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ON A LINEAR COMBINATION OF CLASSES OF HARMONIC $p-$VALENT FUNCTIONS DEFINED BY CERTAIN MODIFIED OPERATOR

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Abstract. In this paper we obtain coefficient characterization, extreme points and distortion bounds for the classes of harmonic $p-$valent functions defined by certain modified operator. Some of our results improve and generalize previously known results.
Keywords: Analytic functions, harmonic functions, extreme points, distortion bounds.
MSC(2010): Primary: 30C45.

1. Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \overline{g} \ , \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$ in D (see [10]).

Recently, Jahangiri and Ahuja [15] defined the class $\mathcal{H}_p(p \in \mathbb{N} = \{1, 2, 3, ...\})$, consisting of all harmonic $p-$valent functions $f = h + \overline{g}$

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that are sense preserving in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and h and g are of the form:

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k\, z^k\ ,\ g(z) = \sum_{k=p}^{\infty} b_k\, z^k\ ,\ |b_p| < 1\ . \tag{1.2}$$

For complex parameters $\alpha_1, .., \alpha_q$ and $\beta_1, .., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ..\}$, $j = 1, 2, ..., s$), $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\ell, \gamma \geq 0$, $\lambda \geq 0$ and $z \in \mathbb{U}$, let $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ denote the family of harmonic $p-$valent functions $f = h + g$, where h and g of the form (1.2) such that

$$\Re\left\{(1-\gamma)\,\frac{I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)f(z)}{z^p} + \gamma\frac{\left(I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)f(z)\right)'}{pz^{p-1}}\right\} > \frac{\delta}{p}, \tag{1.3}$$

where $0 \leq \delta < p$ and the operator $I_{p,q,s,\lambda}^{m,\ell}(\alpha_1)f\,(z)$ is defined as follows (see El-Ashwah and Aouf [14]):

$$I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)f(z) = I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)h(z) + (-1)^n\, I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)g(z), \tag{1.4}$$

$$I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)h(z) = z^p + \sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \Gamma_k\,(\alpha_1)\, a_k z^k, \tag{1.5}$$

$$I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)g(z) = z^p + \sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \Gamma_k\,(\alpha_1)\, b_k z^k, \tag{1.6}$$

where

$$\Gamma_k\,(\alpha_1) = \frac{(\alpha_1)_{k-p}...(\alpha_q)_{k-p}}{(\beta_1)_{k-p}...(\beta_s)_{k-p}(1)_{k-p}}, \tag{1.7}$$

and $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta+1)....(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

Let the subclass $\mathcal{H}_{p,q,s}^{-}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ consist of harmonic functions $f_n = h + \overline{g}_n$ in $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ so that h and g_n are of the form:

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k\,|\, z^k\ ,\ g_n(z) = (-1)^{n-1}\sum_{k=p}^{\infty} |b_k|\ z^k\ \ ,\ |b_p| < 1\ . \tag{1.8}$$

We note that, by the special choices of α_i $(i = 1, 2, ..., q)$ and β_j ($j = 1, 2, ..., s$), n, ℓ, γ and λ, we obtain the following classes studied by various authors:

(i) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$ and $n = 0$, we have $\mathcal{H}_{p,s+1,s}(0, \ell, \lambda, 1; \gamma, \delta) = \mathcal{H}_p\mathcal{R}(\gamma, \delta)$ the class of harmonic multivalent functions f in $\mathbb{U}$ studied by Ahuja and Jahangiri [1];

(ii) For $q = 2, s = 1, p = 1, \alpha_2 = \beta_1, \alpha_1 = m+1\,(m > -1)$ and $l = 0$ we get $\mathcal{H}_{1,2,1}(n, 0, \lambda, m+1; \gamma, \delta) = SHP_\lambda(\gamma, \delta, n, m, k)$, the class of harmonic univalent functions f in $\mathbb{U}$ studied by Darus and Sangle [11];

(iii) For $q = 2, s = 1, \alpha_2 = \beta_1, \alpha_1 = m+p\,(m > -p, p \in \mathbb{N})$ and $l = 0$ we get $\mathcal{H}_{p,2,1}(n, 0, \lambda, m+p; \gamma, \delta) = \mathcal{H}_p(n, \gamma, \delta, \lambda, m)$, the class of harmonic multivalent functions f in $\mathbb{U}$ studied by Atshan et al. [5].

