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PAIRWISE NON-COMMUTING ELEMENTS IN FINITE METACYCLIC 2-GROUPS AND SOME FINITE p-GROUPS

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ABSTRACT. Let G be a finite group. A subset X of G is a set of pairwise non-commuting elements if any two distinct elements of X do not commute. In this paper we determine the maximum size of these subsets in any finite non-abelian metacyclic 2-group and in any finite non-abelian p-group with an abelian maximal subgroup.
Keywords: Metacyclic p-group, powerful 2-group, covering, pairwise non-commuting elements.
MSC(2010): Primary: 20D15; Secondary: 20D60.

1. Introduction

Let G be a finite non-abelian group and let X be a subset of pairwise non-commuting elements of G such that $|X| \geq |Y|$ for any other set of pairwise non-commuting elements Y in G. Then X is said to have the maximum size and the size of such a subset is denoted by $\omega(G)$. Also $\omega(G)$ is the maximum clique size in the non-commuting graph of a finite group G. Let $Z(G)$ be the centre of G. The non-commuting graph of a group G is defined as a graph whose $G \backslash Z(G)$ is the set of vertices and two vertices are joined if and only if they do not commute. Various attempts have been made to find $\omega(G)$ for some groups G, see for example [1–3, 6–8, 13, 14] and [15]. Moreover in [9] for any finite non-abelian metacyclic p-group G with $p > 2$, it is proved that $\omega(G) =$

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$\frac{|G'|}{p}(1+p)$ and also it is shown that this equality is not true for finite non-abelian metacyclic 2-groups. In the present paper continuing the work in [9], we find $\omega(G)$ when G is a finite non-abelian metacyclic 2-group. Also we show that $\omega(G) = |G'|+1$ for any finite non-abelian p-group G with an abelian maximal subgroup, see (Theorem 3.4). Our proofs are based on maximal subgroups and powerful 2-groups. Following [4], we say that a 2-group G is powerful if $G/\mho_2(G)$ is abelian, where $\mho_2(G) = \langle x^4 | x \in G\rangle$. Moreover to prove our main theorem, we use coverings of groups by abelian subgroups. Following [5, §116], we say that a finite group G is covered by the proper subgroups $A_1, \ldots, A_n$ if $G = A_1 \cup \cdots \cup A_n$. As a matter of fact if G is covered by n proper abelian subgroups, then $\omega(G) \leq n$ since two elements that do not commute cannot be in the same abelian subgroup.

In particular for metacyclic 2-groups we prove the following theorem:

Theorem 1.1. *Let G be a finite non-abelian metacyclic 2-group with $G = \langle b, a\rangle$, $b^{-1}ab = a^k$, $1 < k < |a|$ and $k + 1 = 2^n\ell$, where $n \geq 1$ and ℓ is an odd number.*

(i) *If G possesses at least one abelian maximal subgroup, then $\omega(G) = |G'| + 1$.*

(iii) *If G possesses no abelian maximal subgroups, then*

$$\omega(G) = \begin{cases} \frac{|G'|}{2^{2n}}(2^{2n} + 3.2^{n-1} - 1) & |G'| \geq 2^{2n} \\ \frac{|G'|}{2^{n+1}}(2^{n+1} + 3) & |G'| < 2^{2n}. \end{cases}$$

We note that we are able to write $k+1 = 2^n\ell$, where $n \geq 1$ and ℓ is an odd number, by using Lemma 2.1(i). Furthermore the above theorem coincides with Theorem 1.1 in [9] for a metacyclic p-group with $p > 2$, only when G is a powerful metacyclic 2-group with no abelian maximal subgroups, see Corollary 5.8.

Throughout this paper the following notation is used. All groups are assumed to be finite. The centralizer of an element a in a group G is denoted by $\mathcal{C}_G(a)$. The order of an element a in a group G is denoted by $|a|$. The letter p denotes a prime number. In a p-group G, we define $\mho_i(G) = \langle x^{p^i} | x \in G\rangle$ for $i \geq 1$. We write $[a, b]$ for $a^{-1}b^{-1}ab$. Also a minimal non-abelian group is a non-abelian group such that all of its proper subgroups are abelian.

2. **Some basic results**

In this section we give some basic results for metacyclic p-groups that are needed for the main results of the paper.

Let G be a finite metacyclic p-group. We know that there exists a normal cyclic subgroup $\langle a\rangle$ of G such that $G/\langle a\rangle$ is cyclic. Therefore we may choose an element $b \in G$ and a number $1 \leq k < |a|$ such that $G = \langle b, a\rangle$ and $b^{-1}ab = a^k$ and so any element of G has the form $b^j a^i$ for $j, i \geq 0$.

