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SOME PROPERTIES OF A GENERAL INTEGRAL OPERATOR

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ABSTRACT. In this paper, we consider a general integral operator $G_n(z)$. The main object of the present paper is to study some properties of this integral operator on the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{M}(\beta)$, $\mathcal{N}(\beta)$ and $\mathcal{KD}(\mu,\beta)$.
Keywords: Analytic functions, integral operator, starlike functions, convex functions.
MSC(2010): Primary: 30C45; Secondary: 30C75.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

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A function $f \in \mathcal{A}$ is a starlike function by the order α, $0 \leq \alpha < 1$ if f satisfies the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{U}.$$

We denote the class of functions $f \in \mathcal{A}$ satisfying the above condition by $\mathcal{S}^*(\alpha)$.
A function $f \in \mathcal{A}$ is a convex function by the order $\alpha, 0 \leq \alpha < 1$ if f satisfies the inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in \mathbb{U}.$$

We denote the class of functions $f \in \mathcal{A}$ satisfying the above condition by $\mathcal{K}(\alpha)$.
Let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) < \beta, \quad z \in \mathbb{U}; \beta > 1.$$

This class was studied by Owa and Srivastava [8].
Let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \mathbb{U}; \beta > 1.$$

This class was studied by Porwal and Dixit [12].
A function f is said to be in the class $\mathcal{KD}(\mu, \beta)$, if it satisfies the following inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) \geq \mu \left|\frac{zf''(z)}{f'(z)}\right| + \beta \tag{1.1}$$

for some $\mu \geq 0$ and $0 \leq \beta < 1$. This class was studied in [13].
For f_i, $g_i \in \mathcal{A}$ and $\alpha_i > 0$, $\gamma_i > 0$, $i = 1, 2, ..., n$, we define the integral operator $G_n(z)$ by

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(g_i'(t)\right)^{\gamma_i} dt. \tag{1.2}$$

Remark 1.1. *This integral operator is a generalization of the integral operator defined by Pescar in* [10]. *For $n = 1$, $g = f$, from (1.2), we obtain the integral operator defined by Pescar.*

Remark 1.2. *Note that the integral operator $G_n(z)$ generalizes the following operators introduced and studied by several authors:*
(1) If $g_i'(z) = f_i'(z)$, $i = 1, 2, ..., n$, we obtain the integral operator

$$I^{\alpha_i,\gamma_i}(f_1, ..., f_n)(z) = \int_0^z (f_1'(t))^{\gamma_1} \left(\frac{f_1(t)}{t}\right)^{\alpha_1} ...(f_n'(t))^{\gamma_n} \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$$

introduced and studied by Frasin [6] *(see also* [5, 4]*).*
(2) For $\gamma_i = 0$, $i = 1, 2, ..., n$, we obtain the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} ... \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$$

introduced and studied by D. Breaz and N. Breaz [2]*.*
(3) For $\alpha_i = 0$, $i = 1, 2, ..., n$, we obtain the integral operator

$$F_{\gamma_1,...,\gamma_n}(z) = \int_0^z (g_1'(t))^{\gamma_1} ...(g_n'(t))^{\gamma_n} dt$$

introduced and studied by Breaz et. al. [3]*.*
(4) For $n = 1$, $\gamma_1 = 0$, $\alpha_1 = \alpha$ and $f_1 = f$, we obtain the integral operator

$$F_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$$

studied in [7]*. In particular, for $\alpha = 1$, we obtain Alexander integral operator*

$$I(z) = \int_0^z \frac{f(t)}{t} dt.$$

introduced in [1]*.*
(5) For $n = 1$, $\alpha_1 = 0$, $\gamma_1 = \gamma$ and $g_1 = g$, we obtain the integral operator

$$G_\alpha(z) = \int_0^z (g'(t))^{\gamma} dt$$

studied in [9] *(see also* [11]*).*

2. Main results

Theorem 2.1. *Let α_i, γ_i be positive real numbers, $i = 1, 2, ..., n$. If $f_i \in \mathcal{M}(\beta_i)$, $\beta_i > 1$ and $g_i \in \mathcal{N}(\lambda_i)$, $\lambda_i > 1$, $i = 1, 2, ..., n$, then the integral operator $G_n(z)$ defined in (1.2) is in the class $\mathcal{N}(\mu)$, where*

$$\mu = 1 + \sum_{i=1}^{n} [\alpha_i (\beta_i - 1) + \gamma_i (\lambda_i - 1)].$$

