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SUPERCYCLIC TUPLES OF THE ADJOINT WEIGHTED COMPOSITION OPERATORS ON HILBERT SPACES

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ABSTRACT. We give some sufficient conditions under which the tuple of the adjoint of weighted composition operators $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on the Hilbert space $\mathcal{H}$ of analytic functions is supercyclic.
Keywords: Supercyclic, adjoint, weighted composition operators, Hilbert space.
MSC(2010): Primary: 47A16; Secondary: 47B33, 47B38, 46E15.

1. Introduction

An n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, \cdots, T_n$ acting on an infinite dimensional separable Banach space X. If $T = (T_1, T_2, \cdots, T_n)$ is an n-tuple of operators, then we let

$$\mathcal{F} = \mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} : k_i \geq 0, \ i = 1, 2, \cdots, n\}$$

be the *semigroup* generated by T. If $x \in X$, the orbit of x under the tuple T is denoted by

$$Orb(T, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector $x \in X$ is called a *hypercyclic vector* for the tuple T if $Orb(T, x)$ is dense in X and in this case the tuple T is called *hypercyclic*. Also, a vector x is called a *supercyclic vector* for T if $\mathbb{C}Orb(T, x)$ is dense in X and in this case the tuple T is called *supercyclic*. Similarly, a vector x is called a *cyclic vector* for the tuple T if the *linear span* of

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$Orb(T, x)$ is dense in X and in this case the tuple T is called *cyclic*. From the definition, *supercyclicity* is an intermediate property among the *hypercyclicity* and the *cyclicity*.

Moreover, the tuple T becomes a single operator when $n = 1$. Thus the above definitions generalize the hypercyclicity, supercyclicity and cyclicity of a single operator to a tuple of operators.

Supercyclicity was introduced in the sixties by Hilden and Wallen [5]. They proved that every unilateral weighted shift is supercyclic. Since 1991, this property has been studied, for example, see Godefroy and Shapiro' work [4]. The first example of supercyclic operator in infinite dimensional Banach spaces (moreover hypercyclic) was discovered by Rolewicz [9] in 1969. Apart from supercyclicity, the other properties have also been studied in recent years. Such as, Liang and Zhou [7] characterized the *hereditarily hypercyclicity* of the unilateral (or bilateral) weighted shifts and gave some conditions for the supercyclicity of three different weighted shifts. Zhang and Zhou [17] studied *disjoint mixing* weighted backward shifts on the space of all complex valued square summable sequences. We refer the readers to these papers and their references.

In the present paper, we want to extend some properties of supercyclicity from a single operator to a n-tuple of operators. But for simplicity, we only prove our results for the case $n = 2$. That is, we consider the tuple $T = (T_1, T_2)$, a pair of *commuting continuous linear* operators. In this case, we still let

$$\mathcal{F} = \{T_1^{k_1} T_2^{k_2} :\ k_i \geq 0,\ i = 1, 2\}.$$

For $x \in X$, the orbit of x under the tuple T is the set

$$Orb(T, x) = \{Sx :\ S \in \mathcal{F}\} = \{T_1^{k_1} T_2^{k_2} x :\ k_i \geq 0,\ i = 1, 2\}.$$

The notation T_d^2 we will refer to the set of two copies of an element of $\mathcal{F}$, that is,

$$T_d^2 = \{S_1 \oplus S_2 : S_i \in \mathcal{F}, i = 1, 2\} = \{T_1^{k_1} T_2^{k_2} \oplus T_1^{k_3} T_2^{k_4} : k_i \geq 0,\ i = 1, 2, 3, 4\}.$$

We say that T_d^2 is *hypercyclic* provided there are $x_1,\ x_2 \in X$ such that

$$\{W(x_1 \oplus x_2) :\ W \in T_d^2\}$$

is dense in $X \bigoplus X$, and similarly we say that T_d^2 is *supercyclic* provided there are $x_1, x_2 \in X$ such that

$$\mathbb{C}\{W(x_1 \oplus x_2) : W \in T_d^2\}$$

is dense in $X \bigoplus X$. Also, we say that T_d^2 is *cyclic* provided there are $x_1, x_2 \in X$ such that the linear span of $\{W(x_1 \oplus x_2) : W \in T_d^2\}$ is dense in $X \bigoplus X$.

We denote $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$. In the following, let $\mathcal{H}$ be an *infinite dimensional separable Hilbert* space of analytic functions defined on $\mathbb{D}$ such that for each $\lambda \in \mathbb{D}$, the linear functional of point evaluation at λ given by $f \to f(\lambda)$ is bounded. In the following, *a Hilbert space of analytic functions $\mathcal{H}$ we mean one satisfying the above conditions.* Moreover, *the constants and the identity function $f(z) = z$* are in the Hilbert space $\mathcal{H}$.

For any $\lambda \in \mathbb{D}$, let e_λ denote the linear functional of point evaluation at λ on $\mathcal{H}$, that is, $e_\lambda(f) = f(\lambda)$ for every $f \in \mathcal{H}$. Since e_λ is a bounded linear functional, the Riesz representation theorem states that

$$e_\lambda(f) = \langle f, k_\lambda \rangle$$

for some $k_\lambda \in \mathcal{H}$.

The weighted Hardy space is the well-known example of such Hilbert space $\mathcal{H}$. Let $(\beta(n))_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ is defined as the space of analytic functions $f = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ on $\mathbb{D}$ satisfying

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 |\beta(n)|^2 < \infty.$$

From the book [2] we know that the classical Hardy space, the Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{1/2}$, respectively. These spaces are Hilbert spaces with the inner product defined by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)} (\beta(n))^2$$

for each $f, \ g \in H^2(\beta)$.

A complex-valued function ω on $\mathbb{D}$ for which $\omega f \in \mathcal{H}$ for every $f \in \mathcal{H}$ is called a *multiplier* of $\mathcal{H}$ and collection of all multipliers is denoted by $\mathcal{M}(\mathcal{H})$. A multiplication operator M_ω defined on $\mathcal{H}$ is denoted by

$$M_\omega f = \omega f, \ f \in \mathcal{H}.$$

Also, note that for each $\lambda \in \mathbb{D}$,

$$M_\omega^* k_\lambda = \overline{\omega(\lambda)} k_\lambda.$$

It is known that each multiplier M_ω is a bounded analytic function on $\mathbb{D}$ (see, e.g. [6, P552]), that is, $\mathcal{M}(\mathcal{H}) \subseteq H^\infty$.

