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A DEGREE CONDITION FOR GRAPHS TO HAVE CONNECTED (g, f)-FACTORS

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ABSTRACT. Let G be a graph of order n, a and b be integers with $1 \le a < b$ and $b \ge 3$, $g(x)$ and $f(x)$ be two integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$, for each $x \in V(G)$ and $f(V(G)) - V(G)$ even. We prove that G has a connected (g, f)-factor if the minimum degree $\delta(G)$ satisfies $\delta(G) \ge \frac{(b-1)n}{a+b-1}$ and $n \ge \frac{(a+b-1)^2+1}{a}$.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$, the set of vertices and the set of edges, respectively. For a vertex $x \in V(G)$, we write $N_G(x)$ for the set of vertices adjacent to x in G, $N_G[x]$ for $N_G(x) \cup \{x\}$, and $d_G(x) = |N_G(x)|$ for the degree of x in G. The minimum degree of vertices in G is denoted by $\delta(G)$. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S, T)$, the number of edges joining S and T. For a subset $S \subseteq V(G)$, we denote by $G - S$, the subgraph obtained from G by deleting the vertices in S together with the edges incident to the vertices in S.

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Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$, for each $x \in V(G)$. A (g,f)-factor of graph G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$, for each $x \in V(G)$ (where, of course d_F denotes the degree in F). If F is connected, then we call it a connected (g,f)-factor. For convenience, we write $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$, $f(S) = \sum_{x \in S} f(x)$ and $f(T) = \sum_{x \in T} f(x)$. Some terminologies and notations not given here can be found in [1].

Many authors have investigated factors [2,6,8,10,14,15,16], connected factors [5,11], and factorizations [13]. Here, we study conditions on the minimum degree and the order of a graph G which guarantee the existence of a connected (g,f)-factor in G. We begin with some known results.

Theorem 1.1. [9] *A graph G has a (g,f)-factor if and only if*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) - h_G(S,T) \geq 0,$$

for any disjoint subsets S and T of $V(G)$, where $h_G(S,T)$ denotes the number of components C of $G-(S \cup T)$ such that $g(x) = f(x)$, for all $x \in V(C)$ and $e_G(T, V(C)) + f(V(C))$ is odd. Furthermore, if $g(x) = f(x)$, for each $x \in V(G)$, then $\delta_G(S,T) = f(V(G))$ (mod 2).

Theorem 1.2. [12] *Let G be a graph of order $n \geq 3$. If for each pair of nonadjacent vertices x and y of G,*

$$d_G(x) + d_G(y) \geq n,$$

then G has a Hamiltonian cycle.

Theorem 1.3. [7] *Let G be a graph, g and f be two positive integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x) \leq d_G(x)$, for each $x \in V(G)$. If G has both a (g,f)-factor and a Hamiltonian path, then G contains a connected $(g, f+1)$-factor.*

Theorem 1.4. [3] *Let $k \geq 1$ be an integer, G a graph of order n with kn even and $n \geq 4k-5$. If*

$$\delta(G) \geq \frac{n}{2},$$

then G has a k-factor.

Theorem 1.5. [4] *Let G be a graph of order n, a and b be integers with $1 \leq a \leq b$. Let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$, for each $x \in V(G)$ and $f(V(G)) \equiv 0$ (mod 2). If*

$$n > \frac{(a+b)(a+b-3)}{a},$$

and

$$\delta(G) \geq \frac{bn}{a+b},$$

then G has an f-factor.

We prove the following theorem for the existence of a connected (g, f)-factor, which is an extension of Theorem 1.4 and Theorem 1.5.

Theorem 1.6. *Let G be a graph of order n, a and b be two integers with $1 \leq a < b$ and $b \geq 3$. Let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$, for each $x \in V(G)$ and $f(V(G)) - V(G)$ even. If*

$$n \geq \frac{(a+b-1)^2+1}{a},$$

and

$$\delta(G) \geq \frac{(b-1)n}{a+b-1},$$

then G has a connected (g, f)-factor.

2. **Proof of Theorem 1.6**

We now prove Theorem 1.6. We assume that G satisfies the conditions of Theorem 1.6. Since $\delta(G) \geq \frac{(b-1)n}{a+b-1}$, then we have,

$$d_G(x) + d_G(y) \geq 2\frac{(b-1)n}{a+b-1} \geq \frac{(a+b-1)n}{a+b-1} = n,$$

for each pair of nonadjacent vertices x and y of G. By Theorem 1.2, G has a Hamiltonian cycle. Hence, by Theorem 1.3, to prove Theorem 1.6, we need only to prove that G has a $(g, f-1)$-factor.