Proof. We calculate the derivatives of the first and second order of $G_n(z)$. From (1.2), we have:

$$G_n'(z) = \prod_{i=1}^{n} \left(\left(\frac{f_i(z)}{z} \right)^{\alpha_i} \left(g_i'(z) \right)^{\gamma_i} \right)$$

and

$$G_n''(z) = \sum_{i=1}^{n} \left[\alpha_i \left(\frac{f_i(z)}{z} \right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z^2} \right) \left(g_i'(z) \right)^{\gamma_i} \right] \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(\left(\frac{f_k(z)}{z} \right)^{\alpha_k} \left(g_k'(z) \right)^{\gamma_k} \right)$$

$$+ \sum_{i=1}^{n} \left[\left(\frac{f_i(z)}{z} \right)^{\alpha_i} \gamma_i \left(g_i'(z) \right)^{\gamma_i - 1} g_i''(z) \right] \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(\left(\frac{f_k(z)}{z} \right)^{\alpha_k} \left(g_k'(z) \right)^{\gamma_k} \right).$$

By a calculation, we obtain that

$$\frac{z G_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} \left(\alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) + \gamma_i \frac{z g_i''(z)}{g_i'(z)} \right). \tag{2.1}$$

The relation (2.1) is equivalent to

$$\frac{z G_n''(z)}{G_n'(z)} + 1 = \sum_{i=1}^{n} \left(\alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) + \gamma_i \frac{z g_i''(z)}{g_i'(z)} \right) + 1. \tag{2.2}$$

We calculate the real part of both terms of (2.2) and obtain

$$\begin{aligned} \operatorname{Re} \left(\frac{z G_n''(z)}{G_n'(z)} + 1 \right) &= \sum_{i=1}^{n} \left(\alpha_i \operatorname{Re} \frac{z f_i'(z)}{f_i(z)} - \alpha_i + \gamma_i \operatorname{Re} \frac{z g_i''(z)}{g_i'(z)} \right) + 1 \\ &= \sum_{i=1}^{n} \left(\alpha_i \operatorname{Re} \frac{z f_i'(z)}{f_i(z)} - \alpha_i + \gamma_i \operatorname{Re} \left(\frac{z g_i''(z)}{g_i'(z)} + 1 \right) - \gamma_i \right) + 1. \end{aligned}$$

Since $f_i \in \mathcal{M}(\beta_i)$, $\beta_i > 1$ and $g_i \in \mathcal{N}(\lambda_i)$, $\lambda_i > 1$, $i = 1, 2, ..., n$, we obtain

$$\begin{aligned} \operatorname{Re} \left(\frac{z G_n''(z)}{G_n'(z)} + 1 \right) &< \sum_{i=1}^{n} \left(\alpha_i \beta_i - \alpha_i + \gamma_i \lambda_i - \gamma_i \right) + 1 \\ &< 1 + \sum_{i=1}^{n} [\alpha_i (\beta_i - 1) + \gamma_i (\lambda_i - 1)]. \end{aligned}$$

Hence $G_n(z) \in \mathcal{N}(\mu)$, where $\mu = 1 + \sum_{i=1}^{n} [\alpha_i (\beta_i - 1) + \gamma_i (\lambda_i - 1)]$. □

Setting $n = 1$ in Theorem 2.1, we obtain the following

Corollary 2.2. *Let α, γ be positive real numbers. If $f \in \mathcal{M}(\beta)$, $\beta > 1$ and $g \in \mathcal{N}(\lambda)$, $\lambda > 1$, then the integral operator*

$$G(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} \left(g'(t)\right)^{\gamma} dt$$

is in the class $\mathcal{N}(\mu)$, where $\mu = 1 + \alpha(\beta - 1) + \gamma(\lambda - 1)$.

