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Author(s):

B. Bagheri Gh. and B. Omoomi

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ON THE ORIENTED PERFECT PATH DOUBLE COVER CONJECTURE

B. BAGHERI GH. AND B. OMOOMI*

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Abstract. An oriented perfect path double cover (OPPDC) of a graph G is a collection of directed paths in the symmetric orientation G_s of G such that each arc of G_s lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as an end of a path. Maxová and Nešetřil (Discrete Math. 276 (2004) 287-294) conjectured that every graph except two complete graphs K_3 and K_5 has an OPPDC and they claimed that the minimum degree of the minimal counterexample to this conjecture is at least four. In the proof of their claim, when a graph is smaller than the minimal counterexample, they missed to consider the special cases K_3 and K_5. In this paper, among some other results, we present the complete proof for this fact. Moreover, we prove that the minimal counterexample to this conjecture is 2-connected and 3-edge-connected.

Keywords: Perfect path double cover, Oriented perfect path double cover, Oriented cycle double cover.
MSC(2010): Primary: 05C38; Secondary: 05C70.

1. **Introduction**

We denote by $G = (V, E)$ a finite undirected graph with no loops or multiple edges. The symmetric orientation of G, denoted by G_s, is an oriented graph obtained from G by replacing each edge of G by a pair of opposite directed arcs.

A cycle double cover (CDC) of a graph G is a collection of its cycles such that each edge of G lies in exactly two of the cycles. A well-known

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*Corresponding author.

conjecture of Seymour [7] asserts that every simple bridgeless graph has a CDC. This problem also motivated several related conjectures. A small cycle double cover (SCDC) of a graph on n vertices is a CDC with at most $n-1$ cycles. Bondy conjectured that every simple bridgeless graph has an SCDC [1].

An oriented cycle double cover (OCDC) of G is a collection of directed cycles in G_s of length at least 3 such that each arc of G_s lies in exactly one of the cycles. Jaeger [3] conjectured that every bridgeless graph has an oriented cycle double cover. An small oriented cycle double cover (SOCDC) of a graph G on n vertices is an OCDC with at most $n-1$ elements.

A perfect path double cover (PPDC) of a graph G is a collection $\mathcal{P}$ of paths in G such that each edge of G belongs to exactly two members of $\mathcal{P}$ and each vertex of G occurs exactly twice as an end of a path in $\mathcal{P}$ [2]. In [4] it is proved that every simple graph has a PPDC. The existence of a PPDC for graphs in general is equivalent to the existence of an SCDC for bridgeless graphs with a vertex joined to all other vertices.

Definition 1.1. [5] *An* oriented perfect path double cover (OPPDC) *of a graph G is a collection of directed paths in the symmetric orientation G_s such that each arc of G_s lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as an end of a path.*

Similar to the above, it can be seen that the existence of an OPPDC for graphs in general is equivalent to the existence of an SOCDC for bridgeless graphs with a vertex joined to all other vertices. Maxová and Nešetřil in [5] showed that two complete graphs K_3 and K_5 have no OPPDC, and in [6], they conjectured the following statement.

Conjecture 1.2. [6] (OPPDC conjecture) *Every connected graph except K_3 and K_5 has an* OPPDC*.*

In the following theorem, a list of sufficient conditions for a graph to admit an OPPDC is provided.

Theorem A. [5] *Let $G \neq K_3$ be a graph. In each of the following cases, G has an* OPPDC*.*

- **(i)** *Each vertex of G has odd degree.*
- **(ii)** *G arises from a graph G' which has an* OPPDC *by dividing one edge of G'.*
- **(iii)** *$G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$ which G_i is a graph with an* OPPDC*, for $i = 1, 2$.*

(iv) *$G \setminus v$ has an* OPPDC, *for some $v \in V(G)$ of degree less than* 3.

In [5], Maxová and Nešetřil in the following two theorems proved that if a graph of order n with a vertex v of degree 3 has no OPPDC, then there exists a graph of order $n-1$ which has no OPPDC either.

Theorem B. [5] *Let G be a graph, $v \in V(G)$ be a vertex of degree* 3*, and $N(v) = \{x, y, z\}$ induces K_3 in G. If $G \setminus v$ has an* OPPDC*, then G has also an* OPPDC.

