ON bD-SETS AND ASSOCIATED SEPARATION AXIOMS†

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ABSTRACT. Here, first we introduce and investigate *bD-sets* by using the notion of *b*-open sets to obtain some weak separation axioms. Second, we introduce the notion of *gb-closed sets* and then investigate some relations of between *b*-closed and gb-closed sets. We also give a characterization of *b*-$T_{1/2}$ spaces via *gb*-closed sets. We introduce two new weak homeomorphisms which are important keys between General Topology and Algebra. Using the notion of m_X-*structures*, we give a characterization theorem of m_X-T_2 spaces. Finally, we give some examples related to the digital line.

1. Introduction

It is known that open sets play a very important role in General Topology. In [38], Tong introduced the notion of D-sets by using open sets and used the notion to define some separation axioms. Later, the modifications of these notions for α-open (resp. semi-open, preopen, δ-semi-open) sets are introduced and some of their properties investigated in [6] (resp.[4], [20], [7]) and [20], respectively. The notion of b-open sets were introduced by Andrijević [3]. The set was named as γ-open and sp-open by El-Atik [17] and Doncthev and Przemski [13], respectively.

The notion of b-open sets is stronger than the notion of β-open sets and is weaker than the notions of semi-open sets and preopen sets. Since then, these notions has been extensively investigated in the literature (see [32], [33], [36], [14], [11], [37], [8], [9], [15]).

Here, first we introduce the notion of bD-sets as the difference sets of b-open sets. Second, we introduce the notion of gb-closed sets and investigate some relations between b-closed and gb-closed sets. We also give a characterization of b-$T_{1/2}$ spaces via gb-closed sets. Then, we investigate some preservation theorems. We must state that we introduce new two weak homeomorphisms. It is well-known that the notion of homeomorphisms is an important key between General Topology and Algebra. By using the notion of m_X-structures, we give a characterization theorem of m_X-T_2 spaces. Finally, we give some examples. Some applications of our results may relate to the digital line.

2. Preliminaries

Through out the paper, by (X, τ) and (Y, φ) (or X and Y) we always mean topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subset X$. Then A is called b-open [3] if $A \subset Cl(Int(A)) \cup Int(Cl(A))$, where $Cl(A)$ and $Int(A)$ denote the closure and interior of A in (X, τ), respectively. The complement A^c of a b-open set A is called b-closed [3] and the b-closure of a set A, denoted by $bCl(A)$, is defined by the intersection of all b-closed sets containing A. The b-interior of a set A, denoted by $bInt(A)$, is the union of all b-open sets contained in A. The symbols $bCl(A)$ and $bInt(A)$ were first used by Andrijević [3]. The family of all b-open (resp. b-closed) sets in (X, τ) will be denoted by $BO(X, \tau)$ (resp. $BC(X, \tau)$) as in [3]. The family of all b-open sets containing x of X will be denoted by $BO(X, x)$ as in [37]. It was shown that [3, Proposition 2.3(a)] the union of any family of b-open sets is a b-open set.

We recall some definitions used in the sequel.

Definition 2.1. A subset A of a space (X, τ) is said to be
  (a) α-open [34] if $A \subset Int(Cl(Int(A)))$,
  (b) semi-open [27] if $A \subset Cl(Int(A))$,
  (c) preopen [30] if $A \subset Int(Cl(A))$,
  (d) β-open [1] if $A \subset Cl(Int(Cl(A)))$.

Through out the paper, the family of all α-open (resp. semi-open, preopen) sets in a topological space (X, τ) is denoted by $\alpha(X)$ (resp. $SO(X,\tau)$, $PO(X,\tau)$).

Definition 2.2. A subset S of a topological space X is called
   (a) a D-set [38] if there are $U, V \in \tau$ such that $U \neq X$ and $S = U\backslash V$,
   (b) an αD-set [6] if there are $U, V \in \alpha(X)$ such that $U \neq X$ and $S = U\backslash V$,
   (c) a sD-set [4] if there are $U, V \in SO(X,\tau)$ such that $U \neq X$ and $S = U\backslash V$,
   (d) a pD-set [21] if there are $U, V \in PO(X,\tau)$ such that $U \neq X$ and $S = U\backslash V$.

Observe that every open (resp. α-open, semi-open, preopen) set U different from X is a D-set (resp. an αD-set, a sD-set, a pD-set) if $S = U$ and $V = \varnothing$. Furthermore, since every open set is α-open, then every α-open set is semi-open and preopen. We have the following properties.

Proposition 2.3. *(a) Every D-set is an αD-set,*
   (b) every αD-set is an sD-set, and
   (c) every αD-set is a pD-set.

In [6], Caldas et al. showed that the converses of (b) and (c) need not be true, in general. One can see related examples [4, Example 3.1] and [4 , Example 3.2]. Since the notions of semi-open sets and preopen sets are independent, then one can easily obtain that the notions of sD-sets and pD-sets are independent of each other.

3. bD-sets and associated separation axioms

Definition 3.1. A subset S of a topological space X is called a bD-set if there are $U, V \in BO(X,\tau)$ such that $U \neq X$ and $S = U \setminus V$.

It is true that every b-open set U different from X is a bD-set if $S = U$ and $V = \varnothing$. So, we can observe the following.

Remark 3.2. Every proper b-open set is a *bD-set*. But, the converse is false as the next example shows.

Example 3.3. Let X={a,b,c,d} and τ={X,$\varnothing$,{a},{a,d},{a,b,d},{a,c,d}}. Then, {b} is a *bD-set* but it is not a b-open. In really, since BO(X,τ)={X,$\varnothing$,{a},{a,b},{a,c},{a,d},{a,b,c},{a,b,d},{a,c,d}}, then U={a,b}$\neq$ X and V={a,c} are b-open sets in X. For U and V, since $S = U\backslash V$={a,b}$\backslash${a,c}={b}, then we have S={b} is a bD-set but it is not b-open.