We further, observe that, by the special choices of α_i $(i = 1, 2, ..., q)$ and β_j $(j = 1, 2, ..., s), n, \ell$ and λ our class $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ gives rise the following new subclasses of the class $\mathcal{H}_p$:

(i) For $n = 0$ we obtain $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta) = \mathcal{H}_{p,q,s}(\alpha_1; \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re\left\{ (1-\gamma)\, \frac{H_{p,q,s}(\alpha_1)f(z)}{z^p} + \gamma \frac{\left(H_{p,q,s}(\alpha_1)f(z)\right)'}{pz^{p-1}} \right\} > \frac{\delta}{p} \right\},$$

where $H_{p,q,s}(\alpha_1)$ is the modified Dziok-Srivastava operator (see [2], [12] and [13]);

(ii) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$, we get $\mathcal{H}_{p,s+1,s}(n, \ell, \lambda, 1; \gamma, \delta) = \mathcal{H}_p(n, \ell, \lambda; \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re\left\{ (1-\gamma)\, \frac{I_p(n, \lambda, l)f(z)}{z^p} + \gamma \frac{\left(I_p(n, \lambda, l)f(z)\right)'}{pz^{p-1}} \right\} > \frac{\delta}{p} \right\},$$

where $I_p(n, \lambda, \ell)$ is the modified Catas operator (see [7]);

(iii) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$, and $l = 0$, we get $\mathcal{H}_{p,s+1,s}(n, 0, \lambda, 1; \gamma, \delta) = \mathcal{H}_p(n, \gamma, \delta, \lambda)$

$$= \left\{ f \in \mathcal{H}_p : \Re\left\{ (1-\gamma)\, \frac{D^n_{\lambda,p}f(z)}{z^p} + \gamma \frac{\left(D^n_{\lambda,p}f(z)\right)'}{pz^{p-1}} \right\} > \frac{\delta}{p} \right\},$$

where $D^n_{\lambda,p}$ is the modified El-Ashwah-Aouf operator [6];

(iv) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s), \ell = 0$ and $\lambda = 1$, we get $\mathcal{H}_{p,s+1,s}(n, 0, 1, 1; \gamma, \delta) = \mathcal{H}_p(n, \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re \left\{ (1-\gamma) \frac{D_p^n f(z)}{z^p} + \gamma \frac{\left(D_p^n f(z)\right)'}{pz^{p-1}} \right\} > \frac{\delta}{p} \right\},$$

where D_p^n is the modified operator defined BY Kamali and Orhan [16] and Aouf and Mostafa [4];

(v) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$, and $\lambda = 1$, we get $\mathcal{H}_{p,s+1,s}(n, l, 1, 1; \gamma, \delta) = \mathcal{H}_p(n, l; \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re \left\{ (1-\gamma) \frac{I_p(n,\ell) f(z)}{z^p} + \gamma \frac{\left(I_p(n,\ell) f(z)\right)'}{pz^{p-1}} \right\} > \frac{\delta}{p} \right\},$$

where $I_p(n, \ell)$ is the modified operator defined by Kumar et al. [17];

(vi) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$, $p = \lambda = 1$ and $\ell = 0$, we obtain $\mathcal{H}_{1,s+1,s}(n, 0, 1, 1; \gamma, \delta) = \mathcal{H}(n, \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re \left\{ (1-\gamma) \frac{D^n f(z)}{z} + \gamma \left(D^n f(z)\right)' \right\} > \delta \right\},$$

where D^n is the modified Salagean operator (see [18]);

(vii) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$, $p = \lambda = 1$, we get $\mathcal{H}_{1,s+1,s}(n, l, 1, 1; \gamma, \delta) = \mathcal{H}(n, l; \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re \left\{ (1-\gamma) \frac{I_\ell^n f(z)}{z} + \gamma \left(I_\ell^n f(z)\right)' \right\} > \delta \right\},$$

where I_ℓ^n is the modified operator introduced and studied by Cho and Srivastava [8] and Cho and Kim [9];