For the rest of the paper we fix the above notation.

Lemma 2.1. *Let G be a non-abelian metacyclic p-group. Then*

(i) $k \equiv 1 \pmod p$,
(ii) $[a^i, b^j] = [a, b]^{i(1+k+\cdots+k^{j-1})}$ *for* $i, j \geq 1$,
(iii) $G' = \langle [a, b]\rangle$,
(iv) *any two arbitrary elements* $x = b^j a^i$ *and* $y = b^s a^r$ *in* G *commute if and only if* $(1 + k + \cdots + k^{s-1})i \equiv (1 + k + \cdots + k^{j-1})r \pmod{|G'|}$, *where* $i, j, r, s \geq 0$ *and we take* $1+k+\cdots+k^{m-1} = 0$ *when* $m = 0$,
(v) $(ba^i)^n = b^n a^{i(1+k+\cdots+k^{n-1})}$ *for* $i, n \geq 1$,
(vi) $\Phi(G) = \langle b^p, a^p\rangle$.

Proof. See [9, Lemma 2.1]. □

By Lemma 2.1(i), if $p = 2$ then we may write $k+1 = 2^n\ell$, where $n \geq 1$ and ℓ is an odd number. Throughout this paper we use this notation everywhere.

Theorem 2.2. *Let G be a non-abelian metacyclic 2-group. Then*

(i) $H_1 = \langle b, a^2\rangle, H_2 = \langle ba, a^2\rangle$ *and* $H_3 = \langle b^2, a\rangle$ *are all distinct maximal subgroups of* G,
(ii) $|H_1'| = |H_2'| = \frac{|G'|}{2}$ *and* $|H_3'| = \frac{|G'|}{2^n}$ *when* H_1, H_2 *and* H_3 *are not abelian and so* $|G'| \nmid k + 1$ *and* $2^{n+1} \mid |G'|$.

Proof. (i) Obviously G has three maximal subgroups since $d(G) = 2$. Moreover $\Phi(G) < H_i < G$ for $1 \leq i \leq 3$ by Lemma 2.1(vi) and (v), which completes the proof.
(ii) This follows from Lemma 2.1(ii), (iii) and the fact that H_i is metacyclic for $1 \leq i \leq 3$. □

For the rest of the paper we use the notation of Theorem 2.2.

Lemma 2.3. *Let k be an odd number.*

(i) *For $j \geq 1$ we have $1 + k^{2^j} = 2u$, where u is an odd number.*
(ii) *If i is an odd number, then $1 + k + \cdots + k^{i-1}$ is also an odd number.*
(iii) *If $i = 2^s u$ in which $s \geq 1$ and u is an odd number, then* $1 + k + \cdots + k^{i-1} = (1 + k)2^{s-1}u'$, *where u' is an odd number.*

Proof. (i) We see that $1 + k^{2^j} = 1 + (k^{2^{j-1}})^2$ and since k is odd, we may write $k^{2^{j-1}} = 2t + 1$ for some $t \geq 0$, as desired.
(ii) Since $k \equiv 1 \pmod 2$, we see that $1 + k + \cdots + k^{i-1} \equiv i \pmod 2$, as required.
(iii) On setting $s(k, i) = \sum_{j=0}^{i-1} k^j$ and by using (i), we prove that $s(k, i) = v \prod_{j=0}^{s-1}(1 + k^{2^j})$ for any odd number k, where v is an odd number. To prove this we use induction on s. Now (ii) and the fact that $s(k, i) = (k + 1)s(k^2, i/2)$, complete the proof. □

Lemma 2.4. *Let G be a non-abelian metacyclic 2-group. Then $|G'| < |a|$ and $\frac{|G'|}{2^{n-1}} \leq |b|$.*

Proof. First we see that $G' < \langle a \rangle$ since G is not cyclic. Hence $|G'| < |a|$. Moreover we have $[a, b^{|b|}] = 1$ and so $|G'|$ divides $\sum_{j=0}^{|b|-1} k^j = (k+1)|b|/2$ by Lemma 2.1(ii), (iii) and Lemma 2.3(iii). Now the fact that $k+1 = 2^n\ell$ completes the proof. □

Lemma 2.5. *Let G be a metacyclic 2-group and $|G'| = 2$. Then*
(i) *G is minimal non-abelian,*
(ii) *$\omega(G) = 3$.*

Proof. (i) See [9, Corollary 3.2].
(ii) See [5, Lemma 116.1(a)]. □

For the rest of the paper we assume that $|G'| > 2$.