We have diagram I below.

$$
\begin{array}{ccccc}
& & \text{preopen} & \longrightarrow & b\text{-open} \\
& \nearrow & | & \nearrow & | \\
\text{open} \longrightarrow & \alpha\text{-open} & | \longrightarrow \text{semi-open} & | \\
| & | & \downarrow & | & \downarrow \\
| & | & pD\text{-set} \longrightarrow| & bD\text{-set} \\
\downarrow & \downarrow & \nearrow & \downarrow \nearrow \\
D\text{-set} \longrightarrow & \alpha D\text{-set} \longrightarrow & sD\text{-set}
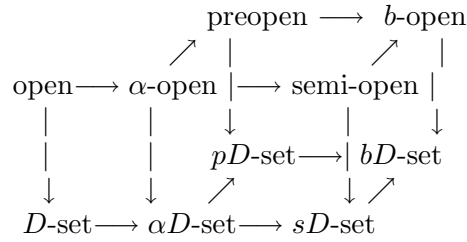\end{array}
$$

Diagram I

Definition 3.4. A topological space (X, τ) is called b-D_0 [10] (resp. b-D_1 [10]) if for any pair of distinct points x and y of X there exists a bD-set of X containing x but not y or (resp. and) a bD-set of X containing y but not x.

Definition 3.5. A topological space (X, τ) is called b-D_2 [10] if for any pair of distinct points x and y of X there exist disjoint bD-sets G and E of X containing x and y, respectively.

Definition 3.6. A space X is called b-T_0 [10] if for every pair of distinct points x and y of X, there exists a b-open set of X containing x but not y or a b-open set of X containing y but not x.

We recall that a topological space (X, τ) is called b-T_1 ([2], [8], [10]) if for each pair of distinct points x and y of X, there exist b-open sets U and V containinig x and y, respectively, such that $y \notin U$ and $x \notin V$. Additionally, in [37], Park introduced the notion of b-T_2 spaces as follows: A topological space (X, τ) is called b-T_2 if for any pair of distinct points x and y in X, there exist $U \in BO(X, x)$ and $V \in BO(X, y)$ such that $U \cap V = \varnothing$.

The following remark and theorem are due to [10].

Remark 3.7. (i) For a topological space (X, τ), the following properties hold:

(i-1) (Caldas and Jafari [10]) (a) If (X, τ) is b-T_i, then it is b-D_i, $i=0,1,2$.

    (b) If (X, τ) is b-D_i, then it is b-D_{i-1}, $i=1,2$.

    (c) If (X, τ) is b-T_i, then it is b-T_{i-1}, $i=1,2$.

(i-2) (Caldas and Jafari [10]) (a) If (X, τ) is b-D_0 if and only if it is b-T_0.

    (b) If (X, τ) is b-D_1 if and only if it is b-D_2.

(ii) In [10], the authors proved that every topological space is b-T_0.

(iii) Using Remark 3.7 (i-1)(a) or (i-2)(a) above, every topological space is b-D_0. The Sierpinski space is not b-D_1.

Definition 3.8. A subset A of a topological space (X, τ) is called a generalized b-closed (briefly gb-closed) set if $bCl(A) \subset U$, whenever $A \subset U$ and U is b-*open* in (X, τ).

The following notion is due to [2].

A topological space (X, τ) is called a B-$T_{1/2}$ space if each singleton is either b-open or b-closed. The authors proved that "every topological space is B-$T_{1/2}$" [2]. Here, we define the concept of "b-$T_{1/2}$-spaces".

Definition 3.9. A topological space (X, τ) is called b-$T_{1/2}$ if every gb-closed set is b-closed.

It is obvious that every b-closed is gb-closed (Definition 3.7). Recall that a topological space (X, τ) is called:

a) b-*symmetric* [15] if for each x and y in X, $x \in bCl(\{y\})$ implies $y \in bCl(\{x\})$;

b) b-R_0 [15] if its every b-open set contains the b-closure of each singleton.

Theorem 3.10. *For a topological space (X, τ), the following properties hold:*

(i) *(Abd El-Monsef, El-Atik and Sharkasy [2]) Let x be a point of (X, τ). Then, $\{x\}$ is b-open or b-closed.*

(ii) *A space (X, τ) is b-$T_{1/2}$ if and only if each singleton is b-open or b-closed in (X, τ).*

(iii) *Every topological space is a b-$T_{1/2}$-space, i.e., a subset A is gb-closed in (X, τ) if and only if A is b-closed.*

(iv) *For a space (X, τ), the following properties are equivalent:*
   (1) (X, τ) is b-symmetric; (2) (X, τ) is b-T_1 ;(3) (X, τ) is b-R_0 .

(v) *For each pair of distinct points x, y of X, $bCl(\{x\}) \neq bCl(\{y\})$.*

Proof. (i) This is obtained in [2; the proof of Lemma 2.3], but here we will give an alternative proof. By [22; Lemma 2], for every point x of any topological space (X, τ), $\{x\}$ is preopen or nowhere dense (i.e., $Int(Cl(\{x\})) = \varnothing$) and so $\{x\}$ is preopen or semi-closed. Therefore, $\{x\}$ is b-open or b-closed.

(ii) *Necessity* : Let $x \in X$. When $\{x\} \notin BC(X, \tau)$, $X \backslash \{x\} \notin BO(X, \tau)$, then for any b-open set U satisfying a property $X \backslash \{x\} \subset U$, we have $U = X$ only and so $bCl(X \backslash \{x\}) \subset U$. This shows that $X \backslash \{x\}$ is gb-closed and, by assumption, the singleton $\{x\}$ is b-open.

   Sufficiency : Let A be a gb-closed set of (X, τ). In order to prove $bCl(A) = A$, let $x \in bCl(A)$. When $\{x\}$ is b-open, $\{x\} \cap A \neq \varnothing$ and so $x \in A$. When $\{x\}$ is b-closed, $X \backslash \{x\} \in BO(X, \tau)$. For this case, suppose that $x \notin A$. Since $A \subset X \backslash \{x\}$ and A is a gb-closed, we have that $x \in bCl(A) \subset X \backslash \{x\}$ and hence $x \in X \backslash \{x\}$. This contradiction shows that $x \in A$ for a point satisfying $x \in bCl(A)$ and $A \in BC(X, \tau)$. Therefore, every gb-closed set is b-closed in (X, τ).

(iii) It follows from (i) and (ii) that every topological space is b-$T_{1/2}$.

(iv) (1) $\implies$ (2). Let $x \in X$. We claim that $bCl(\{x\}) \subset \{x\}$. Let $y \in bCl(\{x\})$. Then, by (i), $x \in bCl(\{y\})$ holds. If $\{x\}$ is b-open, then $\{x\} \cap \{y\} \neq \varnothing$ and so $y \in \{x\}$. If $\{x\}$ is b-closed, $y \in bCl(\{x\}) = \{x\}$ and so $y \in \{x\}$. By using (i), the claim is proved. Therefore, (X, τ) is b-T_1 .
(2) $\implies$ (3). Let $G \in BO(X, \tau)$. For a point $x \in G$, $bCl(\{x\}) = \{x\} \subset G$. Thus, (X, τ) is b-R_0 .
(3) $\implies$ (1). It is similar to [15].