We begin to prove that G has a $(g, f-1)$-factor. Suppose that G satisfies the conditions of Theorem 1.6, but it has no $(g, f-1)$-factor.

Then, according to Theorem 1.1, there exist two disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = f(S) - |S| + d_{G-S}(T) - g(T) - h_G(S,T) \le -1. \tag{2.1}$$

In view of the conditions of Theorem 1.6 and (2.1), we obtain,

$$a|S| + d_{G-S}(T) - (b-1)|T| - \omega \le \delta_G(S,T) \le -1, \tag{2.2}$$

where ω denotes the number of components of $G-(S\cup T)$. Clearly,

$$\omega \le n - |S| - |T|. \tag{2.3}$$

Case 1. $\omega = 0$.

In this case, we have $n = |S| + |T|$. Obviously, $T \neq \varnothing$. Otherwise, by (2.2) we get $a|S| \le -1$, which is a contradiction.

In view of (2.2), we obtain,

$$-1 \ge a|S| + d_{G-S}(T) - (b-1)|T| \ge a(n-|T|) - (b-1)|T| = an - (a+b-1)|T|,$$

and so

$$|T| \ge \frac{an+1}{a+b-1}.$$

For each $x \in T$, we get,

$$d_{G-S}(x) + |S| \ge \delta(G) \ge \frac{(b-1)n}{a+b-1}.$$

Thus, we obtain,

$$d_{G-S}(T) \ge \frac{(b-1)n}{a+b-1}|T| - |S||T| = \frac{(b-1)n}{a+b-1}|T| - (n-|T|)|T|. \tag{2.4}$$

According to (2.2), (2.4) and $n = |S| + |T|$, we get,

$$\begin{aligned}
-1 &\ge \delta_G(S,T) \ge a|S| + d_{G-S}(T) - (b-1)|T| \\
&\ge a(n-|T|) + \frac{(b-1)n}{a+b-1}|T| - (n-|T|)|T| - (b-1)|T| \\
&= |T|^2 - (\frac{an}{a+b-1} + a + b - 1)|T| + an.
\end{aligned}$$

Let $f(|T|) = |T|^2 - (\frac{an}{a+b-1} + a + b - 1)|T| + an$. Since $|T| \ge \frac{an+1}{a+b-1}$ and $n \ge \frac{(a+b-1)^2+1}{a}$, then $f(|T|)$ attains its minimum value at $|T| = \frac{an+1}{a+b-1}$.

Thus, we have,

$$\begin{aligned}
-1 &\geq f(|T|) \geq f(\frac{an+1}{a+b-1}) \\
&= (\frac{an+1}{a+b-1})^2 - (\frac{an}{a+b-1} + a+b-1)(\frac{an+1}{a+b-1}) + an \\
&= (\frac{an+1}{a+b-1})(\frac{an+1}{a+b-1} - (\frac{an}{a+b-1} + a+b-1)) + an \\
&= (\frac{an+1}{a+b-1})(\frac{1}{a+b-1} - (a+b-1)) + an \\
&= \frac{an+1}{(a+b-1)^2} - (an+1) + an \\
&= \frac{an+1}{(a+b-1)^2} - 1 \geq \frac{(a+b-1)^2+2}{(a+b-1)^2} - 1 \\
&= \frac{2}{(a+b-1)^2} > 0,
\end{aligned}$$

a contradiction.

Case 2. $\omega \geq 1$.

Let m denote the minimum order of components of $G-(S\cup T)$. Then,

$$m \leq \frac{n-|S|-|T|}{\omega}, \tag{2.5}$$

and

$$\delta(G) \leq m-1+|S|+|T|. \tag{2.6}$$

Subcase 2.1. $T = \varnothing$.

Claim 1. $S \neq \varnothing$.

Proof. Assume that $S = \varnothing$. Since G has a Hamiltonian cycle and $\delta_G(S,T) \leq -1$, then $h_G(S,T) = 1$. Obviously, $f(x)-1 = g(x)$, for each $x \in V(G)$ by the definition of $h_G(S,T)$. Hence, $\delta_G(S,T) = -1$. On the other hand, since $f(V(G)) - V(G)$ is even, then in view of Theorem 1.1, $\delta_G(S,T)$ is even, which is a contradiction.