Theorem C. [5] *Let G be a graph, $v \in V(G)$ be a vertex of degree* 3*, $N(v) = \{x, y, z\}$, and $e = xz \notin E(G)$. If $(G \setminus v) \bigcup \{e\}$ has an* OPPDC*, then G also has an* OPPDC.

The structure of this paper is as follows. In Section 2, the properties of the minimal counterexample to the OPPDC conjecture are studied and it is proved that such graphs are 2-connected and 3-edge-connected with minimum degree at least four. In Section 3, some sufficient conditions for a graph to admit an OPPDC are provided.

2. **The minimal counterexample to the** OPPDC **conjecture**

In this section, among some other results, we prove that the minimal counterexample to the OPPDC conjecture is 2-connected and 3-edge-connected with minimum degree at least four.

Suppose that G is a minimal counterexample to the OPPDC conjecture and G' is a graph smaller than G. Since G' can not be a counterexample to the conjecture, either G' has an OPPDC or $G' \in \{K_3, K_5\}$. In [5] as a corollary of Theorems A(iv), B and C it is concluded that the minimum degree of the minimal counterexample to the OPPDC conjecture is at least four, but the cases $G' \in \{K_3, K_5\}$ are missed to investigate. In the following theorem along with the missing cases, we give the complete proof for this result.

Theorem 2.1. *If G is the minimal counterexample to the* OPPDC *conjecture, then $\delta(G) \geq 4$.*

Proof. On the contrary, let G be a minimal counterexample to the OPPDC conjecture that contains a vertex t of degree less than three and let $G' = G \setminus t$. Hence, either G' has an OPPDC or $G' \in \{K_3, K_5\}$. In the former case by Theorem A(iv), G has an OPPDC. In the latter case, G is one of the graphs G_1, G_2, G_3 or G_4, shown in Figure 1. In each cases $\mathcal{P}_i$, $1 \leq i \leq 4$, is an OPPDC of G_i, where

$\mathcal{P}_1 = \{tuyvxw, uxywvt, vuwyx, wxuvy, xvwu, yutv\}$,
$\mathcal{P}_2 = \{tvwu, uv, vtuw, wvut\}$, $\mathcal{P}_3 = \{tuw, uv, vwu, wvut\}$ and
$\mathcal{P}_4 = \{tuyxw, ut, vxuwy, wxvyu, xywvut, yvwux\}$. This contradicts our assumption, thus the minimum degree of G is at least three.

Now let $t \in V(G)$ with $\deg(t) = 3$ and $G' = G \setminus t$. If the neighbours of t induce K_3, and G' has an OPPDC, then by Theorem B, G admits an OPPDC. Otherwise, if $G' = K_3$, then $G = K_4$ which has an OPPDC and if $G' = K_5$, then $G = G_5$ and
$\mathcal{P}_5 = \{twvxu, uwyxvt, vuxyw, wtuyv, xwuvy, yutvwx\}$ is an OPPDC of G.

If there are $u, v \in N(t)$ such that $e = uv \notin E(G)$, then $G' = (G \setminus t) \bigcup \{e\}$ is smaller than G. If G' has an OPPDC, then by Theorem C, G admits an OPPDC. Otherwise, $G' \in \{K_3, K_5\}$. In these cases $G \in \{G_6, G_7\}$, where $\mathcal{P}_6 = \{tw, uwvt, vwtu, wutv\}$ and $\mathcal{P}_7 = \{tuxw, utwyv, vwxyu, wuyxvt, xuwvy, ywtvx\}$ are OPPDC of G, respectively.

All above cases contradict our assumption that G has no OPPDC. Therefore, $\delta(G) \geq 4$. □

The complete graphs K_3 and K_5 are the only known examples of connected graphs which have no OPPDC. By Theorem A(i), K_{2n} has an OPPDC. It is known that every symmetric orientation of K_{2n+2}, $n \geq 3$, has a decomposition into $2n + 1$ directed Hamiltonian cycles [8]. This decomposition forms an OPPDC for K_{2n+1}, $n \geq 3$, by deleting a fix vertex from each cycle.

By Theorem A(iii), if every block of graph G has an OPPDC, then G also has an OPPDC. Remind that a block is a maximal connected subgraph of G with no cut-vertex. Let G be the minimal counterexample to the OPPDC conjecture. Therefore, G, either is 2-connected or at least one of its blocks is K_3 or K_5. In the following theorem, we show that the latter can not happen.