(v) Suppose that there exists a pair of distinct points x and y such that $bCl(\{x\}) = bCl(\{y\})$. Then, by using (i), $\{x\}$ is b-open or b-closed.

   If $\{x\}$ is b-open, then $\{x\} \cap \{y\} \neq \varnothing$, because $x \in bCl(\{y\})$. Thus, we have $x = y$.

If $\{x\}$ is *b*-closed, $\{x\} = bCl(\{x\}) = bCl(\{y\})$ and so $\{x\} = \{y\}$. For both cases, we have contradiction. $\qquad\square$

For a subset A of a topological space (X, τ) and a family m_X of subsets of (X, τ) satisfying properties $\varnothing, X \in m_X$, the following subset $\Lambda_m(A)$ is defined in [11]: $\Lambda_m(A) = \cap\{U \mid A \subset U, U \in m_X\}$. Such a family m_X is called an m_X-*structure* on X [35]. For $m_X = \tau$ (resp. $SO(X, \tau), PO(X, \tau), BO(X, \tau)$), the set $\Lambda_m(A)$ is denoted by $\Lambda(A)$ [28] (resp. $\Lambda_s(A)$ [5], $\Lambda_p(A)$ [19], $\Lambda_b(A)$ [14]).

Corollary 3.11. *Let A be a subset of a topological space* (X, τ).

(i) $\Lambda_b(A) \subset \Lambda_s(A) \cap \Lambda_p(A)$ *and* $\Lambda_s(A) \cup \Lambda_p(A) \subset \Lambda(A)$ *hold.*

(ii) *(a) Assume that* $BO(X, \tau)$ *is a topology of X. If* $\Lambda_b(\{x\}) \neq X$ *for a point* $x \in X$, *then* $\{x\}$ *is a bD-set of* (X, τ).
　　　(b) If a singleton $\{x\}$ *is a bD-set of* (X, τ), *then* $\Lambda_b(\{x\}) \neq X$.

(iii) *If* $\Lambda(\{x\}) \neq X$ *for a point* $x \in X$, *then* $\{x\}$ *is a bD-set of* (X, τ).

(iv) *For a topological space* (X, τ) *with at least two points,* (X, τ) *is a b-D_1-space if and only if* $\Lambda_b(\{x\}) \neq X$ *holds for every point* $x \in X$.

(v) *Let X be a set with at least two points. If there exists a point* $x \in X$ *such that* $\Lambda_b(\{x\}) = X$, *then* (X, τ) *is not b-D_1 (thus, it is not b-D_2).*

Proof. (i) According to [3], since $\tau \subset SO(X, \tau) \cap PO(X, \tau)$ and $SO(X, \tau) \cup PO(X, \tau) \subset BO(X, \tau)$, then we have $\Lambda_b(A) \subset \Lambda_s(A)$, $\Lambda_b(A) \subset \Lambda_p(A)$, $\Lambda_s(A) \subset \Lambda(A)$ and $\Lambda_p(A) \subset \Lambda(A)$. This shows that we have the required implications.

(ii) (a) Since $\Lambda_b(\{x\}) \neq X$ for a point $x \in X$, then there exists a subset $U \in BO(X, \tau)$ such that $\{x\} \subset U$ and $U \neq X$. Using Theorem 3.10(i) for the point x, then $\{x\}$ is *b*-open or *b*-closed in (X, τ). When the singleton $\{x\}$ is *b*-open, $\{x\}$ is a *bD*-set of (X, τ). When the singleton $\{x\}$ is *b*-closed, then $(\{x\})^c$ is *b*-open in (X, τ). Put $U_1 = U$ and $U_2 = U \cap (\{x\})^c$. Then, $\{x\} = U_1 \backslash U_2$, $U_1 \in BO(X, \tau)$ and $U_1 \neq X$. It follows from the hypothesis that $U_2 \in BO(X, \tau)$ and so $\{x\}$ is a *bD*-set.
(b) Since $\{x\}$ is a *bD*-set of (X, τ), then there exist two subsets $U_1 \in BO(X, \tau)$ and $U_2 \in BO(X, \tau)$ such that $\{x\} = U_1 \backslash U_2$, $\{x\} \subset U_1$ and $U_1 \neq X$. Thus, we have that $\Lambda_b(\{x\}) \subset U_1 \neq X$ and so $\Lambda_b(\{x\}) \neq X$.

(iii) Since $\Lambda(\{x\}) \neq X$, then there exists a subset $U \in \tau$ such that $\{x\} \subset U$ and $U \neq X$. Using Theorem 3.10(i) for the point x, $\{x\}$ is b-open or b-closed in (X, τ). When the singleton $\{x\}$ is b-open, then $\{x\}$ is a bD-set of (X, τ). When the singleton $\{x\}$ is b-closed, then $(\{x\})^c$ is b-open in (X, τ). Put $U_1 = U$ and $U_2 = U \cap (\{x\})^c$. By [3, Proposition 2.3(b)], the set U_2 is b-open. Therefore, $\{x\} = U_1 \backslash U_2$ and $\{x\}$ is a bD-set, because $U_1 \in BO(X, \tau)$ and $U_1 \neq X$.

(iv) Necessity: Let $x \in X$. For a point $y \neq x$, there exists a bD-set U such that $x \in U$ and $y \notin U$. Say $U = U_1 \backslash U_2$, where $U_i \in BO(X, \tau)$ for each $i \in \{1, 2\}$ and $U_1 \neq X$. Thus, for the point x, we have a b-open set U_1 such that $\{x\} \subset U_1$ and $U_1 \neq X$. Hence, $\Lambda_b(\{x\}) \neq X$.
Sufficiency: Let x and y be a pair of distinct points of X. We prove that there exist bD-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Theorem 3.10(i), we can take the subsets A and B for the following four cases for two points x and y.

Case1. $\{x\}$ is b-open and $\{y\}$ is b-closed in (X, τ). Since $\Lambda_b(\{y\}) \neq X$, then there exists a b-open set V such that $y \in V$ and $V \neq X$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = V$, then $\{y\}^c$, V is a b-open set with $V \neq X$ and $\{y\}^c$ is b-open, and B is a required bD-set containing y such that $x \notin B$. Obviously, A is a required bD-set containing x such that $y \notin A$.

Case 2. $\{x\}$ is b-closed and $\{y\}$ is b-open in (X, τ). The proof is similar to Case 1.