If $\omega \in M(\mathcal{H})$ and φ is an analytic mapping from $\mathbb{D}$ into $\mathbb{D}$ such that $f \circ \varphi \in \mathcal{H}$ for every $f \in \mathcal{H}$, then from the closed graph theorem we obtain that the weighted composition operator $C_{\omega,\varphi}$ defined by

$$C_{\omega,\varphi}(f)(z) = M_\omega C_\varphi(f)(z) = \omega(z) f(\varphi(z))$$

is bounded. The mapping φ is called the composition map and ω is called the weight. From now on, we always *suppose* $\omega_1,\ \omega_2 \in \mathcal{M}(\mathcal{H})$ *and* $\varphi_1,\ \varphi_2$ *satisfy these properties.*

For a positive integer n, the nth iterate of φ_i, denoted by $(\varphi_i)_n$ for $i = 1, 2$, is the function obtained by composing φ_i with itself n times; also, φ_0 is defined to be the identity function. Besides, if φ_i is invertible, we can define the iterates $(\varphi_i)_{-n} = \varphi_i^{-1} \circ \varphi_i^{-1} \circ \ldots \circ \varphi_i^{-1}(n \text{ times})$ for $i = 1, 2$.

Now for $\omega \in \mathcal{M}(\mathcal{H})$ and an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, since $C^*_{\omega,\varphi}(k_\lambda) = \overline{w(\lambda)} k_{\varphi(\lambda)}$ for every $\lambda \in \mathbb{D}$, it follows that

$$C^{*n}_{\omega,\varphi}(k_\lambda) = \left(\Pi_{j=0}^{n-1} \overline{\omega(\varphi_j(\lambda))}\right) k_{\varphi_n(\lambda)}$$

for each $f \in \mathcal{H},\ \lambda \in \mathbb{D}$, where k_λ is the reproducing kernel of the $\mathcal{H}$.

The holomorphic self-maps of the unit disc $\mathbb{D}$ are divided into classes of *elliptic* and *nonellptic.* In this paper, we pay more attention to the *elliptic* type. The *elliptic* type is an automorphism and has a fixed point in $\mathbb{D}$. It is well-known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps that are not elliptic are called of *nonelliptic* type. The iterate of a *nonelliptic* map can be characterized by the *Denjoy-Wolff Iteration Theorem* as following,

Proposition 1.1. *[2, Theorem 2.51] If φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the disc $\mathbb{D}$ into itself, then there is a point $a \in \overline{\mathbb{D}}$ so that the iterates φ_n of φ converge to a uniformly on compact subsets of $\mathbb{D}$.*

Recently, there has been a great interest in studying the dynamical properties of a single adjoint weighted composition operator $C^*_{\omega,\varphi}$ on the Hilbert space $\mathcal{H}$, see for example monographs [3, 6, 8, 14, 16], which are good resources for our understanding. We list a result which characterizes the supercyclicity of a weighted composition operator for the convenience of the readers.

Proposition 1.2. *[6, Theorem 3] Let φ be a disc automorphism. Set*

$$E = \Big\{\lambda \in \mathbb{D} : \Big\{\Pi_{j=0}^{n-1} w \circ \varphi_j(\lambda)\Big\}_n \text{ is a bounded sequence}\Big\},$$

$$F = \Big\{\lambda \in \mathbb{D} : \{(w \circ \varphi_{-n}(\lambda))^{-1}\}_n \text{ is not a Blaschke sequence}\Big\},$$

$$G = \Big\{\lambda \in \mathbb{D} : \{w \circ \varphi_n(\lambda)\}_n \text{ is not a Blaschke sequence}\Big\},$$

$$H = \Big\{\lambda \in \mathbb{D} : \Big\{\Big(\Pi_{j=1}^{n} w \circ \varphi_{-j}(\lambda)\Big)^{-1}\Big\}_n \text{ is a bounded sequence}\Big\}.$$

If one of the following conditions holds then $C^*_{w,\varphi}$ is a supercyclic operator.

(i) The sets E and F have limit points in $\mathbb{D}$; moreover, $\{k_{\varphi_n(\lambda_1)}\}_n$ and $\{k_{\varphi_{-n}(\lambda_2)}\}_n$ are bounded sequences for all $\lambda_1 \in E$ and $\lambda_2 \in F$.

(ii) The sets G and H have limit points in $\mathbb{D}$; furthermore, $\{k_{\varphi_n(\lambda_1)}\}_n$ and $\{k_{\varphi_{-n}(\lambda_2)}\}_n$ are bounded sequences for all $\lambda_1 \in G$ and $\lambda_2 \in H$.

For the tuple of adjoint weighted composition operators $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$, the very recent paper [10] gives the sufficient conditions for its hypercyclicity on the Hilbert space $\mathcal{H}$. In 2011, Yousefi characterized the supercyclity of multiple weighted composition operators in [11]. From [11], we know the pair $((M_{\omega_1}C_\varphi)^*, (M_{\omega_2}C_\varphi)^*)$ can satisfy the Supercyclicity Criterion under some conditions. We list in the following, under the prerequisite $M_{\omega_1}C_\varphi M_{\omega_2}C_\varphi = M_{\omega_2}C_\varphi M_{\omega_1}C_\varphi$.

Proposition 1.3. *[11, Lemma 2.2] Let $\varphi(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi]$ and every $z \in \mathbb{D}$. Also, let $\omega_i : \mathbb{D} \to \mathbb{C}$ be such that the sets*

$$E_1 = \{\lambda \in \mathbb{D} : \lim_{n\to\infty} \Pi_{j=0}^{n-1} \omega_1(e^{(j+n)i\theta}\lambda) \cdot \omega_2(e^{ji\theta}\lambda) = 0\},$$

and

$$E_{-1} = \Big\{\lambda \in \mathbb{D} : \Big\{\Big(\Pi_{j=1}^{n} \omega_1(e^{ji\theta}\lambda) \cdot \omega_2(e^{-(j+n)i\theta}\lambda)\Big)^{-1}\Big\}_n \text{ is a bounded sequence}\Big\},$$

have limit points in $\mathbb{D}$. Then the pair $((M_{\omega_1}C_\varphi)^*, (M_{\omega_2}C_\varphi)^*)$ satisfies the Supercyclicity Criterion.

