K_\omega):\omega\in\Omega\}$ is called a continuous g-frame for H with respect to $(K_\omega)_{\omega\in\Omega}$ if the mapping $\Omega\to\mathbb{C}$, defined by $\omega\to\|\Lambda_\omega f\|$ is a measurable function on Ω for any $f\in H$ and there are also two constants $0<A\le B<\infty$ such that*

$$A\|f\|^2\le\int_\Omega\|\Lambda_\omega f\|^2d\mu(\omega)\le B\|f\|^2,\quad (f\in H).$$

Definition 2.6. *([9]) Let H be a Hilbert space and (Ω,μ) a measure space with positive measure μ. Also, suppose that B_H is the collection of all Bessel sequences in H and I is an at most countable index set. A mapping $F:\Omega\to B_H$ defined by $\omega\to(f_i(\omega))_{i\in I}$ is called a generalized continuous frame with respect to (Ω,μ) if F is weakly measurable, i.e., for all $f\in H, i\in I$, the function $\omega\to\langle f,f_i(\omega)\rangle$ is measurable on Ω and there exist positive constants A,B such that*

$$A\|f\|^2\le\int_\Omega\sum_{i\in I}|\langle f,f_i(\omega)\rangle|^2d\mu(\omega)\le B\|f\|^2,\quad (f\in H).$$

Definition 2.7. *([22]) Let $v : (\Omega, \mu) \to (0, \infty)$ be a measurable function and H_C the collection of all nonzero closed subspaces of H. A mapping $F : \Omega \to H_C$ is called a continuous frame of subspaces with respect to v for H if $\omega \to \pi_{F(\omega)}$ is a measurable function from Ω to $B(H)$ (π_M is the projection to M) and there exist $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \int_\Omega v^2(\omega)\|\pi_{F(\omega)}(f)\|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in H).$$

3. Main results

In this section we first introduce our g-frame and show that it is equivalent to the continuous frames. Then, we show that the g-frames mentioned in Section (2) are equivalent to our g-frame and so are equivalent to the continuous frame defined by Ali et al. in 1993,[3].

Now we give our new generalization of frame.

Definition 3.1. *Let H and K be two Hilbert spaces. A linear operator $\Lambda : H \longrightarrow K$ is called a generalized frame or simply a g-frame for H with respect to K if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \|\Lambda f\|^2 \leq B\|f\|^2$$

for all $f \in H$.

The constants A, B are called a lower and an upper bound, respectively. Note that the right hand inequality shows that Λ is a bounded linear operator. In this case, we call Λ Bessel operator with Bessel bound B. If $A = B$, Λ is a multiple of isometry and we call it a tight g-frame. If $A = B = 1$, Λ is an isometry and is called a normalized tight or Parseval g-frame.

We can consider a discrete frame as a bounded linear operator from some Hilbert space H to some Hilbert space K.

Example 3.2. *Assume that H is a Hilbert space and $(f_i)_{i\in I} \subset H$ is a discrete frame with bounds A, B. Let $K = l^2(I)$ and define $\Lambda : H \to K$ by $f \to (\langle f, f_i\rangle)_{i\in I}$. We have*

$$\|\Lambda f\|^2 = \|(\langle f, f_i\rangle)_{i\in I}\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2,$$

which implies that

$$A\|f\|^2 \leq \|\Lambda f\|^2 \leq B\|f\|^2.$$

So Λ is a g-frame.

Remark 3.3. *From now on throughout this paper we always mean by a g-frame the one defined in Definition 3.1.*

Ali et al. continuous frame can be considered as a special kind of our g-frame.