Case 3. $\{x\}$ and $\{y\}$ are b-open in (X, τ). Put $A = \{x\}$ and $B = \{y\}$.

Case 4. $\{x\}$ and $\{y\}$ are b-closed in (X, τ). Put $A = \{y\}^c$ and $B = \{x\}^c$.

For each case above, the subsets A and B are the required bD-sets. Therefore, (X, τ) is a b-D_1-space.

(v) By (iv) and Remark 3.7 (i-2)(b), (v) is obtained.   □

Remark 3.12. (i) The converse of Corollary 3.11(iii) is not true, in general. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then, $\Lambda_b(\{c\}) = \{c\} \neq X$ and the singleton $\{c\} = \{b, c\} \backslash \{b\}$ is a bD-set; $\Lambda(\{c\}) = X$ holds.

(ii) It follows from Corollary 3.11 (i) that for a point $x \in X$, $\Lambda_b(\{x\}) \neq X$ if $\Lambda(\{x\}) \neq X$; $\Lambda(\{x\}) = X$ if $\Lambda_b(\{x\}) = X$.

4. **Preservation theorems**

Here, first we recall some definitions. Then, we will give several preservation theorems.

Definition 4.1. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be

(a) α-continuous [31] if $f^{-1}(V)$ is α-open in (X, τ), for every open set V of (Y, φ),

(b) α-open [31] if $f(U)$ is α-open in (Y, φ), for every open set U of (X, τ),

(c) γ-irresolute [14] if $f^{-1}(V)$ is γ-open in X, for every γ-open set V of Y,

(d) γ-continuous [17] if $f^{-1}(V)$ is γ-open in (X, τ), for every open set V of (Y, φ).

We note that since the notion of b-open sets and the notion of γ-open sets are the same, then here we will use the term of b-irresolute (resp. b-continuous) functions instead of γ-irresolute (resp. γ-continuous) functions. In [10; Definition 6] the authors used the term of b-continuous functions instead of γ-irresolute functions.

Theorem 4.2. *If $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a b-continuous (resp. b-irresolute) surjective function and S is a D-set (resp. bD-set) of (Y, φ), then $f^{-1}(S)$ is a bD-set of (X, τ).*

Proof. Let $S = O_1 \backslash O_2$ be a D-set (resp. bD-set) of (Y, φ), where $O_i \in \varphi$ (resp. $O_i \in BO(Y, \varphi)$), for each $i \in \{1, 2\}$ and $O_1 \neq Y$. We have that $f^{-1}(O_i) \in BO(X, \tau)$, for each $i \in \{1, 2\}$ and $f^{-1}(O_1) \neq X$. Hence, $f^{-1}(S) = f^{-1}(O_1) \cap (X \backslash f^{-1}(O_2))$. Therefore, $f^{-1}(S)$ is a bD-set. $\square$

Theorem 4.3. *If (Y, φ) is a D_1 space (resp. b-D_1 space) and $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a b-continuous (resp. b-irresolute) bijective function, then (X, τ) is a b-D_1 space.*

Proof. Suppose that Y is a D_1 space (resp. b-D_1 space). Let x and y be any pair of distinct points in X. Since f is injective and Y is D_1 (resp. b-D_1), then there exist D-sets (resp. bD-sets) S_x and S_y of Y containing

$f(x)$ and $f(y)$, respectively, such that $f(x) \notin S_y$ and $f(y) \notin S_x$. By Theorem 4.2, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are bD-sets in X containing x and y, respectively, such that $x \notin f^{-1}(S_y)$ and $y \notin f^{-1}(S_x)$. This implies that X is a b-D_1 space. $\square$

Theorem 4.4. *A topological space (X, τ) is b-D_1 if for each pair of distinct points x, $y \in X$, there exists a b-continuous (resp. b-irresolute) surjective function $f : (X, \tau) \longrightarrow (Y, \varphi)$, where (Y, φ) is a D_1 space (resp. b-D_1 space) such that $f(x)$ and $f(y)$ are distinct.*

Proof. Let x and y be any pair of distinct points in X. By hypothesis, there exists a b-continuous (resp. b-irresolute) surjective function f of a space (X, τ) onto a D_1 space (resp. b-D_1 space) (Y, φ) such that $f(x) \neq f(y)$. It follows from Theorem 4.2 of [37] (resp. Remark 3.7(i-2)(b)) that $D_1 = D_2$ (resp. b-$D_1 = b$-D_2). Hence, there exist disjoint D-sets (resp. bD-sets) S_x and S_y in Y such that $f(x) \in S_x$ and $f(y) \in S_y$. Since f is b-continuous (resp. b-irresolute) and surjective, by Theorem 4.2, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are disjoint bD-sets in X containing x and y, respectively. So, the space (X, τ) is b-D_1 . $\square$

The following notion is due to Hatir and Noiri [20].

A filterbase **B** is called D-convergent to a point $x \in X$ if for any D-set A containing x, there exists $B_1 \in \mathbf{B}$ such that $B_1 \subset A$.

Definition 4.5. Let (X, τ) be a topological space. A filter base **B** is called bD-convergent to a point $x \in X$, if for any bD-set A containing x, there exists $B_1 \in \mathbf{B}$ such that $B_1 \subset A$.

Theorem 4.6. *If a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is b-continuous (resp. b-irresolute) and surjective, then for each point $x \in X$ and each filterbase **B** on (X, τ), bD-converging to x, the filterbase $f(\mathbf{B})$ is D-convergent (resp. bD-convergent) to $f(x)$.*

Proof. Let $x \in X$ and **B** be any filterbase bD-converging to x. Since f is a b-continuous (resp. b-irresolute) surjection, by Theorem 4.2, for each D-set (resp. bD-set) $V \subset Y$ containing $f(x)$, $f^{-1}(V) \subset X$ is a bD-set containing x. Since **B** is bD-converging to x, then there exists $B_1 \in \mathbf{B}$ such that $B_1 \subset f^{-1}(V)$ and hence $f(B_1) \subset V$. It follows that $f(\mathbf{B})$ is D-convergent (resp. bD-convergent) to $f(x)$. $\square$

Recall that a topological space (X, τ) is said to be *D-compact* [20] if every cover of X by D-sets has a finite subcover.

Definition 4.7. A topological space (X, τ) is said to be *bD-compact* if every cover of X by *bD-sets* has a finite subcover.