Lemma 2.6. *Let G be a non-abelian metacyclic p-group. Then $\{a, b, ba, ba^2, \ldots, ba^{|G'|-1}\}$ is a subset of pairwise non-commuting elements in G. Therefore $|G'| + 1 \leq \omega(G)$.*

Proof. This follows from Lemma 2.1(iv). □

Note. In the following lemmas we use the fact that the congruence $ax \equiv b \pmod m$ has a solution c, $0 \leq c < m$ when $(a, m) = 1$, see for example [12, Proposition 3.3.1].

Lemma 2.7. *Let G be a non-abelian metacyclic 2-group. Then*

(i) $\mathcal{C}_G(a)$, $\mathcal{C}_G(b)$ *and* $\mathcal{C}_G(ba^i)$ *for* $i \geq 0$ *are abelian subgroups of* G,

(ii) $\mathcal{C}_G(ba^{2i}) = \mathcal{C}_{H_1}(ba^{2i})$ *and* $\mathcal{C}_G(ba^{2i+1}) = \mathcal{C}_{H_2}(ba^{2i+1})$ *for* $i \geq 0$,

(iii) $H_1 = \bigcup_{i=0}^{\frac{|G'|}{2}-1} \mathcal{C}_G(ba^{2i}) \cup \Phi(G)$,

(iv) $H_2 = \bigcup_{i=0}^{\frac{|G'|}{2}-1} \mathcal{C}_G(ba^{2i+1}) \cup \Phi(G)$.

Proof. (i) If $b^j a^i \in \mathcal{C}_G(a)$, then $b^j \in Z(G)$ and so $\mathcal{C}_G(a)$ is abelian. Similarly $\mathcal{C}_G(b)$ is abelian. Now by the above argument we deduce that $\mathcal{C}_G(ba^i)$ is abelian for $i \geq 0$ since $G = \langle a, ba^i \rangle$.

(ii) First we see that $\mathcal{C}_{H_1}(ba^{2i}) \leq \mathcal{C}_G(ba^{2i})$ and if $b^r a^s \in \mathcal{C}_G(ba^{2i})$, then $2i \sum_{t=0}^{r-1} k^t \equiv s \pmod{|G'|}$ by Lemma 2.1(iv). Therefore $2 \mid s$ since 2 divides $|G'|$, as desired. Now to prove the second equation, we see that $b^r a^s \in H_2$ if and only if either both r and s are odd or both r and s are even by the fact that $\Phi(G) \leq H_2$. Therefore we can complete the proof by using Lemma 2.1(iv) and Lemma 2.3(ii).

(iii) If $b^r a^{2s} \in H_1 \setminus \Phi(G)$, then r is odd. Hence there exists $0 \leq i < \frac{|G'|}{2}$ such that $i \sum_{t=0}^{r-1} k^t \equiv s \pmod{\frac{|G'|}{2}}$ since $\sum_{t=0}^{r-1} k^t$ and $\frac{|G'|}{2}$ are coprime. Hence $b^r a^{2s} \in \mathcal{C}_G(ba^{2i})$ by Lemma 2.1(iv), as required.

(iv) By the argument in the proof of (ii), if $b^r a^s \in H_2 \setminus \Phi(G)$, then both r and s are odd. Hence there exists $0 \leq j < |G'|$ such that $j \sum_{t=0}^{r-1} k^t \equiv s \pmod{|G'|}$. Moreover we may write $j = 2i + 1$ since s is odd and so $b^r a^s \in \mathcal{C}_G(ba^j)$, as desired. $\square$

3. Non-abelian p-groups with an abelian maximal subgroup

In this section we consider a general case. Let G be a finite non-abelian p-group with an abelian maximal subgroup. Then we show that $\omega(G) = |G'| + 1$. First we give the following definition. A group G is called an AC-group if the centralizer of every non-central element of G is abelian.

Lemma 3.1. *[8, Lemma 2.2] Let G be an AC-group.*

(i) *If $a, b \in G \setminus Z(G)$ with different centralizers, then* $\mathcal{C}_G(a) \cap \mathcal{C}_G(b) = Z(G)$.

(ii) *If* $G = \cup_{i=1}^{k} \mathcal{C}_G(a_i)$, *where* $\mathcal{C}_G(a_i)$ *and* $\mathcal{C}_G(a_j)$ *are distinct for* $1 \leq i < j \leq k$, *then* $\{a_1 \ldots a_k\}$ *is a maximal set of pairwise non-commuting elements in* G.