Proposition 1.4. *[11, Theorem 2.3] Let φ be an elliptic automorphism with interior fixed point p and $\omega_i : \mathbb{D} \to \mathbb{C}$ satisfies the inequality: $|\omega_i(p)| < 1 < \lim_{|z|\to 1^-} \inf |\omega_i(z)|$ for $i = 1, 2$. Then the pair $((M_{\omega_1}C_\varphi)^*, (M_{\omega_2}C_\varphi)^*)$ satisfies the Supercyclicity Criterion.*

Building on these foundations, we continue to investigate the supercyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on the Hilbert space $\mathcal{H}$. We generalize the results in [6] and [11] to a certain extent. The proofs of the present paper are partially based on the work, but some properties are not easily managed, we need some new methods and calculating techniques. The paper is organized as follows. In Section 2, we list some lemmas. In Section 3, we show some sufficient conditions for the supercyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$.

As we all know, linear continuous operators T and S on a separable infinite dimensional Banach space X are quasiconjugate (quasisimilar), if there exists a continuous map ϕ on X with dense range such that $T\circ\phi = \phi\circ S$. Moreover, if ϕ can be chosen to be a homeomorphism, then T and S are called conjugate (similar). The quasisimilarity and similarity preserve supercyclicity and hypercyclicity. In this paper, we mainly use the similarity preserves supercyclicty. For general case, S satisfies the Supercyclicity Criterion if and only if T satisfies the Supercyclicity Criterion when T is similar to S.

2. **Some lemmas**

Firstly, we give a necessary and sufficient condition for two weighted composition operators C_{ω_1,φ_1} and C_{ω_2,φ_2} to commute. In the proof of the Lemma, we use the fact that *the constant and the identity function* $f(z) = z$ *are in the Hilbert space* $\mathcal{H}$.

Lemma 2.1. *[10, Lemma 1] If $\omega_1(z)$ and $\omega_2(z)$ are nonzero for all $z \in \mathbb{D}$, then C_{ω_1,φ_1} and C_{ω_2,φ_2} can commute if and only if*

$$\varphi_1\circ\varphi_2 = \varphi_2\circ\varphi_1 \quad \textit{and} \quad \omega_1\cdot(\omega_2\circ\varphi_1) = \omega_2\cdot(\omega_1\circ\varphi_2). \tag{2.1}$$

Remark 2.2. *In the following, we will always assume that $\omega_1(z)$ and $\omega_2(z)$ are nonzero for all $z \in \mathbb{D}$ and φ_1, φ_2 satisfy*

$$\varphi_1\circ\varphi_2 = \varphi_2\circ\varphi_1, \quad \omega_1 = \omega_1\circ\varphi_2 \quad \textit{and} \quad \omega_2 = \omega_2\circ\varphi_1. \tag{2.2}$$

It is clear that the condition (2.2) is a special case of (2.1). Thus, the weighted composition operators C_{ω_1,φ_1} and C_{ω_2,φ_2} can commute under the assumption (2.2). There are some examples in [10] satisfying the condition (2.2). We show them for the convenience of the readers.

Suppose that $\varphi_r(z) = e^{ir\pi}z$ where $r = \frac{p}{q}$, p and q are integers so that $(p,q) = 1$. Define the weight $w_r(z) = \sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} \frac{1}{2^n}, & (n = \frac{2kq}{p} \text{ for some } k \in \mathbb{Z}), \\ 0, & \text{otherwise;} \end{cases}$$

then $w_r \in H^\infty$. Moreover, $w_r \circ \varphi_r(z) = w_r(z)$ for all $z \in \mathbb{D}$ and $\varphi_r \circ \varphi_s = \varphi_s \circ \varphi_r$.

In the following we denote $T_i = C^*_{\omega_i,\varphi_i}$ for $i = 1, 2$. By easy computation

$$T_i^n k_z = \left(\Pi_{j=0}^{n-1}\overline{(\omega_i \circ (\varphi_i)_j)(z)}\right) k_{(\varphi_i)_n(z)}, \quad i = 1, 2, \quad n \geq 1;$$

Thus using (2.2) it follows that

$$\begin{aligned}
(2.3)\quad & T_2^n T_1^n k_z \\
&= \left(\Pi_{k=0}^{n-1}\overline{(\omega_2 \circ (\varphi_2)_k)(z)}\right)\left(\Pi_{j=0}^{n-1}\overline{(\omega_1 \circ (\varphi_1)_j \circ (\varphi_2)_n)(z)}\right) k_{(\varphi_1)_n \circ (\varphi_2)_n(z)} \\
&= \left(\Pi_{k=0}^{n-1}\overline{(\omega_2 \circ (\varphi_2)_k)(z)}\right)\left(\Pi_{j=0}^{n-1}\overline{(\omega_1 \circ (\varphi_2)_n \circ (\varphi_1)_j)(z)}\right) k_{(\varphi_1)_n \circ (\varphi_2)_n(z)} \\
&= \left(\Pi_{k=0}^{n-1}\overline{(\omega_2 \circ (\varphi_2)_k)(z)}\right)\left(\Pi_{j=0}^{n-1}\overline{(\omega_1 \circ (\varphi_1)_j)(z)}\right) k_{(\varphi_1)_n \circ (\varphi_2)_n(z)} \\
&= \left[\Pi_{j=0}^{n-1}\left(\overline{(\omega_2 \circ (\varphi_2)_j)(z)} \cdot \overline{(\omega_1 \circ (\varphi_1)_j)(z)}\right)\right] k_{(\varphi_1)_n \circ (\varphi_2)_n(z)} \\
&= \left[\Pi_{j=0}^{n-1}\left(\overline{(\omega_1 \circ (\varphi_1)_j)(z)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(z)}\right)\right] k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}.
\end{aligned}$$

Next, we first present the Supercyclicity Criterion for a single operator, similarly we list the Supercyclicity Criterion for tuples.

Proposition 2.3. *(Supercyclicity Criterion for a single operator) Let X be a separable infinite dimensional Banach space and T be a continuous linear mapping on X. Suppose that there exist two dense subsets Y and Z in X, a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, and also there exist mappings $S_{n_k} : Z \to X$ such that*

(1) $T^{n_k} S_{n_k} z \to z$, *for every* $z \in Z$.