Theorem 4.8. *Let a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ be b-continuous (resp.b-irresolute) and surjective. If (X, τ) is bD-compact, then (Y, φ) is D-compact (resp. bD-compact).*

Proof. It is proved by using Theorem 4.2. □

Recall that a topological space (X, τ) is said to be *D-connected* [20] if (X, τ) cannot be expressed as the union of two disjoint nonempty D-sets.

Definition 4.9. A topological space (X, τ) is said to be *bD-connected* if (X, τ) cannot be expressed as the union of two disjoint nonempty *bD-sets*.

Theorem 4.10. *If $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a b-continuous (resp. b-irresolute) surjection and (X, τ) is bD-connected, then (Y, φ) is D-connected (resp.bD-connected).*

Proof. It is proved by using Theorem 4.2. □

Remark 4.11. Theorems 4.2, 4.3, 4.4, 4.6, 4.8 and 4.10 are true for an α-continuous and α-open function f instead of a *b-irresolute* function f. For an α-continuous and α-open function f, the inverse image $f^{-1}(S)$ of each b-open set S is b-open (see El-Atik [17]).

It is well known that the notion of homeomorphisms is very important in General Topology. The following definition provides two new weak forms of homeomorphisms.

Definition 4.12. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called a *br*-homeomorphism (resp. *b*-homeomorphism) if f is a *b*-irresolute bijection (resp. *b*-continuous bijection) and $f^{-1} : (Y, \varphi) \longrightarrow (X, \tau)$ is a *b*-irresolute (resp. *b*-continuous).

Now, we can give the following definition by taking the space (X, τ), instead of the space (Y, φ).

Definition 4.13. For a topological space (X, τ), we define the following two collections of functions:

$br\text{-}h(X, \tau) = \{f \mid f : (X, \tau) \longrightarrow (X, \tau)$ is a *b-irresolute bijection*, $f^{-1} : (X, \tau) \longrightarrow (X, \tau)$ is *b-irresolute*$\}$;

$b\text{-}h(X, \tau) = \{f \mid f : (X, \tau) \longrightarrow (X, \tau)$ is a *b-continuous bijection*, $f^{-1} : (X, \tau) \longrightarrow (X, \tau)$ is *b-continuous*$\}$.

Theorem 4.14. *For a topological space (X, τ), the following properties hold:*

(i) $h(X, \tau) \subset br\text{-}h(X, \tau) \subset b\text{-}h(X, \tau)$, where $h(X, \tau) = \{f \mid f : (X, \tau) \longrightarrow (X, \tau)$ is a homeomorphism $\}$.

(ii) The collection $br\text{-}h(X, \tau)$ forms a group under the composition of functions.

(iii) The group $h(X, \tau)$ of all homeomorphisms on (X, τ) is a subgroup of $br\text{-}h(X, \tau)$.

Proof. *(i)* First we show that every homeomorphism $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a br-homeomorphism. Indeed, for a subset $A \in BO(Y, \varphi)$, $f^{-1}(A) \subset f^{-1}(Cl(Int(A)) \cup Int(Cl(A))) = Cl(Int(f^{-1}(A))) \cup Int(Cl(f^{-1}(A)))$ and so $f^{-1}(A) \in BO(X, \tau)$. Thus, f is b-irresolute. In a similar way, it is shown that f^{-1} is b-irresolute. Hence, we have that $h(X, \tau) \subset br\text{-}h(X, \tau)$.

Finally, it is obvious that $br\text{-}h(X, \tau) \subset b\text{-}h(X, \tau)$, because every b-irresolute function is b-continuous.

(ii) If $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \eta)$ are br-homeomorphisms, then their composition $g \circ f : (X, \tau) \longrightarrow (Z, \eta)$ is a br-homeomorphism. It is obvious that for a bijective br-homeomorphism $f : (X, \tau) \longrightarrow (Y, \varphi)$, $f^{-1} : (Y, \varphi) \longrightarrow (X, \tau)$ is also a br-homeomorphism and the identity $1 : (X, \tau) \longrightarrow (X, \tau)$ is a br-homeomorphism. A binary operation $\alpha : br\text{-}h(X, \tau) \times br\text{-}h(X, \tau) \longrightarrow br\text{-}h(X, \tau)$ is well defined by $\alpha(a, b) = b \circ a$, where $a, b \in br\text{-}h(X, \tau)$ and $b \circ a$ is the composition of a and b. By using the above properties, the set $br\text{-}h(X, \tau)$ forms a group under composition of functions.

(iii) For any $a, b \in h(X, \tau)$, we have $\alpha(a, b^{-1}) = b^{-1} \circ a \in h(X, \tau)$ and $1_X \in h(X, \tau) \neq \varnothing$. Thus, using *(i)* and *(ii)*, it is obvious that the group $h(X, \tau)$ is a subgroup of $br\text{-}h(X, \tau)$. $\square$

For a topological space (X, τ), we can construct a new group $br\text{-}h(X, \tau)$ satisfying the property: if there exists a homeomorhism $(X, \tau) \cong (Y, \varphi)$, then there exists a group isomorphism $br\text{-}h(X, \tau) \cong br\text{-}h(Y, \varphi)$.

Corollary 4.15. *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \eta)$ be two functions between topological spaces.*

(i) For a br-homeomorphism $f : (X, \tau) \longrightarrow (Y, \varphi)$, there exists an isomorphism, say $f_* : br\text{-}h(X, \tau) \longrightarrow br\text{-}h(Y, \varphi)$, defined by $f_*(a) = f \circ a \circ f^{-1}$, for any element $a \in br\text{-}h(X, \tau)$.

(ii) For two br-homeomorphisms $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \eta)$, $(g \circ f)_* = g_* \circ f_* : br\text{-}h(X, \tau) \longrightarrow br\text{-}h(Z, \eta)$ holds.

(iii) For the identity function $1_X : (X, \tau) \longrightarrow (X, \tau)$, $(1_X)_* = 1 : br\text{-}h(X, \tau) \longrightarrow br\text{-}h(X, \tau)$ holds, where 1 denotes the identity isomorphism.

Proof. Straightforward. $\square$

Remark 4.16. (i) The following example shows that $h(X, \tau)$ is a proper subgroup of $br\text{-}h(X, \tau)$. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$. We note that $BO(X, \tau) = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$. It is shown that $h(X, \tau) = \{1_X\}$ and $br\text{-}h(X, \tau) = \{1_X, h_a\}$, where 1_X is the identity on (X, τ) and $h_a : (X, \tau) \longrightarrow (X, \tau)$ is a bijection defined by $h_a(a) = a$, $h_a(b) = c$ and $h_a(c) = b$.