Lemma 3.2. *Let G be a finite non-abelian p-group with an abelian maximal subgroup A and let $x \in G \setminus A$. Then*

(i) $Z(G) \leq A$, $A \cap \mathcal{C}_G(x) = Z(G)$ *and* $|\mathcal{C}_G(x) : Z(G)| = p$,
(ii) G *is an* AC*-group.*

Proof. (i) We see that $A \leq AZ(G) < G$ since G is not abelian and so $Z(G) \leq A$. Therefore $A \cap \mathcal{C}_G(x) = Z(G)$ since $G = A\langle x \rangle$. Hence $|\mathcal{C}_G(x) : Z(G)| = p$ by the fact that $G = A\mathcal{C}_G(x)$.
(ii) If $a \in A \setminus Z(G)$, then $\mathcal{C}_G(a) = A$ and if $x \in G \setminus A$, then $|\mathcal{C}_G(x) : Z(G)| = p$ by (i), as desired. □

Lemma 3.3. *If* G *is a finite group with* $A \trianglelefteq G$ *and* G/A *is cyclic, then* $|A| = |G'||A \cap Z(G)|$.

Proof. See [11, Aufgabe III. 1.2]. □

Theorem 3.4. *Let* G *be a finite non-abelian* p*-group with an abelian maximal subgroup* A*. Then* $\omega(G) = |G'| + 1$.

Proof. First by Lemma 3.2(ii), G is an AC-group. Now since G is finite and $\mathcal{C}_G(a) = A$ for any $a \in A \setminus Z(G)$, we may write $G = \mathcal{C}_G(x_1) \cup \mathcal{C}_G(x_2) \cup \cdots \cup \mathcal{C}_G(x_m) \cup \mathcal{C}_G(a)$ for some m, where $a \in A \setminus Z(G)$, $x_i \in G \setminus A$ for $1 \leq i \leq m$ and all elements of the union are distinct. Moreover the intersection of any two elements of the union is $Z(G)$ by Lemma 3.1(i). Hence by Lemma 3.2(i), $|G| = m(p|Z(G)| - |Z(G)|) + |A|$ and so $m = |A|/|Z(G)| = |G'|$ by Lemma 3.3. This completes the proof by using Lemma 3.1(ii). □

4. Powerful metacyclic 2-groups

In this section we give some results for powerful metacyclic 2-groups that will be used in the sequel. In fact we prove that H_3 is powerful and we use it to prove our main theorem in the next section. Following [4], a 2-group G is said to be powerful if $G/\mho_2(G)$ is abelian. We note that if G is a 2-group, then $\Phi(G) = \mho_1(G)$. Next two lemmas show that most results about powerful p-groups with $p > 2$ are true for powerful 2-groups as well.

Lemma 4.1. *Let* G *be a powerful* 2*-group and* $G = \langle a_1, a_2, \ldots, a_d \rangle$*. Then*

(i) $\Phi(G) = \langle a_1^2, a_2^2, \ldots, a_d^2 \rangle$,
(ii) $G = \langle a_1 \rangle \langle a_2 \rangle \ldots \langle a_d \rangle$,
(iii) $\mho_i(G) = \{x^{2^i} | x \in G\}$ *for* $i \geq 1$.

Proof. (i) See [4, Lemma 26.24].
(ii) See [4, Corollary 26.25].
(iii) See [4, Proposition 26.23]. □

Lemma 4.2. *Let G be a powerful 2-group and $G = M_1 \cup \cdots \cup M_t \cup \Phi(G)$, where M_i's subgroups of G. Then $G = M_1 \cup \cdots \cup M_t$.*

Proof. The proof is the same as [9, Lemma 4.1]. □

Following [4, §26], a metacyclic 2-group G is called ordinary metacyclic if G has a cyclic normal subgroup A such that G/A is cyclic and $[G, A] \leq \mho_2(A)$.

Theorem 4.3. *[4, Proposition 26.27] A two generator 2-group is powerful if and only if it is ordinary metacyclic.*

Lemma 4.4. *Let G be a non-abelian metacyclic 2-group. Then G is powerful if and only if $n = 1$. Moreover in this case $|H_3'| = \frac{|G'|}{2}$.*