For every OPPDC of a connected graph G, say $\mathcal{P}$, and every vertex $v \in V$, let P^v and P_v denote the paths in $\mathcal{P}$ beginning and ending with v, respectively. Also note that we can assume, in an OPPDC, directed paths of length zero are presented only at isolated vertices.

Theorem 2.2. *The minimal counterexample to the* OPPDC *conjecture is* 2*-connected.*

Proof. Let G be the minimal counterexample to the OPPDC conjecture. On the contrary, suppose that, $G = B_1 \cup \ldots \cup B_k$ and B_i's are blocks of

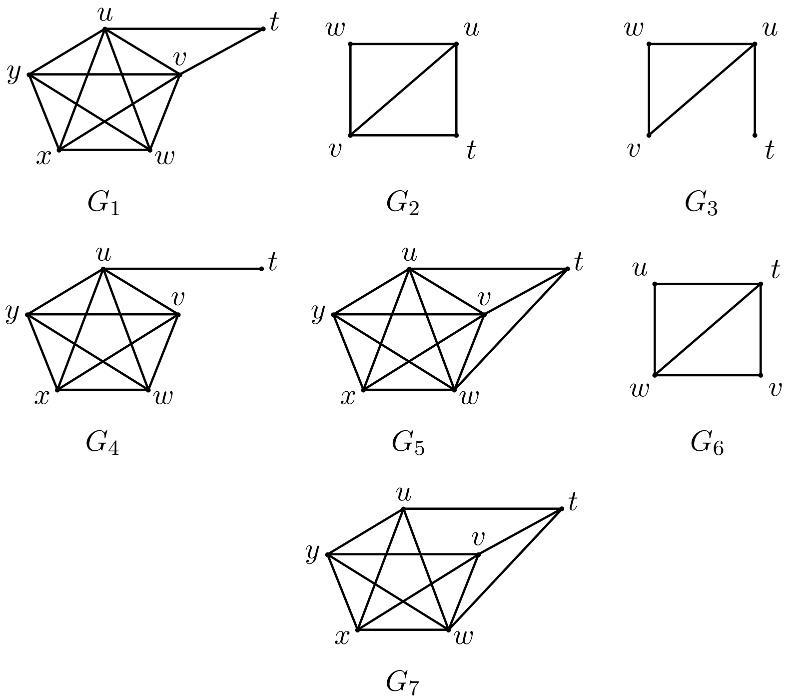

Figure 1. Special Cases.

G. If every block of G has an OPPDC, then by Theorem A(iii), G also has an OPPDC, which is a contradiction. Otherwise, at least one of the B_i's is K_3 or K_5. Since by Theorem 2.1, $\delta(G) \geq 4$, we need to consider the following cases.

If $k = 2$ and $B_1 = B_2 = K_5$, where $V(B_1) = \{u, v, w, x, y\}$ and $V(B_2) = \{u', v, w', x', y'\}$. Then the following is an OPPDC of G,

$$\begin{aligned}\mathcal{P} \;=\; & \{uxwyvy'w'x'u', ywxuvu'x'w'y', x'vx, u'vu, xyuwvw'u'y'x', y'v,\\ & wuyxvx'y'u'w', w'vw, vy\}.\end{aligned}$$

If $G = G_1 \cup G_2$, where $G_1 = K_5$ and G_2 has an OPPDC, then assume that $V(K_5) = \{u, v, w, x, y\}$, v is a cut vertex, and $\widetilde{\mathcal{P}}$ is an OPPDC of G_2. Also, let $\widehat{\mathcal{P}}$ be the OPPDC of $K_5 \setminus \{e = uv\}$ as given above. Consider two new directed paths $P_w = \widetilde{P}_v vu\widehat{P}^u$ and $P^u = uv\widetilde{P}^v$. Thus,

$$\mathcal{P} = \widehat{\mathcal{P}} \cup \widetilde{\mathcal{P}} \cup \{P_w, P^u\} \setminus \left\{\widetilde{P}^v, \widetilde{P}_v, \widehat{P}^u\right\}$$

is an OPPDC of G. In all above cases, we get a contradiction.

For $k \geq 3$, by the induction on k and Theorem A(iii), we find an OPPDC of G, which is a contradiction. □

Theorem 2.2 concludes the following corollaries. A block graph is a graph for which each block is a clique.