(2) $\|T^{n_k} y\| \|S_{n_k} z\| \to 0$ *for every* $y \in Y$ *and every* $z \in Z$.

Then T is supercyclic.

If an operator T holds in the assumptions of Proposition 2.3, then we will say that T satisfies the Supercyclicity Criterion.

Lemma 2.4. *(Supercyclicity Criterion for tuples) [12, Definition 2.1] Suppose X is a separable infinite dimensional Banach space and $T = (T_1, T_2)$ is a pair of continuous linear mappings on X. We say that T satisfies the Supercyclicity Criterion if there exist two dense subsets Y and Z in X, and a pair of strictly increasing positive integer sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$, and a sequence of mappings $S_k : Z \to X$ such that*

(1) $T_1^{m_k}T_2^{n_k}S_kz \to z$, *for every* $z \in Z$.
(2) $\|T_1^{m_k}T_2^{n_k}y\|\|S_kz\| \to 0$ *for every* $y \in Y$ *and every* $z \in Z$.

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, we refer to

$$\bigcup_{n=1}^{\infty} \text{Ker}(T^n)$$

as the generalized kernel of T, where $\text{Ker}(T^n) = \{f \in \mathcal{H} : T^n f = 0\}$. The following lemma comes from [1, Corollary 3.3].

Lemma 2.5. *Let T be a bounded linear operator on a separable Hilbert space $\mathcal{H}$ with dense generalized kernel. Then, the following conditions are equivalent:*

(1) T *has a dense range.*
(2) T *is supercyclic.*
(3) T *satisfies the Supercyclic Criterion.*

Remark 2.6. *(1) We refer the interested readers to [15, Theorem 2.3] to get the proof for this lemma.*
(2) The generalized kernel of the tuple $T = (T_1, T_2)$ is defined as follows (see, e.g. [13, P392]), by a polynomial $p(.,.)$ we will mean

$$p(z, w) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}z^i w^j, \; z, w \in \mathbb{C}.$$

We will denote the generalized kernel of the pair $T = (T_1, T_2)$ by $GK(T)$, that is defined as,

$$GK(T) = \bigcup\{Ker(p(T_1, T_2)) : \; p(.,.) \quad \textit{is a polynomial}\,\}.$$

It is obvious that the set $\bigcup_{n=1}^{\infty} Ker(T_2^n T_1^n)$ is a subset of $GK(T)$.

Similarly, the range of the tuple $T = (T_1, T_2)$ on $\mathcal{H}$ can be represented as follows,

$$\bigcup\{p(T_1, T_2)g : \; p(.,.) \quad \text{is a polynomial }, g \in \mathcal{H}\}.$$

Lemma 2.7. *[12, Theorem 2.2] Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be a pair of operators T_1, T_2. Then,* *the following are equivalent,*

(i) T *satisfies the Supercyclicity Criterion.*

(ii) T_d^2 *is supercyclic on* $X \bigoplus X$.

Remark 2.8. *In the following, we will use these operators:*

$$T = (T_1, T_2) = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2}),$$
$$T_d^2 = \{T_1^{k_1} T_2^{k_2} \oplus T_1^{k_3} T_2^{k_4} :\ k_i \geq 0,\ i = 1, 2, 3, 4\}.$$

Then from Lemma 2.7, T_d^2 *is supercyclic on* $\mathcal{H} \bigoplus \mathcal{H}$*, when* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *satisfies the Supercyclicity Criterion.*

3. Supercyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$

In this section, we give some sufficient conditions for the supercyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on the Hilbert space $\mathcal{H}$. Firstly we give the following four sets A, B, C, D,

$$A = \Big\{z \in \mathbb{D} : \text{the sequence} \Big\{\Pi_{j=0}^{n-1}\Big(\omega_1 \circ (\varphi_1)_j(z) \cdot \omega_2 \circ (\varphi_2)_j(z)\Big)\Big\}_n \text{ is bounded}\Big\},$$
$$B = \Big\{z \in \mathbb{D} : \lim_{n\to\infty} \Pi_{j=1}^{n}\Big(\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)\Big)^{-1} = 0\Big\},$$
$$C = \Big\{z \in \mathbb{D} : \lim_{n\to\infty} \Pi_{j=0}^{n-1}\Big(\omega_1 \circ (\varphi_1)_j(z) \cdot \omega_2 \circ (\varphi_2)_j(z)\Big) = 0\Big\},$$
$$D = \Big\{z \in \mathbb{D} : \text{the sequence} \Big\{\Pi_{j=1}^{n}\Big(\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)\Big)^{-1}\Big\}_n \text{is bounded }\Big\}.$$

Theorem 3.1. *Let* $\omega_1(z), \omega_2(z)$ *be two nonzero complex-valued functions for all* $z \in \mathbb{D}$ *and* $\varphi_1(z), \varphi_2(z)$ *be two automorphisms on the unit disc* $\mathbb{D}$ *satisfying (2.2). Suppose*

$$M := \sup_{z\in\mathbb{D}} \sup_{n\in\mathbb{Z}} \|k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\| < \infty. \tag{3.1}$$

If one of the following holds:

(i) *The sets* A *and* B *have limit points in* $\mathbb{D}$.

(ii) *The sets* C *and* D *have limit points in* $\mathbb{D}$.

Then the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is supercyclic on* $\mathcal{H}$*. Moreover,* T_d^2 *is supercyclic on* $\mathcal{H} \bigoplus \mathcal{H}$.

Proof. We will use Lemma 2.3 to prove the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is supercyclic. Firstly, we suppose the condition (i) is true.

Take $S_A = \text{span}\{k_z :\ z \in A\}$ and $S_B = \text{span}\{k_z :\ z \in B\}$. Then, the sets S_A and S_B are dense in Hilbert space $\mathcal{H}$, that is, $\overline{S_A} = \overline{S_B} = \mathcal{H}$.

In fact, if $f \in \mathcal{H}$ is orthogonal to k_z for every $z \in S_A$, then $f(z) = \langle f, k_z\rangle$. From the condition (i), the set A has the limit point in $\mathbb{D}$, hence the identity theorem for holomorphic functions implies that f vanishes identically on $\mathcal{H}$. Thus $(S_A)^{\perp} = \{0\}$. That is $\overline{S_A} = \mathcal{H}$. By the similar argument, we can obtain that $\overline{S_B} = \mathcal{H}$.

