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ON GRADED CLASSICAL PRIME AND GRADED PRIME SUBMODULES

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce several results concerning graded classical prime submodules. For example, we give a characterization of graded classical prime submodules. Also, the relations between graded classical prime and graded prime submodules of M are studied.

Keywords: Graded submodules, graded prime submodule, graded classical prime submodule.
MSC(2010): Primary: 13A02; Secondary: 16W50.

1. Introduction and preliminaries

Graded prime submodules of graded modules over graded commutative rings have been introduced and studied in [1,2,7]. Also, the concept of graded classical prime submodules of graded modules over graded commutative rings was introduced in [9]. Here, we study several results concerning graded classical prime submodules of graded modules over graded commutative rings. We will characterize graded classical prime submodules. Also, we will find some relations between graded classical prime submodules and graded prime submodules. For example, it is proved that if M is a graded multiplication or a graded P-secondary, then every graded classical prime submodule of M is a graded prime submodule. Moreover, the relations between graded classical prime and graded irreducible submodules of a graded module M are studied.

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Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is called a *graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M) = \{r \in R : rM \subseteq N\}$ or $m \in N$. In case $P = (N :_R M)$ is a graded prime ideal of R we say that N is a *graded P-prime submodule* of M (see [1].) A proper graded submodule N of M is called a *graded classical prime submodule* if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ (see [9].) Of course, every graded prime submodule is a graded classical prime submodule, but the converse is not true in general (see [9].)

Before we state some results, let us introduce some notations and terminologies. Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a *G-graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G) (see [6].) The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded.

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \underset{g\in G}{\oplus} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \underset{g\in G}{\cup} M_g$ and the elements of $h(M)$ are called to be *homogeneous*. Let $M = \underset{g\in G}{\oplus} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \underset{g\in G}{\oplus} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N (see [6].)

2. Graded prime submodules and graded classical prime submodules

The following results give us a characterization of a graded classical prime submodule of a graded module.

Theorem 2.1. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then the following statements are equivalent.*

(i) *N is a graded classical prime submodule of M.*

(ii) *For every graded submodule K of M and every pair of elements $r, s \in h(R)$ such that $rsK \subseteq N$, either $rK \subseteq N$ or $sK \subseteq N$.*

(iii) *For every pair of graded ideals I, J of R and every graded submodule K of M with $IJK \subseteq N$, either $IK \subseteq N$ or $JK \subseteq N$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N is a graded classical prime submodule of M. Let $rsK \subseteq N$ for some graded submodule K of M and $r, s \in h(R)$. Assume that $rK \nsubseteq N$ and $sK \nsubseteq N$. Since K is generated by $K \cap h(M)$, there exist k_1, $k_2 \in K \cap h(M)$ such that rk_1, $sk_2 \notin N$. Since N is a graded classical prime submodule, we conclude that $rk_2 \in N$ and $sk_1 \in N$. Since $rs(k_1+ k_2) \in N$, we have either $r(k_1+ k_2) \in N$ or $s(k_1+ k_2) \in N$ and hence either $rk_1 \in N$ or $sk_2 \in N$, which is a contradiction. Therefore $rK \subseteq N$ or $sK \subseteq N$.

$(ii) \Rightarrow (iii)$ Let I, J be graded ideals of R and K be a graded submodule of M such that $IJK \subseteq N$. Suppose that $IK \nsubseteq N$ and $JK \nsubseteq N$. Then there exists $r \in I \cap h(R), s \in J \cap h(R)$ such that $rK \nsubseteq N$ and $sK \nsubseteq N$. By (ii), $rsK \nsubseteq N$, a contradiction.

$(iii) \Rightarrow (i)$ Let $rsm \in N$ for some $m \in h(M)$ and $r, s \in h(R)$. So we have $(r)(s)(m) \subseteq N$ where $(r), (s)$ are graded ideals of R and (m) is a graded submodule of M. By our assumption we obtain either $(r)(m) \subseteq N$ or $(s)(m) \subseteq N$ and hence either $rm \in N$ or $sm \in N$. Thus, N is a graded classical prime submodule of M.

□

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module, N a proper graded submodule of M and $S = M \backslash N$. Then the following statements are equivalent.*

(i) *N is a graded classical prime submodule of M.*

(ii) *If I, J are graded ideals of R, K is a graded submodule of M which is contained in N, and V is a graded submodule of M such that $(K + IV) \cap S \neq \phi$ and $(K + JV) \cap S \neq \phi$, then $(K + IJV) \cap S \neq \phi$.*

Proof. $(i) \Rightarrow (ii)$ Let I, J be graded ideals of R and K, V be graded submodules of M such that $(K + IV) \cap S \neq \phi$ and $(K + JV) \cap S \neq \phi$. If $(K + IJV) \cap S = \phi$, then $IJV \subseteq N$. Since N is graded classical prime, it follows from Theorem 2.1 that we have either $IV \subseteq N$ or

$JV \subseteq N$, so that $K+IN \subseteq N$ or $K+JV \subseteq N$. Hence, either $(K+IV)\cap S = \phi$or$(K+JV)\cap S = \phi$, which is a contradiction. Therefore, $(K+IJV)\cap S \neq \phi$.