Theorem 2.3. *Let α_i, γ_i be positive real numbers, $i = 1, 2, ..., n$. We suppose that the functions f_i are starlike functions by order $\frac{1}{\alpha_i}$, that is $f_i \in \mathcal{S}^*(\frac{1}{\alpha_i})$ and $g_i \in \mathcal{KD}(\mu_i, \lambda_i)$, $\mu_i \geq 0$, $0 \leq \lambda_i < 1$, $i = 1, 2, ..., n$. If*

$$\sum_{i=1}^{n} [\alpha_i + \gamma_i (1 - \lambda_i)] - n < 1,$$

then the integral operator $G_n(z)$ defined by (1.2) is in the class $\mathcal{K}(\delta)$, where

$$\delta = 1 + n + \sum_{i=1}^{n} [\gamma_i (\lambda_i - 1) - \alpha_i].$$

Proof. Following the same steps as in Theorem 2.1, we obtain

$$\begin{aligned} \frac{zG_n''(z)}{G_n'(z)} &= \sum_{i=1}^{n} \left(\alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \gamma_i \frac{zg_i''(z)}{g_i'(z)}\right) \\ &= \sum_{i=1}^{n} \left(\alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_i + \gamma_i \frac{zg_i''(z)}{g_i'(z)}\right). \end{aligned} \tag{2.3}$$

The relation (2.3) is equivalent to

$$\frac{zG_n''(z)}{G_n'(z)} + 1 = \sum_{i=1}^{n} \left(\alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_i + \gamma_i \frac{zg_i''(z)}{g_i'(z)}\right) + 1.$$

Taking the real part of the above expression, we obtain

$$\begin{aligned} \mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) &= \sum_{i=1}^{n} \left(\alpha_i \mathrm{Re}\frac{zf_i'(z)}{f_i(z)} - \alpha_i + \gamma_i \mathrm{Re}\frac{zg_i''(z)}{g_i'(z)}\right) + 1 \\ &= \sum_{i=1}^{n} \left(\alpha_i \mathrm{Re}\frac{zf_i'(z)}{f_i(z)} - \alpha_i + \gamma_i \mathrm{Re}\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) - \gamma_i\right) + 1. \end{aligned} \tag{2.4}$$

But $f_i \in \mathcal{S}^*(\frac{1}{\alpha_i})$, so $\mathrm{Re}\frac{zf_i'(z)}{f_i(z)} > \frac{1}{\alpha_i}$ and since $g_i \in \mathcal{KD}(\mu_i, \lambda_i)$, for $\mu_i \geq 0$ and $0 \leq \lambda_i < 1$, $i = 1, 2, ..., n$, from (2.4), we get

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) > 1 + \sum_{i=1}^{n}\left(\alpha_i \cdot \frac{1}{\alpha_i} - \alpha_i + \gamma_i\left(\mu_i\left|\frac{zg_i''(z)}{g_i'(z)}\right| + \lambda_i\right) - \gamma_i\right)$$
$$> 1 + n - \sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\gamma_i\mu_i\left|\frac{zg_i''(z)}{g_i'(z)}\right| + \sum_{i=1}^{n}\gamma_i\left(\lambda_i - 1\right).$$

Since $\gamma_i\mu_i\left|\frac{zg_i''(z)}{g_i'(z)}\right| > 0$, we obtain

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) > 1 + n - \sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\gamma_i\left(\lambda_i - 1\right)$$
$$> 1 + n + \sum_{i=1}^{n}[\gamma_i\left(\lambda_i - 1\right) - \alpha_i]. \tag{2.5}$$

Using the hypothesis $\sum_{i=1}^{n}[\alpha_i + \gamma_i\left(1 - \lambda_i\right)] - n < 1$ in (2.5), we obtain that the integral operator $G_n(z)$ is in the class $\mathcal{K}(\delta)$, where

$$\delta = 1 + n + \sum_{i=1}^{n}[\gamma_i\left(\lambda_i - 1\right) - \alpha_i].$$

□

Setting $n = 1$ in Theorem 2.3, we obtain the following

Corollary 2.4. *Let α, γ be positive real numbers. We suppose that the function f is a starlike function of order $\frac{1}{\alpha}$, that is $f \in \mathcal{S}^*(\frac{1}{\alpha})$ and the function $g \in \mathcal{KD}(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$. If*

$$\alpha + \gamma\left(1 - \lambda\right) < 2,$$

then the integral operator

$$G(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha}\left(g'(t)\right)^{\gamma} dt$$

is in the class $\mathcal{K}(\delta)$, where

$$\delta = 2 + \gamma\left(\lambda - 1\right) - \alpha.$$

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