Proof. First recall that $k+1 = 2^n\ell$, $b^{-1}ab = a^k$ and $k > 1$ since G is not abelian. If G is powerful, then we have $a^{k-1} = [a, b] \in \mho_2(\langle a\rangle) = \langle a^4\rangle$ by Theorem 4.3. Hence we deduce that $|a| \mid k-1-4t$ for some t. Moreover $[a, b] \in \langle a^4\rangle$ shows that $a^4 \neq 1$ or equivalently $4 \mid |a|$. Therefore $4 \mid k-1$, as desired. Now if $n = 1$, then $4 \mid k-1$, completing the proof. The rest follows from Theorem 2.2(ii) when H_3 is not abelian. Moreover if H_3 is abelian, then $[a, b^2] = 1$. This yields that $|G'| = 2$ by Lemma 2.1(ii), (iii), as required. □

Lemma 4.5. *Let G be a non-abelian metacyclic 2-group. Then H_3 is powerful.*

Proof. We see that $H_3' = \langle a^{k^2-1}\rangle \leq \langle a^4\rangle \leq \mho_2(H_3)$ by Lemma 2.1(i),(ii) which completes the proof. □

Lemma 4.6. *If G is a powerful metacyclic 2-group with $|G'| = 2^m$, where $m \geq 2$, then H_3 is covered by 2^{m-1} abelian subgroups and $\Phi(G)$.*

Proof. We use induction on m. For $m = 2$, we see that H_3 is minimal non-abelian by Lemma 4.4 and Lemma 2.5. Obviously H_3 is covered by its three abelian maximal subgroups in which $\Phi(G)$ is one of them by Theorem 2.2(i). Now suppose that $m \geq 3$, $|G'| = 2^m$ and the result holds for all powerful metacyclic 2-groups in which the order of the derived subgroup is 2^{m-1}. We see that H_3 is powerful and $|H_3'| = 2^{m-1}$ by Lemmas 4.5, 4.4. Also by Theorem 2.2, $M_1 = \langle b^2, a^2\rangle = \Phi(G)$,

$M_2 = \langle b^2a, a^2\rangle$ and $M_3 = \langle b^4, a\rangle$ are all distinct maximal subgroups of H_3. Hence by the induction hypothesis, M_3 is covered by 2^{m-2} abelian subgroups and $\Phi(H_3)$. Moreover by Lemma 2.7, we see that M_2 is covered by $\Phi(H_3)$ and 2^{m-2} abelian subgroups of H_3. Therefore we can complete the proof by the fact that $\Phi(H_3) \leq \Phi(G)$ and $H_3 = M_1 \cup M_2 \cup M_3$. $\square$

Lemma 4.7. *Let G be a non-abelian metacyclic 2-group. Then*

(i) $\mathcal{C}_{H_3}(b^{2i}a) = \mathcal{C}_G(b^{2i}a)$ *for* $i \geq 0$,

(ii) $\mathcal{C}_G(b^{2i}a)$ *is abelian for* $i \geq 0$,

(iii) $H_3 = \bigcup_{i=0}^{\frac{|G'|}{2}-1} \mathcal{C}_G(b^{2i}a) \cup \Phi(G)$ *when G is powerful and* $|G'| > 2$.

Proof. (i) This follows from Lemma 2.1(iv) and Lemma 2.3(ii).
(ii) Assume that b^ra^s, $b^{r'}a^{s'} \in \mathcal{C}_G(b^{2i}a)$. Thus by Lemma 2.1(iv), $\sum_{t=0}^{r-1} k^t \equiv s\sum_{t=0}^{2i-1} k^t \pmod{|G'|}$ and $\sum_{t=0}^{r'-1} k^t \equiv s'\sum_{t=0}^{2i-1} k^t \pmod{|G'|}$. Therefore
$s'\sum_{t=0}^{r-1} k^t \equiv s\sum_{t=0}^{r'-1} k^t \pmod{|G'|}$, as desired.
(iii) By Lemma 4.6, we may write $H_3 = \bigcup_{t=0}^{\frac{|G'|}{2}-1} M_t \cup \Phi(G)$, where M_t is abelian for $0 \leq t \leq \frac{|G'|}{2} - 1$. Moreover $|G'| \leq |b|$ by Lemma 2.4 and Lemma 4.4 and so we see that $\{a, b^2a, b^4a, b^6a, \ldots, b^{|G'|-2}a\}$ is a subset of pairwise non-commuting elements of $H_3 \setminus \Phi(G)$. For otherwise if $[b^{2i}a, b^{2j}a] = 1$ for $0 \leq j < i < \frac{|G'|}{2}$, then $|G'|$ divides $\sum_{t=0}^{2(i-j)-1} k^t$ by Lemma 2.1(iv). Hence $\frac{|G'|}{2} \mid i - j$ by Lemma 2.3(iii) and the fact that $n = 1$, which is impossible. Therefore we may assume that $b^{2i}a \in M_i$ for $0 \leq i \leq \frac{|G'|}{2} - 1$. Thus $M_i \leq \mathcal{C}_{H_3}(b^{2i}a)$ which completes the proof by (i). $\square$