$(ii) \Rightarrow (i)$ Let I, J be graded ideals of R and V be a graded submodule of M such that $IJV \subseteq N$. If $IV \nsubseteq N$ and $JV \nsubseteq N$ gives $IV\cap S \neq \phi$ and $JV\cap S \neq \phi$. By our assumption we obtain $IJV\cap S \neq \phi$, which is a contradiction. Hence either $IV \subseteq N$ or $JV \subseteq N$. By Theorem 2.1 again, N is a graded classical prime submodule of M. □

By definition, every graded prime of a graded module is graded classical prime. However, the converse is not true in general. The example of this is given below.

Example 2.3. Let's take a graded module over a graded G-ring M as in [5, Example 2.4]. Let $P = 3\mathbb{Z}\times 0$. Then P is a graded submodule of M. For $3 \in R_0$ and $(2,0) \in M_0$, $3(2,0) \in P$, but $3(0,1) = (0,3) \notin P$, so $3 \notin P :_R M$. Moreover, $(2,0) \notin P$. Hence, P is not graded prime. But, P is graded classical prime. In fact, assume that $rsx \in P$, where $r,s \in h(R), x \in h(M)$. There are three cases to consider.

Case 1. If $x \in M_0$, then write $x = (b,0)$, where $b \in \mathbb{Z}$. Then $rs(b,0) = (3a,0)$ for some $a \in \mathbb{Z}$, so $rsb = 3a$. $3|rsb$. $3|r$, or $3|s$, or $3|b$. If $3|r$, then $rx = r(b,0) \in P$. If $3|s$, then $sx = s(b,0) \in P$. If $3|b$, then $rx = r(b,0) = b(r,0) \in P$ and $sx = s(b,0) = b(s,0) \in P$.

Case 2. If $x \in M_1 = 0\times\mathbb{Z}$, then write $x = (0,c)$, where $c \in \mathbb{Z}$. $rs(0,c) = (3a,0)$, $rsc = 0$, so $r = 0$, or $s = 0$, or $c = 0$. If $r = 0$, then $rx \in P$. If $s = 0$, then $sx \in P$. If $c = 0$, then $rx \in P$ and $sx \in P$.

Case 3. If $x \in M_g$ for $g \neq 0$ and $g \neq 1$, then $x \in M_g = 0\times 0$; hence $rx \in P$ and $sx \in P$. This shows that P is graded classical prime.

Now, it is natural to ask under what conditions every graded classical prime submodule is graded prime.

3. Graded irreducible, graded classical primary, graded prime, graded classical prime, and graded semiprime submodules

Let R be a G-graded ring and K, N *be* graded R-modules. Then $(N :_R K) = \{r \in R : rK \subseteq N\}$ is a graded ideal of R (see [1].) Also, a proper graded ideal P of R is said to be a *graded prime ideal* if whenever $r,s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in P$(see [8].)

Lemma 3.1. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then, N is a graded classical prime* *submodule of M if and only if every graded submodule V of M such that $V \nsubseteq N$, $(N :_R V)$ is a graded prime ideal of R. In particular, if N is a graded classical prime submodule of M, then $(N :_R M)$ is a graded prime ideal of R.*

Proof. Suppose that N is a graded classical prime submodule of M. Let V be a graded submodule of M such that $V \nsubseteq N$. Then $(N :_R V) \neq R$. Let r, $s \in h(R)$ with $rs \in (N :_R V)$, i.e., $rsV \subseteq N$. Since N is a graded classical prime, $rV \subseteq N$ or $sV \subseteq N$ and hence either $r \in (N :_R V)$ or $s \in (N :_R V)$. Thus $(N :_R V)$ is a graded prime ideal.

Conversely, let $abK \subseteq N$ where a, $b \in h(R)$ and K is a graded submodule of M. If $K \subseteq N$, then $aK \subseteq aN \subseteq N$ and $bK \subseteq bN \subseteq N$, we are done. If $K \nsubseteq N$, then $ab \in (N :_R K)$. Since $(N :_R K)$ is a graded prime ideal of R, we have either $a \in (N :_R K)$ or $b \in (N :_R K)$ and hence either $aK \subseteq N$ or $bK \subseteq N$. Thus N is a graded classical prime submodule of M.