If we take $Y = S_A$ and $Z = S_B$. Then Y and Z are two dense subsets of the Hilbert space $\mathcal{H}$.

Since φ_1 and φ_2 are two automorphisms on the unit disc $\mathbb{D}$, thus φ_1^{-1} and φ_2^{-1} exist on $\mathbb{D}$. Then from (2.2), it follows that

$$\varphi_1^{-1} \circ \varphi_2^{-1} = \varphi_2^{-1} \circ \varphi_1^{-1}, \ \ \omega_1 = \omega_1 \circ \varphi_2^{-1} \text{ and } \omega_2 = \omega_2 \circ \varphi_1^{-1}. \tag{3.2}$$

We still denote $T_i = C^*_{\omega_i,\varphi_i}$ for $i = 1, 2$. From (2.3), we have that

$$\begin{aligned} T_2^n T_1^n k_z \ &= \ \left[\Pi_{j=0}^{n-1}\left(\overline{(\omega_1 \circ (\varphi_1)_j)(z)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(z)}\right)\right] \\ &\quad \cdot k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}, \ \ n \geq 1. \end{aligned} \tag{3.3}$$

To find the desired right inverse of T_2T_1. Next, we divide the proof into two cases by the fact that the set $G_B = \{k_z : \ z \in B\}$ is linearly independent or not.

Case (I) Suppose that G_B is a linearly independent set. Define the operator $S : G_B \to \mathcal{H}$ by

$$Sk_z \ = \ \overline{[(\omega_1 \circ \varphi_1^{-1}(z)) \cdot (\omega_2 \circ \varphi_2^{-1}(z))]^{-1}} k_{\varphi_2^{-1} \circ \varphi_1^{-1}(z)}, \ z \in \mathbb{D}.$$

Thus, we can define S^n on G_B for all $n \geq 1$ by (3.2). That is,

$$S^n k_z = \Pi_{j=1}^{n} \overline{[\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)]^{-1}} k_{(\varphi_2)_{-n} \circ (\varphi_1)_{-n}(z)}. \tag{3.4}$$

Since G_B is linearly independent, then we can extend S by linearity on $S_B = \text{span}\{k_z : \ z \in B\}$. Therefore S^n is well-defined on S_B for all $n \geq 1$.

In this case, by the following conditions from (3.2)

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1, \ \ \omega_2 = \omega_2 \circ \varphi_1,$$

it is clear that

$$\begin{aligned} T_2T_1Sk_z &= T_2T_1\Big(\overline{[(\omega_1\circ\varphi_1^{-1}(z))\cdot(\omega_2\circ\varphi_2^{-1}(z))]^{-1}}k_{\varphi_2^{-1}\circ\varphi_1^{-1}(z)}\Big)\\ &= T_2\Big(\overline{[\omega_2\circ\varphi_2^{-1}\circ\varphi_1(z)]^{-1}}k_{\varphi_2^{-1}(z)}\Big)\\ &= \overline{\omega_2(z)[\omega_2\circ\varphi_2^{-1}\circ(\varphi_1\circ\varphi_2)(z)]^{-1}}k_z\\ &= \overline{\omega_2(z)[\omega_2\circ\varphi_2^{-1}\circ(\varphi_2\circ\varphi_1)(z)]^{-1}}k_z\\ &= \overline{\omega_2(z)[\omega_2(\varphi_1(z))]^{-1}}k_z\\ &= \overline{\omega_2(z)[\omega_2(z)]^{-1}}k_z\\ &= k_z. \end{aligned}$$

From which it follows that T_2T_1S is the identity on S_B. Therefore, $T_2^nT_1^nS^n$ is the identity on S_B for every $n\geq 1$. That is

$$T_2^nT_1^nS^nz\to z \ \text{ for every } \ z\in Z=S_B. \tag{3.5}$$

On the other hand, by condition (i) and (3.1) it follows that

$$\begin{aligned} &\lim_{n\to\infty}\|T_2^nT_1^nk_y\|\|S^nk_z\| \\ &=\lim_{n\to\infty}\|\Big[\Pi_{j=0}^{n-1}\Big(\overline{(\omega_1\circ(\varphi_1)_j)(y)}\cdot\overline{(\omega_2\circ(\varphi_2)_j)(y)}\Big)\Big]k_{(\varphi_1)_n\circ(\varphi_2)_n(y)}\| \\ &\cdot\|\Pi_{j=1}^{n}\overline{[\omega_1\circ(\varphi_1)_{-j}(z)\cdot\omega_2\circ(\varphi_2)_{-j}(z)]^{-1}}k_{(\varphi_2)_{-n}\circ(\varphi_1)_{-n}(z)}\| \\ &\leq M^2\sup_{n\in\mathbb{N}}|\Pi_{j=0}^{n-1}\Big(\overline{(\omega_1\circ(\varphi_1)_j)(y)}\cdot\overline{(\omega_2\circ(\varphi_2)_j)(y)}\Big)| \\ &\cdot\lim_{n\to\infty}|\Pi_{j=1}^{n}\overline{[\omega_1\circ(\varphi_1)_{-j}(z)\cdot\omega_2\circ(\varphi_2)_{-j}(z)]^{-1}}| \\ &=0,\ \text{ for } \ \forall y\in Y,\ \ \forall z\in Z. \end{aligned} \tag{3.6}$$

From (3.5), (3.6) and Lemma 2.3, it follows the tuple $(C^*_{\omega_1,\varphi_1},C^*_{\omega_2,\varphi_2})$ satisfies the Supercyclicity Criterion. By Lemma 2.5, T_d^2 is supercyclic on $\mathcal{H}\bigoplus\mathcal{H}$.