Corollary 2.3. *Every block graph $G \neq K_3, K_5$ has an* OPPDC.

Proof. In the proof of Theorem 2.2, we show that, if $G = B_1 \cup \ldots \cup B_k$ and B_i's are blocks of G, $\delta(G) \geq 4$ and for each i, $1 \leq i \leq k$, B_i has an OPPDC or $B_i = K_3$ or K_5, then G has an OPPDC. To complete the proof we need to consider the following remained cases.

If $k = 2$ and $B_1 = B_2 = K_3$, where $V(B_1) = \{u, v, w\}$ and $V(B_2) = \{w, x, y\}$, then, $\mathcal{P} = \{uwxy, ywvu, xw, wuv, vwyx\}$ is an OPPDC of G.

If $k = 2$, $B_1 = K_5$ and $B_2 = K_3$, where $V(B_1) = \{u, v, w, x, y\}$ and $V(B_2) = \{v, s, t\}$. Let $G' = B_1 \setminus \{e = uv\}$. Then the following is an OPPDC of G',

$$\widehat{\mathcal{P}} = \{uyxw, yvwux, wxvyu, xywv, vxuwy\}.$$

Consider four new directed paths. $P^t = tsvu\widehat{P}^u$, $P_t = vt$, $P_s = uvs$, and $P^s = stv\widehat{P}^v$. The following is an OPPDC of G,

$$\mathcal{P} = \widehat{\mathcal{P}} \cup \left\{P^t, P^s, P_t, P_s\right\} \setminus \left\{\widehat{P}^u, \widehat{P}^v\right\}.$$

Now, let $G = G_1 \cup G_2$, where $G_1 = K_3$ and G_2 has an OPPDC. Assume that $V(K_3) = \{u, v, w\}$, v is a cut vertex, and $\widetilde{\mathcal{P}}$ is an OPPDC of G_2. Now we define four new directed paths $P_u = \widetilde{P}_v vwu$, $P^u = uv$, $P^v = vuw$, and $P^w = wv\widetilde{P}^v$. Therefore,

$$\mathcal{P} = \widetilde{\mathcal{P}} \cup \{P_u, P^u, P^v, P^w\} \setminus \left\{\widetilde{P}^v, \widetilde{P}_v\right\}$$

is an OPPDC of G. Thus, the statement concludes. □

Since the line graph of every tree is a block graph, we have the following corollary.

Corollary 2.4. *For every tree $T \neq K_{1,3}, K_{1,5}$, $L(T)$ has an* OPPDC.

For line graphs, the following result is also obtained from Theorem A(iii).

Corollary 2.5. *If the degree of no adjacent vertices in G have the same parity, then the line graph $L(G)$ has an* OPPDC.

The following lemmas are necessary to prove our next theorem.

Lemma 2.6. *If* $G_1 = G_2 = K_5$ *and* $G = G_1 \cup G_2 \cup \{uu', vv'\}$, *where* $\{u, v\} \in V(G_1)$ *and* $\{u', v'\} \in V(G_2)$, *then* G *has an* OPPDC.

Proof. Let $G_1 = G_2 = K_5$, $V(G_1) = \{u, v, w, x, y\}$, and $V(G_2) = \{u', v', w', x', y'\}$. Then the following is an OPPDC of $G = G_1 \cup G_2 \cup \{uu', vv'\}$.

$$\begin{aligned} \mathcal{P} \;=\; & \{uxywvv'y'u'x'w', xvwu, wxuvy, yuu', vuwyx, v'x'y'w'u'uyvxw, \\ & x'u'w'v', w'x'v'u'y', y'v'v, u'v'w'y'x'\}. \end{aligned}$$

□

Lemma 2.7. *Let* $G_1 = K_5$ *and* G_2 *be a graph with an* OPPDC. *If* $G = G_1 \cup G_2 \cup \{uu', vv'\}$, *where* $\{u, v\} \in V(G_1)$ *and* $\{u', v'\} \in V(G_2)$, *then* G *has an* OPPDC.