(viii) For $q = s+1, \alpha_i = 1$ $(i = 1, ..., s+1), \beta_j = 1$ $(j = 1, ..., s)$, $p = 1$ and $\ell = 0$, we obtain $\mathcal{H}_{1,s+1,s}(n, 0, \lambda, 1; \gamma, \delta) = \mathcal{H}(n, \lambda; \gamma, \delta)$

$$= \left\{ f \in \mathcal{H}_p : \Re \left\{ (1-\gamma) \frac{D_\lambda^n f(z)}{z} + \gamma \left(D_\lambda^n f(z)\right)' \right\} > \delta \right\},$$

where D_λ^n is the modified Al-Oboudi operator [3].

In this paper we obtain coefficient characterization of the classes $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ and $\mathcal{H}^-_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$. We also obtain extreme points and distortion bounds for the class $\mathcal{H}^-_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$.

2. **Coefficient characterization**

Unless otherwise mentioned, we assume throughout this paper that $n \in \mathbb{N}_0,\ 0 \le \delta < p,\ \ell, \gamma \ge 0, \lambda > 0$ and $\Gamma_k(\alpha_1)$ is given by (1.7). We begin with a necessary condition for functions in $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.2). Then $f \in \mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ if*

$$\sum_{k=p+1}^{\infty} [(k-p)\gamma + p] \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1) a_k| + \sum_{k=p}^{\infty} |(k+p)\gamma - p| \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1) b_k| \le p - \delta. \tag{2.1}$$

Proof. Let

$$\omega(z) = (1-\gamma)\frac{I^{n,\ell}_{p,q,s,\lambda}(\alpha_1) f(z)}{z^p} + \gamma \frac{\left(I^{n,\ell}_{p,q,s,\lambda}(\alpha_1) f(z)\right)'}{pz^{p-1}}.$$

To prove $Re\{\omega(z)\} > \frac{\delta}{p}$, it suffices to show that $|p - \delta + p\omega(z)| \ge |p + \delta - p\omega(z)|$. Substituting for $\omega(z)$ and making use of (1.5) to (1.7), we find that

$$|p - \delta + p\omega(z)| \ge 2p - \delta - \sum_{k=p+1}^{\infty} [(k-p)\gamma + p] \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1) a_k| |z|^{k-p} - \sum_{k=p}^{\infty} |(k+p)\gamma - p| \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1) b_k| |z|^{k-p} \tag{2.2}$$

and

$$|p + \delta - p\omega(z)| \le \delta + \sum_{k=p+1}^{\infty} [(k-p)\gamma + p] \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1) a_k| |z|^{k-p} + \sum_{k=p}^{\infty} |(k+p)\gamma - p| \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1) b_k| |z|^{k-p}. \tag{2.3}$$

Evidently, the inequalities (2.2) and (2.3) in conjunction with (2.1) yield

$$\begin{aligned}
& |p-\delta+p\omega(z)|-|p+\delta-p\omega(z)| \\
\geq\ & 2\left[p-\delta-\sum_{k=p+1}^{\infty}[(k-p)\gamma+p]\left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}|\Gamma_k(\alpha_1)a_k|\,|z|^{k-p}\right. \\
& \left.-\sum_{k=p}^{\infty}|(k+p)\gamma-p|\left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}|\Gamma_k(\alpha_1)b_k|\,|z|^{k-p}\right]\geq 0.
\end{aligned}$$

The harmonic functions

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{x_k}{[(k-p)\gamma+p]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}|\Gamma_k(\alpha_1)|}z^k+\sum_{k=p}^{\infty}\frac{\bar{y}_k}{|(k+p)\gamma-p|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}|\Gamma_k(\alpha_1)|}\bar{z}^k, \tag{2.4}$$

where $\sum\limits_{k=p+1}^{\infty}|x_k|+\sum\limits_{k=p}^{\infty}|y_k|=p-\delta$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.4) are in $\mathcal{H}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ because in view of (2.1), we have

$$\begin{aligned}
& \sum_{k=p+1}^{\infty}[(k-p)\gamma+p]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}|\Gamma_k(\alpha_1)a_k| \\
& +\sum_{k=p}^{\infty}|(k+p)\gamma-p|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}|\Gamma_k(\alpha_1)b_k| \\
=\ & \sum_{k=p+1}^{\infty}|x_k|+\sum_{k=p}^{\infty}|y_k|=p-\delta.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