Case (II). Now suppose that $G_B=\{k_z:\ z\in B\}$ is not necessarily linearly independent. In this case, we use the method which has been used by Godefroy and Shapiro in [4, Theorem 4.5]. For the convenience of the readers, we include this method. Consider a countable dense subset

$$B_1=\{w_n\in\mathbb{D}:\ n\geq 1\}$$

of the set B. Next we will use induction to choose a sequence z_n. Take $z_1=w_1$, denote

$$B_2=B_1\setminus\{w\in B_1:\ k_w\in\operatorname{span}\{k_{z_1}\}\}.$$

Denote the first element of B_2 by z_2 and let

$$B_3 = B_2 \setminus \{w \in B_2 : \ k_w \in \text{span}\{k_{z_1}, k_{z_2}\} \ \}.$$

The infinite dimensionality of $\mathcal{H}$ insures the process never terminates. Then we can obtain an infinite subset $L = \{z_n \in \mathbb{D} : \ n \geq 1\}$ of the set B, for which the corresponding set of kernel functions $H_L = \{k_z : \ z \in L\}$ is linearly independent and is dense in $\mathcal{H}$. Now the operator S can be defined exactly as above, just with H_L in place of G_B. Therefore, the Supercyclicity Criterion holds too in this case.

To sum up, in both cases, the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ satisfies the Supercyclicity Criterion, thus it is supercyclic and T_d^2 is supercyclic on $\mathcal{H} \bigoplus \mathcal{H}$.

Similarly, if the condition (ii) holds, we can also give the proof for the supercyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$. This completes the proof. □

From Theorem 3.1, we can easily obtain the supercyclicity of the tuple $(M^*_{\omega_1}, M^*_{\omega_2})$.

Corollary 3.2. *Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$. Denote the sets*

$$\widetilde{A} = \Big\{ z \in \mathbb{D} : \ \text{the sequence } \{(\omega_1(z)\omega_2(z))^n\}_n \text{ is bounded} \Big\},$$

$$\widetilde{B} = \Big\{ z \in \mathbb{D} : \ \lim_{n\to\infty} \frac{1}{(\omega_1(z)\omega_2(z))^n} = 0 \Big\},$$

$$\widetilde{C} = \Big\{ z \in \mathbb{D} : \ \lim_{n\to\infty} (\omega_1(z)\omega_2(z))^n = 0 \Big\},$$

$$\widetilde{D} = \Big\{ z \in \mathbb{D} : \ \text{the sequence } \Big\{ \frac{1}{(\omega_1(z)\omega_2(z))^n} \Big\}_n \text{ is bounded} \Big\}.$$

If (i) or (ii) holds,

(i) The sets $\widetilde{A}$ and $\widetilde{B}$ have limit points in $\mathbb{D}$.

(ii) The sets $\widetilde{C}$ and $\widetilde{D}$ have limit points in $\mathbb{D}$.

then the tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ is supercyclic on $\mathcal{H}$. Moreover,

$$\{M^{*k_1}_{\omega_1} M^{*k_2}_{\omega_2} \oplus M^{*k_3}_{\omega_1} M^{*k_4}_{\omega_2}; \ k_i \geq 0, \ i = 1, 2, 3, 4\}$$

is supercyclic on $\mathcal{H} \bigoplus \mathcal{H}$.

Proof. Let $\varphi_1(z) = \varphi_2(z) = z$ in Theorem 3.1. It is clear that $M := \sup_{z\in\mathbb{D}} \|k_z\| < \infty$ defined in (3.1) holds. Then the desired result easily follows from Theorem 3.1. □

We give a simple example for understanding the Corollary 3.2.

Example 3.3. *Let $w_1(z) = z$ and $w_2(z) = z + 4$. It is obvious that*

$$\{x : 0 \le x < \sqrt{5} - 2\} \subseteq \{z \in \mathbb{D} : \textit{the sequence } \{(z(z+4))^n\}_n \textit{ is bounded}\};$$

and

$$\{x : -1 < x < -2 + \sqrt{3}\} \subseteq \{z \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{(z(z+4))^n} = 0\}.$$

That is, the sets $\widetilde{A}$ and $\widetilde{B}$ have limit points in $\mathbb{D}$. Hence by Corollary 3.2, it follows that the tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ is supercyclic on $\mathcal{H}$.

If φ_1 and φ_2 are two elliptic disc automorphisms (note every one of them only has unique fixed point in $\mathbb{D}$) satisfying $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, then they have the same interior fixed points. In fact, suppose that $\varphi_1(z_1) = z_1 \in \mathbb{D}$ and $\varphi_2(z_2) = z_2 \in \mathbb{D}$. Then

$$\varphi_1 \circ \varphi_2(z_2) = \varphi_2 \circ \varphi_1(z_2) \Rightarrow \varphi_1(z_2) = \varphi_2(\varphi_1(z_2)) \Rightarrow \varphi_1(z_2) = z_2 \Rightarrow z_1 = z_2.$$

Remark 3.4. *For general case, when both φ_1 and φ_2 have interior fixed points in $\mathbb{D}$ and satisfy $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, the interior fixed points are the same one.*

For $a \in \mathbb{D}$, an automorphism $\phi_a(z)$ of $\mathbb{D}$ is defined by

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D}. \tag{3.7}$$

As we all know that there are so many spaces that contain ϕ_a, such as the Hardy space, Bergman space and Dirichlet spaces and so on. We call such spaces the automorphism invariant.

Theorem 3.5. *Suppose that $\mathcal{H}$ is automorphism invariant. Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two elliptic disc automorphisms with an interior fixed point $a \in \mathbb{D}$ satisfying (2.2). If one of the conditions (i) and (ii) in Theorem 3.1 holds, then the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is supercyclic on $\mathcal{H}$. Moreover, T_d^2 is supercyclic on $\mathcal{H} \bigoplus \mathcal{H}$.*

Proof. **Case (I)** Suppose that $a = 0$. Then there are $\theta_1,\ \theta_2 \in [0, 2\pi]$ such that

$$\varphi_1(z) = e^{i\theta_1} z,\ \varphi_2(z) = e^{i\theta_2} z.$$

It is obvious that

$$(\varphi_2)_n \circ (\varphi_1)_n(z) = e^{in\theta_1} e^{in\theta_2} z.$$

Thus the iterate $\{(\varphi_2)_n \circ (\varphi_1)_n : n \in \mathbb{Z}\} \subseteq z\partial\mathbb{D}$. Since $z\partial\mathbb{D}$ is compact subset of $\mathbb{D}$, thus for every $f \in \mathcal{H}$, f is analytic on the unit disc $\mathbb{D}$, then

$$\Big(f((\varphi_2)_n \circ (\varphi_1)_n)\Big)_{n\in\mathbb{Z}}$$

is a bounded sequence. Thus, by the uniform boudedness principle, it follows that

$$M := \sup_{z\in\mathbb{D}} \sup_{n\in\mathbb{Z}} \|k_{(\varphi_2)_n \circ (\varphi_1)_n}\| < \infty. \tag{3.8}$$

Employing (3.8) and Theorem 3.1, it follows that the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ satisfies the Supercyclicity Criterion.