Proof. Let $V(G_1) = \{u, v, w, x, y\}$, $\widehat{\mathcal{P}}$ be the OPPDC of $G_1 \setminus \{e = uv\}$ given in the proof of Theorem 2.2, and $\widetilde{\mathcal{P}}$ be an OPPDC of G_2. Now set four new directed paths. $P^u = uu'$, $P_v = \widetilde{P}_{u'}u'uv$, $P_w = \widetilde{P}_{v'}v'vu\widehat{P}^u$, and $P_{v'} = \widehat{P}_v vv'$. Thus,

$$\mathcal{P} = \widehat{\mathcal{P}} \cup \widetilde{\mathcal{P}} \cup \{P^u, P_v, P_w, P_{v'}\} \setminus \left\{\widehat{P}^u, \widehat{P}_v, \widetilde{P}_{u'}, \widetilde{P}_{v'}\right\}$$

is an OPPDC of G. □

By Theorem 2.2, the minimal counterexample to the OPPDC conjecture is bridgeless, therefore if G has an edge cut F of size 2, then the edges of F are vertex disjoint. In the next theorem, we show that G has no vertex disjoint edge cut of size 2.

Theorem 2.8. *The minimal counterexample to the* OPPDC *conjecture is* 3*-edge-connected.*

Proof. Let G be the minimal counterexample to the OPPDC conjecture. Suppose, on the contrary, that G has an edge cut of size 2, say F. By Theorems 2.1 and 2.2, F is vertex disjoint. Let $F = \{uv, wx\}$, and G_1 and G_2 be the components of $G \setminus F$ such that $u, w \in V(G_1)$.

If G_1 and G_2 have no OPPDC, then by minimality of G and by Theorem 2.1, G_1 and G_2 are isomorphic to K_5. Therefore by Lemma 2.6, G has an OPPDC which is a contradiction. Now without loss of generality, suppose that only G_1 has an OPPDC. By minimality of G and Theorem 2.1, G_2 is isomorphic to K_5; thus by Lemma 2.7, G has an OPPDC which is a contradiction.

It remains to consider the case that, G_1 and G_2 have an OPPDC, $\widehat{P}$ and $\widetilde{P}$, respectively. Now we define four new directed paths $P = \widehat{P}_u uv \widetilde{P}^v$, $P^v = vu$, $Q = \widehat{P}_w wx \widetilde{P}^x$, and $P^x = xw$. Therefore,

$$\mathcal{P} = \widehat{\mathcal{P}} \cup \widetilde{\mathcal{P}} \cup \{P, Q, P^v, P^x\} \setminus \left\{\widehat{P}_u, \widehat{P}_w, \widetilde{P}^v, \widetilde{P}^x\right\}$$

is an OPPDC of G. This contradiction implies that G is 3-edge-connected. □

3. Some sufficient conditions for existence of an OPPDC

In this section, we prove some sufficient conditions for a graph to admit an OPPDC. Since the minimal counterexample to the OPPDC conjecture is 2-connected, first we consider the OPPDC conjecture for 2-connected graphs.

An ear-decomposition of a 2-connected graph G is a decomposition of $E(G)$ to subgraphs $G_0 = C_0 \subset G_1 \subset \ldots \subset G_k = G$ such that C_0 is a cycle and for i, $2 \leq i \leq k$, $G_i \setminus G_{i-1}$ is a simple path in G_i, with only two distinct end vertices in G_{i-1}.

Theorem 3.1. *If a 2-connected graph G has an ear-decomposition $G_0 = C_0 \subset G_1 \subset \ldots \subset G_k = G$ such that $G_i \setminus G_{i-1} = P_i$ is a path of length at least 2, for $i = 1, \ldots, k$, and $C_0 \neq K_3$, then G has an* OPPDC.

Proof. We prove the statement by induction on k. For $k = 0$, G is a cycle and the following is an OPPDC of cycle $C = [v_1, v_2, \ldots, v_n]$.

$$\mathcal{P} = \{v_n v_{n-1}, v_{n-1} v_{n-2} \ldots v_2 v_1 v_n, v_{n-2} v_{n-1} v_n v_1\} \cup \left(\cup_{i=1}^{n-3} \{v_i v_{i+1}\}\right).$$

Now by induction on k and by Theorem A(iv) and (ii), an OPPDC of G is obtained. □

The following corollary provides a condition for every ear decomposition of the minimal counterexample to the OPPDC conjecture.

Corollary 3.2. *Every ear-decomposition of the minimal counterexample to the* OPPDC *conjecture has at least one ear of length* 1.