Theorem 4.8. *Let G be a powerful metacyclic 2-group with $|G'| > 2$. Then $G = (\bigcup_{i=0}^{|G'|-1} \mathcal{C}_G(ba^i)) \cup (\bigcup_{t=0}^{\frac{|G'|}{2}-1} \mathcal{C}_G(b^{2t}a))$, where the elements of the union are abelian subgroups of G.*

Proof. This follows from Lemmas 2.7, 4.7 and 4.2. $\square$

Corollary 4.9. *Let G be a metacyclic 2-group with $|H_3'| > 2$. Then $H_3 = (\bigcup_{i=0}^{|H_3'|-1} \mathcal{C}_{H_3}(b^2a^i)) \cup (\bigcup_{t=0}^{\frac{|H_3'|}{2}-1} \mathcal{C}_{H_3}(b^{4t}a))$, where the elements of the union are abelian subgroups of H_3.*

Proof. This is a consequence of Lemma 4.5, Theorem 4.8 and the fact that $H_3 = \langle a, b^2\rangle$. $\square$

By the above argument in this section, we may deduce the following corollary, however this result can be obtained directly from the main theorem of this paper, see Corollary 5.8. We note that a powerful metacyclic 2-group G has no abelian maximal subgroups if and only if $|G'| > 2$ by using Lemma 4.4.

Corollary 4.10. *Let G be a powerful metacyclic 2-group with $|G'| > 2$, then $\omega(G) = \frac{3}{2}|G'|$.*

Proof. This follows from Theorem 4.8 and the fact that $\{ba^i | 0 \leq i \leq |G'| - 1\} \cup \{b^{2t}a | 0 \leq t \leq \frac{|G'|}{2} - 1\}$ is a subset of pairwise non-commuting elements in G. $\square$

5. Metacyclic 2-groups with no abelian maximal subgroups

In this section we determine $\omega(G)$ when G is a metacyclic 2-group with no abelian maximal subgroups. Recall that $G = \langle b, a \rangle$, $\langle a \rangle \trianglelefteq G$, $b^{-1}ab = a^k$, $1 < k < |a|$ and $k + 1 = 2^n \ell$, where $n \geq 1$ and ℓ is an odd number. Moreover by Theorem 2.2(ii), $|G'| \geq 2^{n+1}$ since G has no abelian maximal subgroups. Also $|a| > |G'|$ and $|b| \geq \frac{|G'|}{2^{n-1}}$ by Theorem 2.2(ii) and Lemma 2.4.

On setting $A = \{a, b, ba, ba^2, \ldots, ba^{|G'|-1}\}$, we know that A is a subset of pairwise non-commuting elements in any metacyclic 2-group G by Lemma 2.6. Now in the two following lemmas we add more pairwise non-commuting elements to this subset.

Lemma 5.1. *Let G be a metacyclic 2-group with no abelian maximal subgroups. Then on setting $B = \{b^2a^r \mid 1 \leq r \leq \frac{|G'|}{2^n}, 2^n \nmid r\}$, we have $A \cup B$ is a subset of pairwise non-commuting elements in G. Moreover*

$$|B| = \begin{cases} \frac{|G'|}{2^n} - \frac{|G'|}{2^{2n}} & |G'| \geq 2^{2n} \\ \frac{|G'|}{2^n} & |G'| < 2^{2n}. \end{cases}$$

Proof. First we see that the elements of B are pairwise non-commuting. For otherwise if $[b^2a^r, b^2a^{r'}] = 1$, then $\frac{|G'|}{2^n} \mid r - r'$ by Lemma 2.1(iv), which is a contradiction. Also the elements of A and B do not commute since $2^n \nmid r$. Now we see that the number of r in B such that $2^n | r$ is $\frac{|G'|/2^n}{2^n}$ when $|G'| \geq 2^{2n}$ and is 0 when $\frac{|G'|}{2^n} < 2^n$, which completes the proof. $\square$

Lemma 5.2. *Let G be as in Lemma 5.1. Then on setting $C = \{b^{4t}a \mid 1 \leq t < \frac{|G'|}{2^{n+1}}\}$, we have that $A \cup B \cup C$ is a subset of pairwise non-commuting elements in G. Moreover $|C| = \frac{|G'|}{2^{n+1}} - 1$.*