Case (II) The general case $a \neq 0$ is a fixed point of φ. We notice that $\mathcal{H}$ is automorphism invariant. Let

$$\widetilde{\varphi_1} = \phi_a \circ \varphi_1 \circ \phi_a^{-1},\ \widetilde{\varphi_2} = \phi_a \circ \varphi_2 \circ \phi_a^{-1}$$

be two automorphisms with the interior fixed point zero, and let

$$\widetilde{\omega_1} = \omega_1 \circ \phi_a^{-1},\ \widetilde{\omega_2} = \omega_2 \circ \phi_a^{-1}$$

be two multipliers of $\mathcal{H}$, where ϕ_a is the automorphism defined in (3.7). It is clear that the tuple $(C^*_{\widetilde{\omega_1},\widetilde{\varphi_1}}, C^*_{\widetilde{\omega_2},\widetilde{\varphi_2}})$ is supercyclic on $\mathcal{H}$ from **Case (I)**, where $C_{\widetilde{\omega_i},\widetilde{\varphi_i}} = C^{-1}_{\phi_a} \circ C_{\omega_1,\varphi_1} \circ C_{\phi_a}$ for $i = 1, 2$. Finally, taking into account that C_{ω_i,φ_i} is similar to $C_{\widetilde{\omega_i},\widetilde{\varphi_i}}$ for $i = 1, 2$ and the similarity preserves supercyclicity, the result follows. This completes the proof. □

Example 3.6. *Take two elliptic disc automorphisms $\varphi_1(z) = iz$, $\varphi_2(z) = -iz$ with an interior fixed point $a = 0 \in \mathbb{D}$ and $w_1(z) = z^4$, $w_2(z) = z^4 + 3$. It is obvious that the conditions of Theorem 3.5 are true. The sets A and B mentioned in Theorem 3.1 are*

$$A = \Big\{z \in \mathbb{D} : \text{ the sequence } \Big\{z^{4n}(z^4 + 3)^n\Big\}_n \text{ is bounded}\Big\},$$

and

$$B = \Big\{z \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{z^{4n}(z^4 + 3)^n} = 0\Big\}.$$

It is easily to show that $[0, \frac{1}{2}) \subseteq A$ and $(\frac{1}{\sqrt[4]{2}}, 1) \subseteq B$. Hence $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is supercyclic on $\mathcal{H}$ from Theorem 3.5.

Theorem 3.7. *Suppose that $\mathcal{H}$ is automorphism invariant. Let $\omega_1(z)$, $\omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two elliptic automorphism with an interior fixed point $a \in \mathbb{D}$ satisfying (2.2). Moreover $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{C}$ satisfy the inequality $|\omega_1(a)\omega_2(a)| < 1$ and there is $0 < \delta < 1$ satisfying $|\omega_1(z)\omega_2(z)| \geq 1$ for all $|z| > 1 - \delta$, then the* *tuple* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is supercyclic on* $\mathcal{H}$. *Moreover,* T_d^2 *is supercyclic on* $\mathcal{H}\bigoplus\mathcal{H}$.

Proof. The same argument as in Theorem 3.5 can be applied. Since $\mathcal{H}$ is automorphism invariant, we can assume $a = 0$. Thus

$$\varphi_1(z) = e^{i\theta_1}z,\ \varphi_2(z) = e^{i\theta_2}z$$

for some $\theta_1, \theta_2 \in [0, 2\pi]$. By the similar proof in **Case (I)** in Theorem 3.5, (3.8) is true.

On the other hand, since $|\omega_1(0)\omega_2(0)| < 1$, there is a constant $0 < r < 1$ and a positive number $\widetilde{\delta} \in (0,1)$ such that

$$|\omega_1(z)\omega_2(z)| < r < 1, \quad \text{whenever} \quad |z| < \widetilde{\delta}.$$

Since $|\varphi_i(z)| = |z|$ for $i = 1, 2$. Thus if $|z| < \widetilde{\delta}$, it follows that

$$\left|\Pi_{j=0}^{n-1}\omega_1 \circ (\varphi_1)_j(z) \cdot \omega_2 \circ (\varphi_2)_j(z)\right| < r^n \to 0, \quad n \to \infty.$$

Thus the set $\{z \in \mathbb{D} :\ |z| < \widetilde{\delta}\}$ is a subset of C in Theorem 3.1.

On the other hand, there is $0 < \delta < 1$ satisfying $|\omega_1(z)\omega_2(z)| \geq 1$ for all $|z| > 1 - \delta$. And since $|\varphi_i^{-1}(z)| = |z|$ for $i = 1, 2$, then we have that

$$\left|\Pi_{j=1}^{n}\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)\right|^{-1} \leq 1,$$

for all $n \geq 1$.

Therefore, the set $\{z \in \mathbb{D} : |z| > 1 - \delta\}$ is a subset of D in Theorem 3.1. Since both $\{z \in \mathbb{D} :\ |z| < \widetilde{\delta}\}$ and $\{z \in \mathbb{D} : |z| > 1 - \delta\}$ have limit points in $\mathbb{D}$, then both C and D have limit points in $\mathbb{D}$. Besides by (3.8) and from Theorem 3.1, we obtain that the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ satisfies the Supercyclicity Criterion. This completes the proof. □

Remark 3.8. *It is easy to check that Example 3.6 holds for Theorem 3.7. Since* $|w_1(0)w_2(0)| = 0 < 1$ *and there is* $0 < \delta = 1 - \frac{1}{\sqrt[4]{2}} < 1$ *satisfying* $|w_1(z)w_2(z)| = |z|^4|z^4 + 3| \geq |z|^4(3 - 1) = 2|z|^4 \geq 1$ *for all* $|z| > 1 - \delta$. *Hence the tuple* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is supercyclic on* $\mathcal{H}$ *from Theorem 3.7.*

Now if φ is an elliptic automorphism, a rotation through a rational multiple of π, then there is $m \in \mathbb{N}$ such that $\varphi_m(z) = z$ for all $z \in \mathbb{D}$. Now, we consider two general analytic self-maps φ_1, φ_2 on $\mathbb{D}$ with the properties (3.9) or (3.10).