(ii) The following example shows that $br\text{-}h(X, \tau)$ is a proper subset of $b\text{-}h(X, \tau)$. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a, b\}, X\}$. Then, $BO(X, \tau) = P(X) \backslash \{\{c\}\}$. There exists an element $h_b \in b\text{-}h(X, \tau)$ such that $h_b \notin br\text{-}h(X, \tau)$, where $h_b : (X, \tau) \longrightarrow (X, \tau)$ is a bijection defined by $h_b(b) = b$, $h_b(a) = c$ and $h_b(c) = a$.

(iii) The converse of Corollary 4.15(i) is not always true. Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a, b\}, X\}$ and $\varphi = \{\varnothing, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a bijection between topological spaces defined by

$f(a) = b$, $f(b) = c$ and $f(c) = a$. Then, it is shown that $f_* : br\text{-}h(X,\tau) \longrightarrow br\text{-}h(Y,\varphi)$ is an isomorphism and the function f is not a $br\text{-}homeomorphism$. Indeed, $BO(X,\tau) = P(X)\backslash\{\{c\}\}$, $BO(Y,\varphi) = \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, Y\}$, $br\text{-}h(X,\tau) = \{1_X, h_b\}$, $br\text{-}h(Y,\varphi) = \{1_Y, h_a\}$, where h_b (resp. h_a) is defined in (ii) (resp. (i)). Moreover, $f_*(h_b) = h_a$ holds and for a set $\{a\} \in BO(Y,\varphi)$, $f^{-1}(\{a\}) = \{c\} \notin BO(X,\tau)$ and so f is not a $br\text{-}homeomorphism$.

5. Some properties of $b\text{-}T_2$spaces

Since the notion of b-open sets and the notion of γ-open sets are the same, then in this paper we will use the term of b-open functions instead of γ-open functions. Recall that a function is called γ-open [14] if the image of every γ-open set is γ-open.

In the following theorems, for a non-empty topological space (Y,φ), we consider a family m_Y of subsets of (Y,φ) such that $m_Y \in \{SO(Y,\varphi), PO(Y,\varphi), BO(Y,\varphi)\}$. Namely, the family m_Y is only one element of $\{SO(Y,\varphi), PO(Y,\varphi), BO(Y,\varphi)\}$. We recall that (Y, m_Y) is called $m_Y\text{-}T_2$ [36] if for each pair of distinct points $x, y \in Y$, there exist $U, V \in m_Y$ containing x and y, respectively, such that $U \cap V = \varnothing$. A topological space (Y,φ) is called semi-T_2 [29] (resp. pre-T_2 [23], b-T_2 [37]) if (Y, m_Y) is m_Y-T_2, where $m_Y = SO(Y,\varphi)$ (resp. $PO(Y,\varphi)$, $BO(Y,\varphi)$). A function $f : (X, m_X) \longrightarrow (Y, m_Y)$ is called M-open [11], if for each set $A \in m_X$, $f(A) \in m_Y$. For topological spaces (X,τ), (Y,φ) with m_X-structure and m_Y-structure, respectively, here we call, a function $f : (X,\tau) \longrightarrow (Y,\varphi)$ to be (m_X, m_Y)-open if $f : (X, m_X) \longrightarrow (Y, m_Y)$ is M-open in the sense of [11] given above.

Theorem 5.1. *Let R be an equivalance relation, $R \subset X \times X$, in a topological space (X,τ) and $(X/R, \Psi)$ an identification space. Let $(m_X, m_{X/R}) = (SO(X,\tau), SO(X/R, \Psi))$ (resp. $(PO(X,\tau), PO(X/R, \Psi))$, $(BO(X,\tau), BO(X/R, \Psi)))$.*
Assume that:

(a) the identification function $\rho : (X,\tau) \longrightarrow (X/R, \Psi)$ is $(m_X, m_{X/R})$-open, and

(b) for each point $(x,y) \in (X \times X)\backslash R$, there exist subsets $U_x, U_y \in m_X$ such that $x \in U_x$, $y \in U_y$ and $U_x \times U_y \subset (X \times X)\backslash R$.

Then, $(X/R, m_{X/R})$ is $m_{X/R}$-T_2 .

Proof. Let $\rho(x)$ and $\rho(y)$ be distinct members of X/R. Since x and y are not equivalent, then $(x,y) \in (X \times X) \backslash R$. By assumption, there exists two subsets $U_x \in m_X$, $U_y \in m_X$ such that $x \in U_x$, $y \in U_y$ and $U_x \times U_y \subset (X \times X) \backslash R$. Then, we have that $U_x \cap U_y = \varnothing$, because $\{(z,z) \in X \times X \mid z = z\} \subset R$. By the further assumption, $\rho(U_x)$ and $\rho(U_y)$ are the required subsets containing $\rho(x)$ and $\rho(y)$, respectively, i.e., $\rho(U_x), \rho(U_y) \in m_{X/R}$ and $\rho(U_x) \cap \rho(U_y) = \varnothing$. $\qquad\square$

Theorem 5.2. *For a topological space (X, τ) and each family $m_X \in \{SO(X,\tau), PO(X,\tau), BO(X,\tau)\}$, the following properties are equivalent:*

(1) (X, m_X) is m_X-T_2 .

(2) For distinct points x and $y \in X$, there exists a subset $U \in m_X$ such that $x \in U$, $y \notin m_X$ -$Cl(U)$, where m_X -$Cl(U)$ is defined by $\cap \{F \mid U \subset F, X \backslash F \in m_X \}$.

(3) For each $x \in X$, $\cap \{m_X$ -$Cl(U) \mid U \in m_X$, $x \in U\} = \{x\}$.

(4) For each pair $(x,y) \in (X \times X) \backslash \Delta$, there exist two subsets $U_x, V_y \in m_X$ such that $x \in U_x$, $y \in V_y$ and $U_x \times V_y \subset (X \times X) \backslash \Delta$, where $\Delta = \{(x,x) \mid x \in X\}$.

Proof. (1) $\Longrightarrow$ (2). Let $x, y \in X$ with $x \neq y$. Then, there exist two subsets $U, V \in m_X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. It is obvious that $y \notin V^c$, m_X -$Cl(U) \subset m_X$ -$Cl(V^c) = V^c$ and therefore $y \notin m_X$ -$Cl(U)$.

(2) $\Longrightarrow$ (3). Assume that $y \notin \{x\}$. There exists a subset $U \in m_X$ such that $x \in U$ and $y \notin m_X$ -$Cl(U)$. So, we have that $y \notin \cap \{m_X$ -$Cl(U) \mid U \in m_X$, $x \in U\}$.