Proof. First we see that if $|G'| = 2^{n+1}$, then $|C| = 0$ and $A \cup B \cup C$ is a subset of pairwise non-commuting elements. Therefore we assume that $|G'| > 2^{n+1}$. Suppose to the contrary that two elements of C commute. For example if $[b^{4t}a, b^{4t'}a] = 1$ with $1 \leq t' < t < \frac{|G'|}{2^{n+1}}$, then $|G'|$ divides $\sum_{i=0}^{4(t-t')-1} k^i$ by Lemma 2.1(iv). On setting $t - t' = 2^s u$, where $s \geq 0$ and u is odd, we see that $\sum_{i=0}^{4(t-t')-1} k^i = 2^{s+1}u'(1+k)$, where u' is odd by Lemma 2.3(iii). Therefore $\frac{|G'|}{2^{n+1}} \mid t - t'$ which is a contradiction. Now we prove that the elements of A and C do not commute. For otherwise if $[b^{4t}a, ba^i] = 1$ for $0 \leq i \leq |G'| - 1$, then $i \sum_{j=0}^{4t-1} k^j \equiv 1 \pmod{|G'|}$. This yields that $4ti \equiv 1 \pmod 2$ by Lemma 2.1(i), a contradiction. Also if $[b^{4t}a, a] = 1$, then by Lemmas 2.1(ii) and 2.3(iii), we see that $\frac{|G'|}{2^{n+1}} \mid t$ which is impossible. Moreover the elements of B and C do not commute. For otherwise if $[b^{4t}a, b^2a^r] = 1$, where $t = 2^s u$ in which $s \geq 0$ and u is odd, then $r \sum_{i=0}^{4t-1} k^i \equiv 1 + k \pmod{|G'|}$. Therefore by Lemma 2.3(iii), $r2^{s+1}u' \equiv 1 \pmod{\frac{|G'|}{2^n}}$, where u' is odd, which is a contradiction, since $2 \mid \frac{|G'|}{2^n}$ and ℓ is odd. The rest is clear. $\square$

Corollary 5.3. *Let G be as in Lemma 5.1. Then $\omega(G) \geq \lambda$, where*

$$\lambda = \begin{cases} \frac{|G'|}{2^{2n}}(2^{2n} + 3.2^{n-1} - 1) & |G'| \geq 2^{2n} \\ \frac{|G'|}{2^{n+1}}(2^{n+1} + 3) & |G'| < 2^{2n}. \end{cases}$$

Proof. This is an immediate consequence of Lemmas 5.1, 5.2 and 2.6. $\square$

Lemma 5.4. *Let G be as in Lemma 5.1 and $|H_3'| = 2$.*

(i) *If $n > 1$, then $\omega(G) = |G'| + 3$.*
(ii) *If $n = 1$, then $\omega(G) = 6$.*

Proof. (i) By Lemma 2.5, we see that H_3 is covered by three abelian maximal subgroups in which $\Phi(G)$ is one of them. Hence G is covered by $|G'| + 3$ abelian subgroups by Lemma 2.7. Moreover $|G'| = 2^{n+1}$ by Theorem 2.2(ii) and so $|G'| < 2^{2n}$. Therefore Corollary 5.3 completes the proof.
(ii) We see that G is powerful and $|G'| = 4$ by Lemma 4.4. Then $\omega(G) = 6$ by Corollary 4.10. $\square$

Lemma 5.5. *Let G be as in Lemma 5.1 with $|G'| \geq 2^{2n}$, $0 \leq r < \frac{|G'|}{2^n}$, $2^n \mid r$ and $b^{2j}a^s \in \mathcal{C}_{H_3}(b^2a^r)$. Then*

(i) $2^n \mid s$,
(ii) *if j is odd, then there exists $0 \leq i < |G'|$ such that $b^{2j}a^s \in \mathcal{C}_G(ba^i)$,*
(iii) *if j is even, then $b^{2j}a^s \in \Phi(H_3)$.*

Proof. (i) This follows from Lemmas 2.1(iv) and 2.3(iii).
(ii) By Lemma 2.3(iii), we have $1 + k + \cdots + k^{2j-1} = (1+k)u$, where u is an odd number. Recall that $(k+1) = 2^n\ell$, where ℓ is an odd number. Therefore there exists $0 \leq i < \frac{|G'|}{2^n}$ such that $\ell ui \equiv \frac{s}{2^n} \pmod{\frac{|G'|}{2^n}}$ since ℓu and $\frac{|G'|}{2^n}$ are coprime by the note in page 4. Thus $(1+k+\cdots+k^{2j-1})i \equiv s \pmod{|G'|}$ which completes the proof by using Lemma 2.1(iv).
(iii) This is evident from (i). □