Theorem 3.9. *Let $\omega_1(z)$, $\omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two analytic self-maps of $\mathbb{D}$ satisfying (2.2). For $\lambda \in \mathbb{D}$, let $\{\lambda_m\}_{m\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ satisfying the following conditions*

$$(3.9) \qquad (\varphi_1)_m(\lambda_m) = \lambda, \ m = 1, 2, 3..., \ \textit{and} \ \omega_1(\lambda) = 0,$$

or

$$(3.10) \qquad (\varphi_2)_m(\lambda_m) = \lambda, \ m = 1, 2, 3..., \ \textit{and} \ \omega_2(\lambda) = 0.$$

Also, suppose that the set $\{\lambda_m : \ m \geq 1\}$ has a limit point in $\mathbb{D}$ and (3.1) *holds. Then, the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is supercyclic on $\mathcal{H}$. Moreover, T_d^2 is supercyclic on $\mathcal{H} \bigoplus \mathcal{H}$.*

Proof. It is clear that the set $K = \text{span}\{k_{\lambda_m} : m = 1, 2, 3, ...\}$ is a dense set in $\mathcal{H}$. In fact, suppose that $\langle f, k_{\lambda_m}\rangle = f(\lambda_m) = 0$, $m = 1, 2, 3, ...,$ since the set $\{\lambda_m : \ m \geq 1\}$ has a limit point in $\mathbb{D}$, then the identity theorem for holomorphic functions implies that $f \equiv 0$. Thus $\overline{K} = \mathcal{H}$. On the other hand, we still denote $T_i = C^*_{\omega_i,\varphi_i}$ for $i = 1, 2$. From (2.3) it follows that

$$T_2^n T_1^n k_z = \Big[\Pi_{j=0}^{n-1}\Big(\overline{(\omega_1 \circ (\varphi_1)_j)(z)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(z)}\Big)\Big] k_{(\varphi_1)_n \circ (\varphi_2)_n (z)}, \quad z \in \mathbb{D}, \ n \geq 1.$$

We suppose that (3.9) holds. Hence, using (3.1) and (3.9), it follows that, for every positive integer n

$$(3.11) \qquad T_2^n T_1^n k_{\lambda_m} = 0, \ m = 0, 1, 2, ..., n-1,$$

where $\lambda_0 = \lambda$. Then $k_{\lambda_m} \in Ker(T_2^n T_1^n), \ m = 0, 1, 2, ..., n-1.$

Since $K = \text{span}\{k_{\lambda_m} : \ m = 1, 2, 3, ...\}$ is a dense set in $\mathcal{H}$, then the set

$$\bigcup_{n=1}^{\infty} Ker(T_2^n T_1^n)$$

is dense in $\mathcal{H}$. As we all know that the set $\bigcup_{n=1}^{\infty} Ker(T_2^n T_1^n)$ is the subset of $GK(T)$, which is the generalized kernel of the tuple (T_1, T_2). Hence, the generalized kernel of the tuple (T_1, T_2) is dense in $\mathcal{H}$. By Lemma 2.5 we only need to prove the tuple $(T_1, T_2) = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ has a dense range.

If for every $g \in \mathcal{H}$ such that

$$T_2^n T_1^n (g) = \Big[\Pi_{j=0}^{n-1}\Big(\overline{\omega_1 \circ (\varphi_1)_j} \cdot \overline{\omega_2 \circ (\varphi_2)_j}\Big)\Big] g \circ (\varphi_1)_n \circ (\varphi_2)_n = 0.$$

Since ω_1 and ω_2 are two nonzero complex-valued functions for all $z \in \mathbb{D}$. Then we have $g \equiv 0$. Hence the set $\{T_2^n T_1^n g : g \in \mathcal{H}\}$ is a dense set in $\mathcal{H}$. However, $\{T_2^n T_1^n g : \ g \in \mathcal{H}\}$ is a subset of the range of the tuple (T_1, T_2). Therefore the tuple $(T_1, T_2) = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ has a dense range. Employing Lemma 2.5, the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ satisfies the Supercyclicity Criterion on $\mathcal{H}$. This completes the proof. □

Theorem 3.10. *Suppose that $\mathcal{H}$ is automorphism invariant. Let ω_1, ω_2 be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two elliptic automorphisms with interior fixed point $a \in \mathbb{D}$ satisfying (2.2). Further suppose that (3.1) is true. If (i) or (ii) holds for some $\lambda \in \mathbb{D} \setminus \{a\}$,*

(*i*) *φ_1 is conjugate to a rotation through an irrational multiple of π and $\omega_1(\lambda) = 0$.*

(*ii*) *φ_2 is conjugate to a rotation through an irrational multiple of π and $\omega_2(\lambda) = 0$.*

Then the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is supercyclic on $\mathcal{H}$. Moreover, T_d^2 is supercyclic on $\mathcal{H} \bigoplus \mathcal{H}$.

Proof. Since $\mathcal{H}$ is automorphism invariant. Similarly, we suppose that $a = 0$. Assume (i) holds, then $\varphi_1(z) = e^{i\pi\theta_1} z$ for some irrational number $\theta_1 \in [0, 2\pi]$. Let

$$\lambda_m = e^{i(-m)\pi\theta_1}\lambda, \ m = 1, 2, 3....,$$

Then $(\varphi_1)_m(\lambda_m) = \lambda$. Note that the set

$$\overline{\{e^{i(-m)\pi\theta_1} : \ \theta_1 \text{ is irrational number }, \ m \geq 0\}} = \partial\mathbb{D}.$$

Since $\lambda\partial\mathbb{D}$ is a compact subset of $\mathbb{D}$. Thus $\{\lambda_m\}_{m\in\mathbb{N}}$ has a limit point in $\mathbb{D}$. Since similarity preserves supercyclicity, then by Theorem 3.9 it follows that the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ satisfies the Supercyclicity Criterion under the condition (i).

Similarly, the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ also satisfies the Supercyclicity Criterion under the condition (ii). This completes the proof. □

Remark 3.11. *Our results are also valid for n-tuples of the adjoint of the weighted composition operators on $\mathcal{H}$. The interested readers can try to prove them.*

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