The restriction imposed in Theorem 2.1 on the moduli of the coefficients of $f=h+\bar{g}$ implies that for arbitrary rotation of the coefficients of f, the resulting functions would still be harmonic multivalent and $f\in\mathcal{H}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$. In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n=h+\overline{g}_n$, where h and g_n are of the form (1.8).

Theorem 2.2. *Let* $f_n = h + \overline{g}_n$, *where* h *and* g_n *are of the form (1.8). Then* $f_n \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ *if and only if*

$$\sum_{k=p+1}^{\infty} \left[(k-p)\gamma + p\right] \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \left|\Gamma_k(\alpha_1)\, a_k\right| \tag{2.5}$$

$$+\sum_{k=p}^{\infty} \left|(k+p)\gamma - p\right| \left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \left|\Gamma_k(\alpha_1)\, b_k\right| \leq p-\delta.$$

Proof. Since $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta) \subset \mathcal{H}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f_n = h + \overline{g}_n$, where h and g_n are of the form (1.8), we notice that the condition

$$R\left\{(1-\gamma)\frac{I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)}{z^p} + \gamma\frac{\left(I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)f(z)\right)'}{pz^{p-1}}\right\} > \frac{\delta}{p}$$

is equivalent to

$$R\left\{(1-\gamma)\frac{I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)h(z) + (-1)^n I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)g(z)}{z^p}\right.$$

$$\left. +\gamma\frac{\left(I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)h(z) + (-1)^n I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)g(z)\right)'}{pz^{p-1}}\right\}$$

$$\geq 1 - \frac{1}{p}\sum_{k=p+1}^{\infty} \left[(k-p)\gamma + p\right] \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \left|\Gamma_k(\alpha_1)\, a_k\right| |z|^{k-p}$$

$$-\sum_{k=p}^{\infty} \left|(k+p)\gamma - p\right| \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \left|\Gamma_k(\alpha_1)\, b_k\right| |z|^{k-p} \geq \frac{\delta}{p}.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$1 - \frac{1}{p}\sum_{k=p+1}^{\infty} \left[(k-p)\gamma + p\right] \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \left|\Gamma_k(\alpha_1)\, a_k\right| r^{k-p}$$

$$-\sum_{k=p}^{\infty} \left|(k+p)\gamma - p\right| \left[\tfrac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n \left|\Gamma_k(\alpha_1)\, b_k\right| r^{k-p} \geq \frac{\delta}{p}.$$

Letting $r \to 1^-$, we obtain the inequality (2.5) and so the proof of Theorem 2.2 is completed. $\square$

3. Extreme points and distortion theorem

The next theorem is on the extreme points of convex hulls of the class $\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ denoted by $clco\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$.

Theorem 3.1. *Let $f_n = h + \overline{g}_n$, where h and g_n are of the form (1.8). Then $f_n \in cloc\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ if and only if*

$$f_n(z) = \sum_{k=p}^{\infty} \left[x_k\, h_k(z) + y_k g_k(z)\right],$$

where $h_p(z) = z^p$,

$$h_k(z) = z^p - \frac{p-\delta}{[(k-p)\gamma+p]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n|\Gamma_k(\alpha_1)|}\ z^k \ \ (k = p+1, p+2, ...),$$

and

$$g_k(z) = z^p - (-1)^n \frac{p-\delta}{|(k+p)\gamma-p|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n|\Gamma_k(\alpha_1)|}\ \overline{z}^k \ (k = p, p+1, ...),$$

$$x_k,\, y_k \geq 0\,, x_p = 1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k.$$