Theorem 3.3. *Let G be a connected graph. If $E(G)$ is partitioned to a cycle C of length at least* 4 *and a connected graph G' such that G' has an* OPPDC *and $|V(C) \setminus V(G')| \geq 2$, then G also has an* OPPDC.

Proof. If $|V(C) \cap V(G')| = 1$, then by Theorem A(iii), G has an OPPDC. Now, suppose that $|V(C) \cap V(G')| \geq 2$. Let $\widehat{\mathcal{P}}$ be an OPPDC of G' and $C = [v_1, v_2, \ldots, v_k]$.

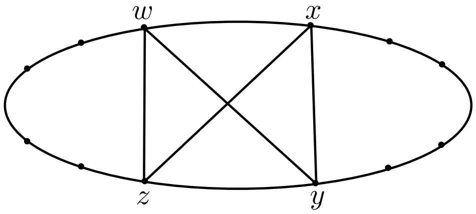

FIGURE 2. Every ear-decomposition of G has at least one ear of length 1.

If there exist two vertices v_i and v_j , $i<j$, in $V(C)\setminus V(G')$ and two vertices v_r and v_s in $V(C)\cap V(G')$, both of which in the same segment of C divided by v_i and v_j, then without loss of generality, we can assume that $1\le i<j<r<s\le k$. Thus, we can find an OPPDC for G as follows. Let $P^{v_i}=v_iv_{i-1}v_{i-2}\dots v_s\widehat{P}^{v_s}, P^{v_s}=v_sv_{s-1}\dots v_i$, $P^{v_j}=v_jv_{j+1}\dots v_r\widehat{P}^{v_r}$, and $P^{v_r}=v_rv_{r+1}\dots v_j$. Now, let $\widetilde{\mathcal{P}}^{v_i}$ and $\widetilde{\mathcal{P}}^{v_s}$ be the collections of directed paths obtained by breaking the paths P^{v_i} and P^{v_s} on the vertices of $V(C)\setminus(V(G')\cup\{v_j\})$. Thus, the following is an OPPDC of G,

$$\mathcal{P}=\widehat{\mathcal{P}}\cup\widetilde{\mathcal{P}}^{v_i}\cup\widetilde{\mathcal{P}}^{v_s}\cup\{P^{v_j},P^{v_r}\}\setminus\{\widehat{P}^{v_r},\widehat{P}^{v_s}\}.$$

Otherwise, $C=[v_1,v_2,v_3,v_4]$ and $V(C)\cap V(G')=\{v_1,v_3\}$. In this case, we define four new directed paths $P_{v_2}=v_1v_4v_3v_2$, $P^{v_2}=v_2v_1\widehat{P}^{v_1}$, $P^{v_4}=v_4v_1v_2v_3$, and $P_{v_4}=\widehat{P}_{v_3}v_3v_4$. Now, the following is an OPPDC of G.

$$\mathcal{P}=\widehat{\mathcal{P}}\cup\{P_{v_2},P^{v_2},P^{v_4},P_{v_4}\}\setminus\{\widehat{P}^{v_1},\widehat{P}_{v_3}\}.$$

□

Corollary 3.4. *Let G be a connected graph. If $E(G)$ is partitioned to a collection of cycles $\{C_1,C_2,\dots,C_k\}$ such that for each i, $2\le i\le k$, $|V(C_i)\setminus\cup_{j<i}V(C_j)|\ge 2$ and $C_1\ne K_3$, then G has an* OPPDC.

Example 1. *The graph G in Figure 2 is a 2-connected even graph that every ear-decomposition of G has at least one ear of length 1. In fact, in every ear-decomposition of G, at least one of the edges of the clique $\langle\{w,x,y,z\}\rangle$ is an ear. So the condition of Theorem 3.1 does not hold. On the other hand, let $C_1=[wxyz]$ and $C_2=E(G)\setminus E(C_1)$. In the cycle decomposition $\{C_1,C_2\}$ of G, $|V(C_2)\setminus V(C_1)|\ge 2$. Thus by Corollary 3.4, G has an* OPPDC.

In the following theorem, we give a sufficient condition for the existence of an OPPDC in graphs of minimum degree at most three.