(3) $\Longrightarrow$ (4). Let $(x,y) \in (X \times X) \backslash \Delta$. Since $y \notin \cap \{m_X$ -$Cl(U) \mid U \in m_X$, $x \in U\}$, then there exists a subset $U \in m_X$ such that $x \in U$, $y \in (m_X$ -$Cl(U))^c$ and $(m_X$ -$Cl(U))^c \in m_X$ ([36, Lemma 3.2, Lemma 3.1(6), Remark 3.1(2)]). Set $U_x = U$ and $U_y = (m_X$ -$Cl(U))^c$. Then, it is shown that $x \in U_x$, $y \in U_y$ and $U_x, U_y \in m_X$. Besides, we have that $(U_x \times U_y) \cap \Delta = \varnothing$, because $U_x \cap U_y = \varnothing$. Therefore, we have $U_x \times U_y \subset (X \times X) \backslash \Delta$.

(4) $\Longrightarrow$ (1). Let $x \neq y$. Then $(x,y) \in (X \times X) \backslash \Delta$, and by (4) there exist two subsets $U_x, U_y \in m_X$ such that $(x,y) \in U_x \times U_y \subset (X \times X) \backslash \Delta$. Hence, we have that $(U_x \times U_y) \cap \Delta = \varnothing$, i.e., $U_x \cap U_y = \varnothing$. $\qquad\square$

6. **Applications**

Here, we are able to apply Theorems 5.1 and 5.2 to investigate properties on the digital line using alternative construction [26; p. 908] of digital lines. In Example 6.2 (a)-(d), we use Theorem 5.1 to prove the b-T_2ness of the digital line. Morever, in Example 6.3, we use Theorem 5.2 to observe an alternative proof on the non-pre-T_2ness of the digital line. We recall the Khalimsky line or so called the digital line $(\mathbb{Z}, \kappa)$ is the set of the integers $\mathbb{Z}$ with the topology κ having $S = \{\{2m - 1, 2m, 2m + 1\} \mid m \in \mathbb{Z}\}$ as a subbase ([24],[25],[26]; eg.,[12],[18],[17]). In $(\mathbb{Z}, \kappa)$, for examples, each singleton $\{2m + 1\}$ is open and each singleton $\{2m\}$ is closed, where $m \in \mathbb{Z}$. A subset U is open in $(\mathbb{Z}, \kappa)$ if and only if whenever $x \in U$ and x is an even integer, then $x - 1,\ x + 1 \in U$. A subset $\{2m - 1, 2m, 2m + 1\}$ is the smallest open set containing $2m$, where $m \in \mathbb{Z}$. It is shown directly that $(\mathbb{Z}, \kappa)$ is semi-T_2 ([17; Theorem 2.3]) and so it is b-T_2 . However, it is not pre-T_2 ([17; Theorem 4.8(ii)]) and so it is not T_2 , because $\kappa = PO(\mathbb{Z}, \kappa)$ holds([18; Theorem 2.1(i)(a)]).

Example 6.1. Let $(\mathbb{R}, \epsilon)$ be the Euclidan line and $q : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{Z}, \kappa)$ a function defined by $q(x) = 2n + 1$ for every point x with $2n < x < 2n + 2$, $q(2n) = 2n$, where $n \in \mathbb{Z}$([26; p. 908]). Let R be an equivalence relation in $(\mathbb{R}, \epsilon)$ defined by $R = (\cup\{V(2n, 2n + 2) \times V(2n, 2n + 2) \mid n \in \mathbb{Z}\}) \cup (\cup\{(2n, 2n) \mid n \in \mathbb{Z}\})$, where $V(2n, 2n + 2) = \{t \in \mathbb{R} \mid 2n < t < 2n + 2\}$. For points t and x of $\mathbb{R}$, t is equivalent to x if and only if $(t, x) \in R$. We denote the set of all equivalence classes by $\mathbb{R}/R = \{[t] \mid t \in \mathbb{R}\}$, where $[t] = \{x \in \mathbb{R} \mid (x, t) \in R\}$ is an equivalence class including t. Then, the projection $p : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{R}/R, \Psi)$ is well defined by $p(t) = [t]$, for every $t \in \mathbb{R}$; Ψ is the identification topology induced by the function p; a subset U_1 of $\mathbb{R}/R$ is open in $(\mathbb{R}/R, \Psi)$ (i.e., $U_1 \in \Psi$) if and only if $p^{-1}(U_1)$ is open in $(\mathbb{R}, \epsilon)$. It is shown that $p(t) = [q(t)]$, for every $t \in \mathbb{R}$

(a) The digital line $(\mathbb{Z}, \kappa)$ and $(\mathbb{R}/R, \Psi)$ are homeomorphic.

(b) For $m_{\mathbb{R}} = SO(\mathbb{R}, \epsilon)$ (resp. $(BO(\mathbb{R}, \epsilon))$ and R above, one of the assumptions in Theorem 5.1, i.e., (b), holds.

(c) The function $p : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{R}/R, \Psi)$ is $(SO(\mathbb{R}, \epsilon), SO(\mathbb{R}/R, \Psi))$-open, $(PO(\mathbb{R}, \epsilon), PO(\mathbb{R}/R, \Psi))$-open and $(BO(\mathbb{R}, \epsilon), BO(\mathbb{R}/R, \Psi))$-open.

(d) The digital line $(\mathbb{Z}, \kappa)$ is semi-T_2 and also b-T_2 .

Proof. (a) A continuous bijection $f : (\mathbb{Z}, \kappa) \longrightarrow (\mathbb{R}/R, \Psi)$ is well defined by $f(q(x)) = p(x)$. Then, $f \circ q = p$ and the inverse f^{-1} is continuous.