Proposition 3.4. *The continuous frames in the sense of Ali et al. as in Definitin* (2.2)*, are a special kind of our g-frame.*

Proof. Suppose that (Ω, μ) is a measure space and $F : (\Omega, \mu) \to H$ is a continuous frame with bounds A, B. Let $K = ((K_\omega)_{\omega\in\Omega}, (\Omega, \mu))$ be as Definition (2.3) where $K_\omega = \mathbb{C}$ for all $\omega \in \Omega$ and $\Lambda : H \to K$ given by $f \to (\langle f, F(\omega)\rangle)_{\omega\in\Omega}$. Indeed by Example (2.4) $K = L^2(\Omega, \mu)$. Also, $\Lambda = T_F$ is as Definition (2.2). We have $\|\Lambda f\|^2 = \|(\langle f, F(\omega)\rangle)_{\omega\in\Omega}\|^2 = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2$, so Λ is bounded and $A\|f\|^2 \leq \|\Lambda f\|^2 \leq B\|f\|^2$. □

Theorem 3.5. *Our g-frame is a special kind of the continuous frame defined by Ali et al. as in Definition* (2.2)*.*

Proof. Suppose that $(e_i)_{i\in I}$ is an orthonormal basis for K and μ is the counting measure on I. We define

$$F : (I, \mu) \to H, i \to F(i) = \Lambda^* e_i$$

For $f \in H$ we have

$$\begin{aligned}\int_I |\langle f, F(i)\rangle|^2 d\mu(i) &= \sum_{i\in I} |\langle f, \Lambda^* e_i\rangle|^2 \\ &= \sum_{i\in I} |\langle \Lambda f, e_i\rangle|^2 = \|\Lambda f\|^2,\end{aligned}$$

then $A\|f\|^2 \leq \int_I |\langle f, F(i)\rangle|^2 d\mu(i) \leq B\|f\|^2$. □

Corollary 3.6. *Our g-frame is equivalent to the continuous frames.*

We intend to show that the Sun g-frame defined in Definition 2.1 is a special case of our g-frame.

Proposition 3.7. *The Sun's g-frame as in Definition 2.1, is a special kind of our g-frame.*

Proof. Let $\{\Lambda_j : H \to K_j, j \in J\}$ (J is at most countable) be a Sun's g-frame with bounds A, B, i.e.,

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad (f \in H).$$

Let $K = \bigoplus_{j\in J} K_j$ and let $\Lambda : H \to K$ —be given by $f \to (\Lambda_j f)_{j\in J}$. We have

$$\|\Lambda f\|^2 = \|(\Lambda_j f)_{j\in J}\|^2 = \sum_{j\in J} \|\Lambda_j f\|^2$$

and so $A\|f\|^2 \le \|\Lambda f\|^2 \le B\|f\|^2$.

□

Corollary 3.8. *The Sun's g-frame as in Definition 2.1, is equivalent to the continuous frames.*

Proof. First, we show that our g-frames is a special kind of Sun g-frames. Let $\Lambda : H \to K$ be a g-frame with bounds A, B, i.e., $A\|f\|^2 \le \|\Lambda f\|^2 \le B\|f\|^2$. In Sun definition $\{\Lambda_j : j \in J\}$, J is a subset of Z. Put $J = \{1\}$ and $\Lambda_1 = \Lambda$ and $K_1 = K$. Now, by Proposition 3.7 the Sun g-frames is equivalent to our g-frames and by Corollary 3.6 it is equivalent to the continuous frames. □

We show that some of other g-frames also are equivalent to the continuous frames.

Proposition 3.9. *Continuous g-frame as in Definition 2.5, is a special kind of our g-frame.*

Proof. With assumptions and notations in Definition 2.5 and Definition 2.3, put $K = ((K_\omega)_{\omega\in\Omega}, (\Omega, \omega))$ and define $\Lambda : H \to K$ by $f \to (\Lambda_\omega f)_{\omega\in\Omega}$, then $\|\Lambda f\|^2 = \|(\Lambda_\omega f)_{\omega\in\Omega}\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)$, so$A\|f\|^2 \le \|\Lambda f\|^2 \le B\|f\|^2$. □

Corollary 3.10. *Continuous g-frame as in Definition 2.5, is equivalent to the continuous frames.*

Proof. The continuous g-frame is a special kind of our g-frames, so is special kind of continuous frames. Also, continuous g-frames are a generalization of continuous frames. Thus, the continuous g-frames are equivalent to the continuous frames. □