In particular, the extreme points of the class $\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Suppose that

$$\begin{aligned} f_n(z) &= \sum_{k=p}^{\infty}(x_k\, h_k(z) + y_k g_k(z)) \\ &= \sum_{k=p}^{\infty}(x_k + y_k)z^p - \sum_{k=p+1}^{\infty} \frac{p-\delta}{[(k-p)\gamma+p]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n\Gamma_k(\alpha_1)} x_k\, z^k \\ &\quad -(-1)^n \sum_{k=p}^{\infty} \frac{p-\delta}{|(k+p)\gamma-p|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n\Gamma_k(\alpha_1)}\ y_k\ \overline{z}^k\ . \end{aligned}$$

Then

$$\begin{aligned}
& \sum_{k=p+1}^{\infty}\left[\left[(k-p)\gamma+p\right]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}\left|\Gamma_{k}\left(\alpha_{1}\right)\right|\right] \\
& \cdot\left(\frac{p-\delta}{\left[(k-p)\gamma+p\right]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}\left|\Gamma_{k}(\alpha_{1})\right|}x_{k}\right) \\
& +\sum_{k=p}^{\infty}\left[\left|(k-p)\gamma-p\right|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}\left|\Gamma_{k}\left(\alpha_{1}\right)\right|\right] \\
& \cdot\left(\frac{p-\delta}{\left|(k-p)\gamma-p\right|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}\left|\Gamma_{k}(\alpha_{1})\right|}y_{k}\right) \\
= \; & (p-\delta)\left(\sum_{k=p+1}^{\infty}x_{k}+\sum_{k=p}^{\infty}y_{k}\right)=(p-\delta)\left(1-x_{p}\right) \\
\leq \; & p-\delta.
\end{aligned}$$

and so $f_n \in clco\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$.
Conversely, if $f_n \in clco\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$. Set

$$x_k=\frac{\left[(k-p)\gamma+p\right]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}\left|\Gamma_{k}(\alpha_{1})\right|}{p-\delta}\left|a_k\right| \quad (k=p+1,p+2,....),$$

and

$$y_k=\frac{\left|(k+p)\gamma-p\right|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^{n}\left|\Gamma_{k}(\alpha_{1})\right|}{p-\delta}\left|b_k\right| \quad (k=p,p+1,....).$$

Then note that by Theorem 2.2, $0\leq x_k\leq 1$, $(k=p+1,p+2,...)$, and $0\leq y_k\leq 1$, $(k=p,p+1,...)$. Let $x_p=1-\sum\limits_{k=p+1}^{\infty}x_k-\sum\limits_{k=p}^{\infty}y_k$ and $x_p\geq 0$.

Consequently, we obtain the required representation, since

$$\begin{aligned} f_n(z) &= z^p - \sum_{k=p+1}^{\infty} |a_k|\, z^k - (-1)^n \sum_{k=p}^{\infty} |b_k|\ z^k \\ &= z^p - \sum_{k=p+1}^{\infty} \frac{p-\delta}{[(k-p)\,\gamma+p]\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1)|} x_k z^k \\ &\quad -(-1)^n \sum_{k=p}^{\infty} \frac{p-\delta}{|(k-p)\,\gamma-p|\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n |\Gamma_k(\alpha_1)|}\ y_k\ \overline{z}^k \\ &= z^p - \sum_{k=p+1}^{\infty} (z^p - h_k(z))\, x_k z^k - \sum_{k=p}^{\infty} (z^p - g_k(z))\ y_k\ \overline{z}^k \\ &= \left(1 - \sum_{k=p+1}^{\infty} x_k - \sum_{k=p}^{\infty} y_k\right)\ z^p + \sum_{k=p+1}^{\infty} h_k(z) x_k z^k \\ &\quad + \sum_{k=p}^{\infty} g_k(z)\ y_k\ \overline{z}^k \\ &= \sum_{k=p}^{\infty} \{x_k\, h_k(z) + y_k\ g_k(z)\}\,. \end{aligned}$$

This completes the proof of Theorem 3.1. □

The following theorem gives the distortion bounds for functions in the class $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ which yields a covering result for this class.