Theorem 3.5. *If* $G \neq K_3$ *is a graph with* $\Delta(G) \leq 4$ *and* $\delta(G) \leq 3$*, then* G *has an* OPPDC*.*

Proof. We proceed by induction on the order of graph, n. For $n = 2$ the statement is trivial. For $n \geq 3$, suppose $\deg(v) = \delta(G) \leq 3$. If $d(v) = 1$ or 2, then $G' = G \setminus v$ is a graph of order $n-1$, $\Delta(G') \leq 4$, and $\delta(G') \leq 3$. Therefore, by the induction hypothesis G' has an OPPDC, and by Theorem A(iv), G also has an OPPDC.
Let $\deg(v) = 3$ and $N(v) = \{x, y, z\}$. Now, if $N(v)$ induces K_3, then by the induction hypothesis and by Theorem B, G has an OPPDC. Otherwise, let $e = xz \notin E(G)$. Thus by the induction hypothesis, $G \setminus v \cup \{e\}$ has an OPPDC. Therefore by Theorem C, G admits an OPPDC. $\square$

Corollary 3.6. *Every* 4*-regular graph with a cut-vertex has an* OPPDC*.*

Proof. If G is a 4-regular graph with a cut-vertex, then every block, G', of G is a graph with $\Delta(G') \leq 4$ and $\delta(G') \leq 3$. Therefore, by Theorems 3.5 and A(iii), G has an OPPDC. $\square$

Following theorem guarantees the existence of an OPPDC for a large family of graphs. The Cartesian product, $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent if and only if either $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$. In the following theorem we prove that the existence of an OPPDC for two graphs G and H, provides an OPPDC for the Cartesian product of G and H.

Theorem 3.7. *If* G *and* H *have an* OPPDC*, then* $G \square H$ *also has an* OPPDC*.*

Proof. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are the OPPDC of G and H, respectively. Let $\mathcal{R} = \{P_u Q^v \ : \ (u, v) \in V(G \square H)\}$, where $P_u \in \mathcal{P}$ is the directed path ending with u in the copy of G in $G \square H$ corresponding to the vertex v in H, and $Q^v \in \mathcal{Q}$ is the directed path starting from v in the copy of H in $G \square H$ corresponding to the vertex u in G. It can be seen that every arc of the symmetric orientation of $G \square H$ is covered by one path in $\mathcal{R}$ and every vertex (u, v) appears just once as a beginning and once as an end of a path in $\mathcal{R}$. Therefore, $\mathcal{R}$ is an OPPDC of $G \square H$. $\square$

Theorem 3.7 concludes that the OPPDC conjecture holds for some well-known families of graphs, such as Cartesian products of cycles, paths, wheels, complete graphs, and complete bipartite graphs.

In the following an OPPDC for the complete bipartite graph is given.

Theorem 3.8. *Every $K_{n,m}$ has an* OPPDC*.*

Proof. Let $V(K_{n,m}) = \{v_1, \ldots, v_n; w_1, \ldots, w_m\}$ and $E(K_{n,m}) = \{v_i w_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$. We proceed by induction on m. Suppose first that $m = 1$. Define $P^{v_1}_{n,1} = v_1 w_1,\ P^{w_1}_{n,1} = w_1 v_n$, and $P^{v_i}_{n,1} = v_i w_1 v_{i-1}$, for $2 \leq i \leq n$. Therefore,

$$\mathcal{P}_{n,1} = \{P^{w_1}_{n,1}, P^{v_i}_{n,1} \ :\ 1 \leq i \leq n\}$$

is an OPPDC of $K_{n,1}$.

Now for $m \geq 2$, define $P^{v_1}_{n,m} = v_1 w_m,\ P^{w_m}_{n,m} = w_m v_n P^{v_n}_{n,m-1},\ P^{v_i}_{n,m} = v_i w_m v_{i-1} P^{v_{i-1}}_{n,m-1}$, for $2 \leq i \leq n$, and $P^{w_j}_{n,m} = P^{w_j}_{n,m-1}$, for $2 \leq j \leq m-1$. Thus,

$$\mathcal{P}_{n,m} = \{P^{v_i}_{n,m}, P^{w_j}_{n,m} \ :\ 1 \leq i \leq n,\ 1 \leq j \leq m\},$$

is an OPPDC of $K_{n,m}$. □

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(Behrooz Bagheri Gh.) Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran
E-mail address: `b.bagheri@math.iut.ac.ir`

(Behnaz Omoomi) Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-83111 Isfahan, Iran
E-mail address: `bomoomi@cc.iut.ac.ir`