□

Recall that a graded R-module M is called a *graded multiplication* if for each graded submodule N of M, $N = IM$ for some graded ideal I of R. One can easily show that if N is graded submodule of a graded multiplication module M, then $N = (N :_R M)M$ (see [7].)

The following results provide some conditions under which a graded classical prime is a graded prime.

Theorem 3.2. *Let R be a G-graded ring. If M is a graded multiplication R-module, then every graded classical submodule of M is graded prime.*

Proof. Suppose that N is a graded classical prime submodule of M. Let $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$ and $m \notin N$. Since M is a graded multiplication, $Rm = IM$ for some graded ideal I of R. So $rIM \subseteq N$ and $IM \nsubseteq N$, i.e., $rI \subseteq (N :_R M)$ and $I \nsubseteq (N :_R M)$. By Lemma 3.1, $(N :_R M)$ is a graded prime ideal of R and hence $r \in (N :_R M)$. Thus N is a graded prime submodule of M.

□

The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [8].) Also, recall that a non-zero graded module M is *graded secondary* if for each $r \in h(R)$, $rM = M$ or there exists $n \in \mathbb{N}$ such that $r^n M = 0$. A graded secondary module is said to be *graded P-secondary* if $P = Gr(AnnM)$ and P is a graded prime ideal of R (see [2].)

Theorem 3.3. *Let R be a G-graded ring. If M is a graded P-secondary R-module, then every graded classical submodule of M is graded P-prime.*

Proof. Let N be a graded classical prime submodule of M. By Lemma 3.1, $(N :_R M)$ is a graded prime ideal of R. So $Gr((N :_R M)) = (N :_R M)$. It is easy to see that $(N :_R M) \subseteq P$. Since $AnnM \subseteq (N :_R M)$, we conclude that $P = Gr(AnnM) \subseteq Gr((N :_R M)) = (N :_R M) \subseteq P$. Now, let $rm \in N$, where $r \in h(R)\backslash(N :_R M)$ and $m \in h(M)$. Since M is a graded P-secondary and $r \notin P$, we have $rM = M$. Then, there exists $m' \in h(M)$ with $rm' = m$. Hence $r^2m' = rm \in N$. Since N is a graded classical prime, $m = rm' \in N$. Therefore, N is a graded P-prime.

□

Lemma 3.4. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then the following statements are equivalent.*

(i) *N is a graded classical prime submodule of M.*
(ii) *If $(N :_R m_1) \neq (N :_R m_2)$ for any $m_1, m_2 \in h(M)$, then $N = (N + Rm_1) \cap (N + Rm_2)$.*

Proof. $(i) \Rightarrow (ii)$ Let N be a graded classical prime submodule of M and let m_1, $m_2 \in h(M)$ and $r \in h(R)$ such that $r \in (N :_R m_1)\backslash(N :_R m_2)$, i.e., $rm_1 \in N$ and $rm_2 \notin N$. Since N is graded classical prime, we can get $(N + Rm_1) \cap Rm_2 \subseteq N$. In fact, let $x \in (N + Rm_1) \cap Rm_2$. Then there exists $n \in N, r_1, r_2 \in R$ such that $x = n + r_1m_1 = r_2m_2$. $r_2rm_2 = rr_2m_2 = rx = rn + r_1rm_1 \in N$, so that $x = r_2m_2 \in N$, as required. Hence by the modular law,

$$(N + Rm_1) \cap (N + Rm_2) = N + ((N + Rm_1) \cap Rm_2) \subseteq N.$$

Other side of the inclusion is obvious, so we get $N = (N + Rm_1) \cap (N + Rm_2)$.

$(ii) \Rightarrow (i)$ Let $rsm \in N$ and $rm \notin N$ where r, $s \in h(R)$ and $m \in h(M)$. Then $(N :_R sm) \neq (N :_R m\ )$. By our assumption $sm \in (N + Rsm) \cap (N + Rm) = N$. Thus N is a graded classical prime submodule of M. □

Recall that a proper graded submodule N of a graded module M is called *graded irreducible* if N cannot be expressed as the intersection of two strictly larger graded submodules of M.

Theorem 3.5. *Let R be a G-graded ring and M a graded R-module. Then every graded irreducible, classical prime submodule of M is graded prime.*

Proof. Suppose that N is graded irreducible, classical prime. Let $rm \in N$ where $m \in h(M)$ and $r \in h(R)\backslash(N :_R M)$. Then there exists $m' \in h(M)$ such that $rm' \notin N$. So $(N :_R m) \neq (N :_R m')$. By Lemma 3.4, we have $N = (N + Rm) \cap (N + Rm')$. Since N is graded irreducible and $N \neq N + Rm'$, we conclude that $N = N + Rm$ and hence $m \in N$. Thus, N is graded prime. □

Recall that a G-graded ring R is called *graded noetherian* if it satisfies the ascending chain condition on graded ideals of R (see [6].) Also a proper graded submodule N of a graded module M is said to be a *graded semiprime submodule* if whenever $r \in h(R)$, $m \in h(M)$ and $n \in \mathbb{Z}^+$ with $r^n m \in N$, then $rm \in N$ (see [4].) This can be defined alternatively as in [5] and we can see further results on this topic.