According to (2.2) and (2.3), we obtain,

$$a|S|+1 \leq \omega \leq n-|S|. \tag{2.7}$$

Hence, we have by (2.5), (2.6), (2.7) and $\delta(G) \geq \frac{(b-1)n}{a+b-1}$ that

$$\begin{aligned}
\frac{(b-1)n}{a+b-1} &\leq \delta(G) \leq m-1+|S| \\
&\leq \frac{n-|S|}{\omega} - 1 + |S| \leq \frac{n-|S|}{a|S|+1} - 1 + |S| \\
&= \frac{n-1}{a+1} - \frac{a(|S|-1)(n-1-a|S|-|S|)}{(a+1)(a|S|+1)}.
\end{aligned}$$

Since $n-1-a|S|-|S| \geq 0$ by (2.7), and $|S|-1 \geq 0$ by Claim 1, then it follows:

$$\frac{n}{2} \leq \frac{(b-1)n}{a+b-1} \leq \frac{n-1}{a+1}.$$

This is a contradiction, since $a \geq 1$.

Subcase 2.2. $T \neq \varnothing$.

Let $h = min\{d_{G-S}(x)|x \in T\}$. Then, obviously,

$$\delta(G) \leq h + |S|. \tag{2.8}$$

Subcase 2.2.1. $h = 0$.

According to (2.2) and (2.3), we have,

$$\begin{aligned}
-1 &\geq \delta_G(S,T) \geq a|S| + d_{G-S}(T) - (b-1)|T| - \omega \\
&\geq a|S| - (b-1)|T| - \omega \\
&\geq a|S| - (b-1)|T| - (n-|S|-|T|) \\
&= (a+1)|S| - (b-2)|T| - n.
\end{aligned}$$

Combining this with (2.3) and $\omega \geq 1$, we get,

$$\begin{aligned}
-1 &\geq (a+1)|S| - (b-2)|T| - n \\
&\geq (a+1)|S| - (b-2)(n-1-|S|) - n \\
&= (a+b-1)|S| - (b-2)(n-1) - n.
\end{aligned}$$

Combining this with (2.8), $h = 0$ and $\delta(G) \geq \frac{(b-1)n}{a+b-1}$, we obtain,

$$\begin{aligned}
-1 &\geq (a+b-1)|S| - (b-2)(n-1) - n \\
&\geq (a+b-1)\frac{(b-1)n}{a+b-1} - (b-2)(n-1) - n \\
&= (b-1)n - (b-2)(n-1) - n \\
&= b-2 > 0,
\end{aligned}$$

a contradiction.

Subcase 2.2.2. $1 \le h \le b-2$.
In view of (2.2), (2.3), (2.8) and the fact that $b-1-h \ge 1$, we have,

$$\begin{aligned} \delta_G(S,T) &\ge a|S| + d_{G-S}(T) - (b-1)|T| - \omega \\ &\ge a|S| + h|T| - (b-1)|T| - (b-1-h)(n-|S|-|T|) \\ &= (a+b-1-h)|S| - (b-1-h)n \\ &\ge (a+b-1-h)(\delta(G)-h) - (b-1-h)n \\ &\ge (a+b-1-h)(\frac{(b-1)n}{a+b-1} - h) - (b-1-h)n. \end{aligned}$$

Let $g(h) = (a+b-1-h)(\frac{(b-1)n}{a+b-1} - h) - (b-1-h)n$. Then, according to $n \ge \frac{(a+b-1)^2+1}{a}$ and $1 \le h \le b-2$,

$$\begin{aligned} g'(h) &= -\frac{(b-1)n}{a+b-1} + h - (a+b-1) + h + n \\ &= \frac{an}{a+b-1} + 2h - (a+b-1) \\ &\ge \frac{(a+b-1)^2+1}{a+b-1} + 2 - (a+b-1) \\ &> a+b-1+2-(a+b-1) = 2 > 0. \end{aligned}$$

Hence, $g(h)$ attains its minimum value at $h=1$. Thus, we get,

$$\begin{aligned} \delta_G(S,T) &\ge g(h) \ge g(1) \\ &= (a+b-2)(\frac{(b-1)n}{a+b-1} - 1) - (b-2)n \\ &= \frac{(b-1)(a+b-2)n - (b-2)(a+b-1)n}{a+b-1} - (a+b-2) \\ &= \frac{an}{a+b-1} - (a+b-2) \\ &\ge \frac{(a+b-1)^2+1}{a+b-1} - (a+b-2) \\ &= \frac{a+b}{a+b-1} > 0. \end{aligned}$$

This contradicts (2.1).