Lemma 5.6. *Let G be as in Lemma 5.1. Then $\omega(G) \leq \lambda$, where*

$$\lambda = \begin{cases} \frac{|G'|}{2^{2n}}(2^{2n} + 3.2^{n-1} - 1) & |G'| \geq 2^{2n} \\ \frac{|G'|}{2^{n+1}}(2^{n+1} + 3) & |G'| < 2^{2n}. \end{cases}$$

Proof. By Lemma 5.4, we may assume that $|H_3'| \geq 4$. Therefore we may write $G = \bigcup_{i=0}^{|G'|-1} \mathcal{C}_G(ba^i) \bigcup_{r=0}^{|H_3'|-1} \mathcal{C}_{H_3}(b^2a^r) \bigcup_{t=0}^{\frac{|H_3'|}{2}-1} \mathcal{C}_{H_3}(b^{4t}a)$ by Corollary 4.9, Lemma 2.7 and the fact that $\Phi(G) \leq H_3$, where the elements of the union are all abelian subgroups of G. First suppose that $2^{2n} \leq |G'|$, then by Lemma 5.5, we see that $\mathcal{C}_{H_3}(b^2a^r) \subseteq \bigcup_{i=0}^{|G'|-1} \mathcal{C}_G(ba^i) \cup \Phi(H_3)$ for any r with $2^n|r$. This yields that

$$G = \bigcup_{i=0}^{|G'|-1} \mathcal{C}_G(ba^i) \bigcup_{\substack{r=0 \\ 2^n \nmid r}}^{|H_3'|-1} \mathcal{C}_{H_3}(b^2a^r) \bigcup_{t=0}^{\frac{|H_3'|}{2}-1} \mathcal{C}_{H_3}(b^{4t}a) \cup \Phi(H_3).$$

Hence

$$H_3 = G \cap H_3 = \bigcup_{i=0}^{|G'|-1} \mathcal{C}_{H_3}(ba^i) \bigcup_{\substack{r=0 \\ 2^n \nmid r}}^{|H_3'|-1} \mathcal{C}_{H_3}(b^2a^r) \bigcup_{t=0}^{\frac{|H_3'|}{2}-1} \mathcal{C}_{H_3}(b^{4t}a)$$

by Lemmas 4.2 and 4.5. Also $G = \bigcup_{i=0}^{|G'|-1} \mathcal{C}_G(ba^i) \cup H_3$ by Lemma 2.7. This implies that

$$G = \bigcup_{i=0}^{|G'|-1} \mathcal{C}_G(ba^i) \bigcup_{\substack{r=0\\ 2^n \nmid r}}^{|H_3'|-1} \mathcal{C}_{H_3}(b^2a^r) \bigcup_{t=0}^{\frac{|H_3'|}{2}-1} \mathcal{C}_{H_3}(b^{4t}a).$$

and so G is covered with $|G'| + (\frac{|G'|}{2^n} - \frac{\frac{|G'|}{2^n}}{2^n}) + \frac{\frac{|G'|}{2^n}}{2}$ abelian subgroups since $|H_3'| = \frac{|G'|}{2^n}$, as desired. Now suppose that $|G'| < 2^{2n}$, then $2^n \nmid r$ for $0 \le r < |H_3'|$ since $|H_3'| = \frac{|G'|}{2^n}$ by Theorem 2.2(ii). Thus by using the first union, G is covered with $|G'| + \frac{|G'|}{2^n} + \frac{\frac{|G'|}{2^n}}{2}$ abelian subgroups, as desired. □

Corollary 5.7. *Let G be a metacyclic 2-group with no abelian maximal subgroups. Then*

$$\omega(G) = \begin{cases} \frac{|G'|}{2^{2n}}(2^{2n} + 3.2^{n-1} - 1) & |G'| \ge 2^{2n} \\ \frac{|G'|}{2^{n+1}}(2^{n+1} + 3) & |G'| < 2^{2n}. \end{cases}$$

Proof. This follows from Corollary 5.3 and Lemma 5.6. □

Corollary 5.8. *Let G be a powerful metacyclic 2-group with no abelian maximal subgroups, then $\omega(G) = \frac{3}{2}|G'|$.*

Proof. We see that $n = 1$ and $|G'| > 2$ by Lemma 4.4. Therefore the first part of Corollary 5.7 completes the proof. □

Proof of Theorem 1.1. This is an immediate consequence of Theorem 3.4 and Corollary 5.7. □

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