Let R be a ring and M be a graded R-module. A proper submodule N of M is called a *graded classical primary submodule* of M if $N \neq M$ and if whenever $rsm \in N$, where $r, s \in h(R)$, and $m \in h(M)$, then $r^n m \in N$ for some positive integer n or $sm \in N$.

Let R be a graded G-ring and M be a graded R-module. Then it is easily seen that (1) every graded classical prime of M is graded semiprime and (2) every graded classical primary, semiprime submodule of M is graded classical prime.

Theorem 3.6. *Let R be a G-graded ring and M be a graded R-module. If R is graded Noetherian, then every graded irreducible submodule of M is graded classical primary and hence every graded irreducible, semiprime submodule of M is graded classical prime.*

Proof. Let N be a graded irreducible submodule of M. Suppose that $rsm \in N$ and $sm \notin N$ where r, $s \in h(R)$ and $m \in h(M)$. Now $(N :_R rm) \subseteq (N :_R r^2 m) \subseteq \cdots \subseteq (N :_R r^i m) \subseteq \cdots$ is an ascending chain of graded ideals of R. Since R is a graded noetherian ring, there exists $n \in \mathbb{N}$ such that $(N :_R r^n m) = (N :_R r^{n+i} m)$ for all $i \in \mathbb{N}$. First we show that $(N + Rr^n m) \cap Rsm \subseteq N$. Let $x \in (N + Rr^n m) \cap Rsm$. Write $x = u + ar^n M = bsm, u \in N, a, b \in R$. Then $ru + ar^{n+1}m = brsm$, so $ar^{n+1}m = brsm - ru \in N$. $a \in (N :_R r^{n+1}m) = (N :_R r^n m)$, so $x = u + ar^n m \in N$, as required. Now, by the modular law,

$$(N + Rr^n m) \cap (N + Rsm) = N + ((N + Rr^n m) \cap Rsm) \subseteq N.$$

Hence $(N + Rr^n m) \cap (N + Rsm) \subseteq N$. Other side of the inclusion is obvious, so we get $N = (N + Rr^n m) \cap (N + Rsm)$. Since N is a graded irreducible and $sm \notin N$, we have $N = (N + Rr^n m)$. Hence $r^n m \in N$. Therefore N is graded classical primary. Now, assume that N is graded irreducible, semiprime. Then N is graded classical primary, semiprime. By the statement (2) just prior to this theorem, N is graded classical prime. □

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. The *graded envelop submodule* $RGE_M(N)$ of N in M is a graded submodule of M generated by the set $GE_M(N) = \{rm : r \in h(R), m \in h(M)$ such that $r^n m \in N$ for some $n \in \mathbb{Z}^+\}$ (see [3].)

Proposition 3.7. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded classical prime, then $RGE_M(N) = N$.*

Proof. Let $n \in N$. Then n can be written as follows:

$$n = \sum_{g \in G} n_g = \sum_{g \in G} 1_R n_g.$$

Moreover, $1_R n_g = n_g \in N$, so $1_R n_g \in GE_M(N)$. Hence $n \in RGE(N)$. Thus $N \subseteq RGE(N)$. Conversely, let $t \in RGE_M(N)$. Then there exist elements $r_i \in h(R)$ and $m_i \in h(M)$ $(1 \leq i \leq l)$ such that $t = r_1 m_1 + \cdots + r_l m_l$ with $r_i^{n_i} m_i \in N$ for some $n_i \in \mathbb{Z}^+$. By the statement (1) just prior to Theorem 3.6, $r_i m_i \in N$ for all $i \in \{1, 2, \cdots, l\}$. Hence $t \in N$. This shows $RGE_M(N) \subseteq N$. Therefore $RGE_M(N) = N$.

Let us summarize the results. Let R be a G-graded ring and M be an R-graded module. (1) Every graded prime submodule of M is graded classical prime, but the converse is not true in general. We gave an example of this. (2) Every graded irreducible, classical prime is graded prime. (3) If M is graded multiplication or graded $Gr(Ann(M))$-secondary, then every classical prime submodule of M is graded prime. (4) Every graded classical primary, semiprime submodule of M is graded classical prime and conversely. (5) If R is graded Noetherian, then every graded irreducible submodule of M is graded classical primary.

□

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