Subcase 2.2.3. $h = b-1$.
By (2.2), we get,

$$\omega \ge a|S| + 1. \tag{2.9}$$

According to (2.5) and (2.9), we obtain,

$$1 \leq m \leq \frac{n-|S|-|T|}{\omega} \leq \frac{n-|S|-1}{a|S|+1}, \tag{2.10}$$

and so,

$$|S| \leq \frac{n-2}{a+1}. \tag{2.11}$$

By $\delta(G) \geq \frac{(b-1)n}{a+b-1}$ and (2.8), we have,

$$\frac{(b-1)n}{a+b-1} \leq \delta(G) \leq h+|S| = b-1+|S|,$$

and so,

$$|S| \geq \frac{(b-1)n}{a+b-1} - b + 1. \tag{2.12}$$

Claim 2. $\frac{(b-1)n}{a+b-1} - b + 1 \geq \frac{2n}{a+2} - 2$.

Proof. By $n \geq \frac{(a+b-1)^2+1}{a}$ and $b \geq 3$, we have,

$$\begin{aligned}
\frac{(b-1)n}{a+b-1} - b + 1 - (\frac{2n}{a+2} - 2) &= \frac{(b-1)n}{a+b-1} - \frac{2n}{a+2} - (b-3) \\
&= \frac{(a+2)(b-1)n - 2(a+b-1)n}{(a+2)(a+b-1)} \\
&\quad -(b-3) \\
&= \frac{(b-3)an}{(a+2)(a+b-1)} - (b-3) \\
&\geq \frac{(b-3)[(a+b-1)^2+1]}{(a+2)(a+b-1)} - (b-3) \\
&\geq \frac{(b-3)(a+b-1)^2}{(a+2)(a+b-1)} - (b-3) \\
&= \frac{(b-3)(a+b-1)}{a+2} - (b-3) \geq 0.
\end{aligned}$$

From Claim 2 and (2.12), we get,

$$|S| \geq \frac{2n}{a+2} - 2. \tag{2.13}$$

In view of (2.11) and (2.13), we have,

$$\frac{n-2}{a+1} \geq \frac{2n}{a+2} - 2,$$

which implies,

$$n \leq 2(a+2). \tag{2.14}$$

On the other hand, we obtain by $b \geq 3$ and $b-1 \geq a$ so that

$$n \geq \frac{(a+b-1)^2+1}{a} > \frac{(a+b-1)^2}{a} \geq 2(a+2),$$

which contradicts (2.14).

Subcase 2.2.4. $h \geq b$.

According to (2.2), we get,

$$a|S| + (h-b+1)|T| - \omega \leq -1, \tag{2.15}$$

and so,

$$\omega \geq a|S| + |T| + 1 \geq |S| + |T| + 1. \tag{2.16}$$

Suppose that $m \geq 2$. Then, in view of (2.5), (2.6), (2.16) and $T \neq \varnothing$, we have,

$$\begin{aligned} \delta(G) &\leq m - 1 + |S| + |T| \leq m + \omega - 2 \\ &\leq m + \omega - 2 + \frac{1}{2}(m-2)(\omega-2) = \frac{1}{2}m\omega \\ &\leq \frac{1}{2}(n - |S| - |T|) < \frac{n}{2}. \end{aligned}$$

This contradicts $\delta(G) \geq \frac{(b-1)n}{a+b-1} \geq \frac{n}{2}$. Thus, we may assume that $m = 1$. Then, from (2.3) and (2.16), we obtain,

$$|S| + |T| + 1 \leq \omega \leq n - |S| - |T|.$$

Thus, we have,

$$|S| + |T| \leq \frac{n-1}{2}.$$

From (2.6), we get,

$$\frac{n}{2} \leq \frac{(b-1)n}{a+b-1} \leq \delta(G) \leq |S| + |T| \leq \frac{n-1}{2} < \frac{n}{2},$$

which is a contradiction, completing the proof of Theorem 1.6.

Remark 2.1. In the proof of Theorem 1.6, it is required that $\delta(G) \geq \frac{(b-1)n}{a+b-1}$. We do not know whether the condition can be improved.

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