Theorem 3.2. *Let* $f_n \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ *with* $\frac{p(2\gamma-1)}{p-\delta}\,|b_p| < 1$. *Then for* $|z| = r < 1$, *we have*

$$\begin{aligned} &(1-|b_p|)r^p - \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n |\Gamma_{p+1}(\alpha_1)|}\left(1 - \frac{p(2\gamma-1)}{p-\delta}\,|b_p|\right) r^{p+1} \\ \leq\ & |f_n(z)| \leq (1+|b_p|)r^p + \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n |\Gamma_{p+1}(\alpha_1)|}\left(1 - \frac{p(2\gamma-1)}{p-\delta}\,|b_p|\right) r^{p+1}. \end{aligned}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted.

Let $f_n \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\le (1+|b_p|)\,r^p + \sum_{k=p+1}^{\infty}(|a_k|+|b_k|)\,r^k \\
&\le (1+|b_p|)r^p + r^{p+1}\sum_{k=p+1}^{\infty}(|a_k|+|b_k|) \\
&= (1+|b_p|)r^p + \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n|\Gamma_{p+1}(\alpha_1)|} \\
&\quad .r^{p+1}\sum_{k=p+1}^{\infty}\frac{[\gamma+p]}{p-\delta}\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n|\Gamma_{p+1}(\alpha_1)|\,(|a_k|+|b_k|) \\
&\le (1+|b_p|)r^p + \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n|\Gamma_{p+1}(\alpha_1)|}r^{p+1} \\
&\quad \left\{\sum_{k=p+1}^{\infty}\frac{[(k-p)\gamma+p]}{p-\delta}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n|\Gamma_k(\alpha_1)\,a_k|\right. \\
&\quad \left.+\sum_{k=p+1}^{\infty}\frac{|(k+p)\gamma-p|}{p-\delta}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n|\Gamma_k(\alpha_1)\,b_k|\right\} \\
&\le (1+|b_p|)r^p + \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n|\Gamma_{p+1}(\alpha_1)|}\left(1-\frac{p(2\gamma-1)}{p-\delta}|b_p|\right)r^{p+1}.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

Remark 3.3. *The bounds given in Theorem 3.2 for functions $f_n = h+\overline{g}_n$, where h and g_n are given by (1.8), also hold for functions of the form $f = h+\overline{g}$, where h and g are given by (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for $f \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1;\gamma,\delta)$ is sharp and the equality occurs for the functions*

$$f(z) = z^p + |b_p|\;\overline{z}^p\; - \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n|\Gamma_{p+1}(\alpha_1)|}\left(1-\frac{p(2\gamma-1)}{p-\delta}|b_p|\right)z^{p+1},$$

and

$$f(z) = z^p + |b_p|\;\overline{z}^p\; - \frac{p-\delta}{[\gamma+p]\left[\frac{p+\ell+\lambda}{p+\ell}\right]^n|\Gamma_{p+1}(\alpha_1)|}\left(1-\frac{p(2\gamma-1)}{p-\delta}|b_p|\right)\overline{z}^{p+1},$$

showing that the bounds given in Theorem 3.2 are sharp.

Remark 3.4. *Putting* $q = s + 1, \alpha_i = 1$ $(i = 1, ..., s + 1), \beta_j = 1$ $(j = 1, ..., s)$ *and* $n = 0$ *in the above results, we obtain the results of Ahuja and Jahangiri* [1, Theorems 1,2,3 and 7 and Corollary 3, respectively].

Remark 3.5. *Putting* $q = 2, s = 1, p = 1, \alpha_2 = \beta_1, \alpha_1 = m+1 (m > -1)$ *and* $l = 0$ *in the above results, we improve the results obtained by Darus and Sangle* [11, Theorems 1,2,4 and 3 and Corollary 1, respectively].

Remark 3.6. *Putting* $q = 2, s = 1, \alpha_2 = \beta_1, \alpha_1 = m+p (m > -p, p \in \mathbb{N})$ *and* $l = 0$ *in the above results, we improve the results of Atshan et al.* [5, Theorems 1, 2, 4 and 3 and Corollary 1, respectively].

Remark 3.7. *For special choices of* α_i $(i = 1, 2, ..., q)$ *and* β_j $(j = 1, 2, ..., s), n, \ell$ *and* λ *in the above results, we get new results of novel subclasses of our class* $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1; \gamma, \delta)$ *as stated in the introduction.*

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