Theorem 3.11. *The g-frame of Dao-Xin Ding as in Definition 2.6, is equivalent to the continuous frames.*

Proof. Suppose that the assumptions of Definition 2.6 are satisfied. Let $K_\omega = \bigoplus_{i\in I} \mathbb{C}$ $(= l^2(I))$ and let $\Lambda_\omega : H \to K_\omega$ be given by $f \to (\langle f, f_i(\omega)\rangle)_{i\in I}$ and suppose that

$$\Lambda : H \to ((K_\omega)_{\omega\in\Omega}, (\Omega, \mu)),\ f \to ((\langle f, f_i(\omega)\rangle)_{i\in I})_{\omega\in\Omega}.$$

We have

$$\begin{aligned}\|\Lambda f\|^2 &= \|((\langle f, f_i(\omega)\rangle)_{i\in I})_{\omega\in\Omega}\|^2\\ &= \int_\Omega \|((\langle f, f_i(\omega)\rangle)_{i\in I})_{\omega\in\Omega}\|^2 d\mu(\omega)\\ &= \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega).\end{aligned}$$

Thus, Λ is bounded and

$$A\|f\|^2 \le \|\Lambda f\|^2 \le B\|f\|^2.$$

So, the g-frame of Dao-Xin Ding is a special kind of our g-frames, and thus, by Corollary 3.6 is a special kind of continuous frames. Also, the g-frame of Dao-Xin Ding is a generalization of continuous frames. Therefore, the g-frame of Dao-Xin Ding is equivalent to the continuous frames. □

Proposition 3.12. *The continuous frame of subspaces defined as in Definition 2.7, is equivalent to the continuous frames.*

Proof. Step 1. First, we show that the continuous frame of subspaces defined as in Definition 2.7, is a special kind of continuous frames. With notations as in Definition 2.7 let $K_\omega = \pi_{F(\omega)}(H)$ for any $\omega \in \Omega$; $\Lambda_\omega : H \to K_\omega$ given by $f \to v(\omega)\pi_{F(\omega)}(f)$, $K = ((K_\omega)_{\omega\in\Omega}, (\Omega, \mu))$ and let $\Lambda : H \to K$ be given by $f \to (\Lambda_\omega f)_{\omega\in\Omega} = (v(\omega)\pi_{F(\omega)}(f))_{\omega\in\Omega}$. We have

$$\begin{aligned}\|\Lambda f\|^2 =&\|(\Lambda_\omega f)_{\omega\in\Omega}\|^2\\ =&\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)\\ =&\int_\Omega \|v(\omega)\pi_{F(\omega)}(f)\|^2 d\mu(\omega)\\ =&\int_\Omega v^2(\omega)\|\pi_{F(\omega)}(f)\|^2 d\mu(\omega),\end{aligned}$$

and we obtain

$$A\|f\|^2 \le \|\Lambda f\|^2 \le B\|f\|^2.$$

So this g-frame is a special kind of our g-frames and by Corollary 3.6 is a special kind of continuous frames.

Step 2. We now show that, the continuous frames are special kind of frame of subspaces. Let $F : (\Omega, \mu) \to H$ be a continuous frame. Let $v : (\Omega, \mu) \to (0, \infty)$ be defined by $v(\omega) = 1$ and let $F_1(\omega)$ be the closed subspace generated by $F(\omega)$, that is, the closed subspace generated by the set $\{\alpha F(\omega) : \alpha \in \mathbb{C}\}$. It is easy to verify that

$$\int_\Omega v^2(\omega)\|\pi_{F_1(\omega)}(f)\|^2 d\mu(\omega) = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega), \quad f \in H.$$

So, continuous frames are a special kind of continuous frame of subspaces. Therefore, the continuous frame of subspaces defined as in Definition 2.7, are equivalent to the continuous frames.

$\square$

Acknowledgments

The authors would like to express their sincere thanks to the referee for his/her valuable comments.

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