(b) In Theorem 5.1, let $(X, \tau) = (\mathbb{R}, \epsilon)$ and $R = (\cup\{V(2n, 2n + 2) \times V(2n, 2n + 2) \mid n \in \mathbb{Z}\}) \cup \{(2n, 2n) \mid n \in \mathbb{Z}\})$. We need the following notations: $V(2n, +\infty) = \{x \in \mathbb{R} \mid 2n < x\}$, $V[2n, +\infty) = \{x \in \mathbb{R} \mid 2n \leq x\}$, $V(-\infty, 2n) = \{x \in \mathbb{R} \mid x < 2n\}$ and $V(-\infty, 2n] = \{x \in \mathbb{R} \mid x \leq 2n\}$, where $n \in \mathbb{Z}$. It is shown that $\mathbb{R}^2 \setminus R = [\cup\{(V[2n, +\infty) \times V(-\infty, 2n)) \cup (V(2n, +\infty) \times V(-\infty, 2n]) \mid n \in \mathbb{Z}\}] \cup [\cup\{(V(-\infty, 2n] \times V(2n, +\infty)) \cup (V(-\infty, 2n) \times V[2n, +\infty)) \mid n \in \mathbb{Z}\}]$. Let $(x, y) \in \mathbb{R}^2 \setminus R$. Then, there exist subsets such that $(x, y) \in V[2n, +\infty) \times V(-\infty, 2n)$, $(x, y) \in V(2n, +\infty) \times V(-\infty, 2n]$, $(x, y) \in V(-\infty, 2n] \times V(2n, +\infty)$ or $(x, y) \in V(-\infty, 2n) \times V[2n, +\infty)$. Since $V[2n, +\infty)$, $V(-\infty, 2n)$, then $V(2n, +\infty)$ and $V(-\infty, 2n]$ are semi-open and also b-open in $(\mathbb{R}, \epsilon)$, the condition (b) of Theorem 5.1 holds for $m_\mathbb{R} = SO(\mathbb{R}, \epsilon)$ and also $m_\mathbb{R} = BO(\mathbb{R}, \epsilon)$.

(c) It is obvious that the function $q : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{Z}, \kappa)$ is open and continuous and $f \circ q = p$ holds. First, let $A \in SO(\mathbb{R}, \epsilon)$. Then, there exists an open subset U such that $U \subset A \subset Cl(U)$. Using f in the proof of (a) above, $f(q(U)) \in \Psi$ and $f(q(U)) \subset f(q(A)) \subset f(Cl(q(U))) = Cl(f(q(U)))$. Thus, we have that $p(U) \in \Psi$ and $p(U) \subset p(A) \subset Cl(p(U))$, i.e., $p(A) \in SO(\mathbb{R}/R, \Psi)$.

Second, let $B \in PO(\mathbb{R}, \epsilon)$. Then, there exists an open subset V such that $B \subset V \subset Cl(B)$. It is shown similarly as above that $p(V) \in \Psi$ and $p(B) \subset p(V) \subset Cl(p(B))$. Namely, $p(B) \in PO(\mathbb{R}/R, \Psi)$.

Finally, let $S \in BO(\mathbb{R}, \epsilon)$. It is well known that $S = sInt(S) \cup pInt(S)$ holds [3; Proposition 2.1]. Since $sInt(S) \in SO(\mathbb{R}, \epsilon)$ and $pInt(S) \in PO(\mathbb{R}, \epsilon)$, we have that $p(sInt(S)) \in SO(\mathbb{R}/R, \Psi)$ and $p(pInt(S)) \in PO(\mathbb{R}/R, \Psi)$.
Then, we have that $p(S) = p(sInt(S)) \cup p(pInt(S)) \subset Cl(Int(p(sInt(S)))) \cup Int(Cl(p(pInt(S)))) \subset Cl(Int(p(S))) \cup Int(Cl(p(S)))$. Namely, $p(S) \in BO(\mathbb{R}/R, \Psi)$.

(d) By (b) and (c), all assumptions of Theorem 5.1 are satisfied for $m_\mathbb{R} = SO(\mathbb{R}, \epsilon)$ (resp. $m_\mathbb{R} = BO(\mathbb{R}, \epsilon)$). Thus, $(\mathbb{R}/R, \Psi)$ is semi-T_2 (resp.
b-T_2). Since $f : (\mathbb{Z}, \kappa) \longrightarrow (\mathbb{R}/R, \Psi)$ is a homeomorphism, then we prove that
$(\mathbb{Z}, \kappa)$ is semi-T_2 (resp. b-T_2). $\qquad\qquad\square$

Example 6.2. (a) The digital line is not pre-T_2 ([17; Theorem 4.8(ii)]). Using Theorem 5.2 (3), we have an alternative proof of the property above. Let $(X, \tau) = (\mathbb{Z}, \kappa)$ and $m_X = PO(\mathbb{Z}, \kappa)$ in Theorem 5.2. Then, the condition (3) of Theorem 5.2 is not satisfied. Indeed, let a point $x = 2n \in \mathbb{Z}$ for some integer n, and U_x be any preopen set $(\mathbb{Z}, \kappa)$ such that $x \in U_x$. By using [17; Lemma 3.3], it is shown that $\{2n-1, x, 2n+1\} \subset U_x$. Thus, we have that $\cap \{pCl(U) \mid U \in PO(\mathbb{Z}, \kappa), x \in U\} \supset pCl(\{2n-1, x, 2n+1\}) \supset \{2n-1, x, 2n+1\} \neq \{x\}$. Therefore, by Theorem 5.2, $(\mathbb{Z}, PO(\mathbb{Z}, \kappa))$ is not $PO(\mathbb{Z}, \kappa)$-T_2. Namely, the digital line is not pre-T_2.

(b) Using Theorem 5.2 (3), for $m_X = SO(\mathbb{Z}, \kappa)$, we have an alternative proof of (d) in Example 6.2 above. Let $x = 2n$ and $y = 2m+1$ for some integers n and m. Then, $\{x, 2n+1\}, \{2n-1, x\}$ and $\{y\}$ are semi-open sets of $(\mathbb{Z}, \kappa)$ (e.g., [18]). Since $sCl(\{x, 2n+1\}) = \{x, 2n+1\}$, then $sCl(\{2n-1, x\}) = \{2n-1, x\}, \cap \{sCl(U) \mid U \in SO(\mathbb{Z}, \kappa), x \in U\} \subset sCl(\{x, 2n+1\}) \cap sCl(\{2n-1, x\}) = \{x, 2n+1\} \cap \{2n-1, x\} = \{x\}$. Morever, $\cap \{sCl(U) \mid U \in SO(\mathbb{Z}, \kappa), y \in U\} \subset sCl(\{y\}) = \{y\}$ holds. Therefore, we conclude that $\cap \{sCl(U) \mid U \in SO(\mathbb{Z}, \kappa), z \in U\} = \{z\}$ holds for any point $z \in \mathbb{Z}$. By Theorem 5.2, it is obtained that $(\mathbb{Z}, SO(\mathbb{Z}, \kappa))$ is $SO(\mathbb{Z}, \kappa)$-T_2. Namely, the digital line is semi-T_2 and also it is b-T_2.

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