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SINC-GALERKIN METHOD FOR SOLVING A CLASS OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this article, we develop the Sinc-Galerkin method based on double exponential transformation for solving a class of weakly singular nonlinear two-point boundary value problems with nonhomogeneous boundary conditions. Also several examples are solved to show the accuracy efficiency of the presented method. We compare the obtained numerical results with results of the other existing methods in the literature. The results of this paper confirm that our method is very fast, simple and considerably accurate.
Keywords: Sinc-Galerkin method, double exponential transformation, nonlinear singular boundary value problems, ordinary differential equation.
MSC(2010): 65L10, 65L60, 65H10, 41A30.

1. Introduction

Nonlinear boundary value problems arise in the analysis of many different problems in engineering and mathematical physics such as gas dynamics, fluid dynamics, nuclear physics, atomic structures, atomic calculations and chemical reactions. These problems have been studied by many researchers. But majority of these problems cannot be solved analytically, hens we need to use numerical methods to find approximate solutions. Therefore, there exist many numerical methods [1-11,13-18,26-27] for solving boundary value problems but no precise and widely applicable method is available that can treat all different types of such problems.

In this paper, we consider the following nonlinear two-point boundary value problem:

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)R(u(x)) = f(x), \ a < x < b \\ u(a) = \alpha, \ u(b) = \beta, \end{cases} \tag{1.1}$$

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where $R(u(x))$ is a function of u, and α, β, a, b are real constants. Furthermore, $p(x), q(x)$ and $f(x)$ are analytic functions of x in the neighborhood (a, b), that could be singular at $x = a$ or $x = b$ or both.

The special case of this model is Bratu-type problem in one dimensional planar coordinates, which is used in a large variety of applications such as the fuel ignition model in thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [3,13,15]. This problem is a nonlinear boundary value problem that is used as a benchmark problem to test the accuracy of many numerical methods.

Problem (1) involves regular and singular problems. The standard numerical methods which are designed for regular problems, suffer from a large loss of accuracy or may even fail to converge when they are applied to singular problems. By applying Sinc method there is no difficulty to handel singularities at the end of the interval because this approach depends only on parameters of the problem regardless the problem is singular or regular. The Sinc method based on single exponential transformation (SE) has been developed by Frank Stenger more than thirty years ago. He and his collaborators applied this method to several fields of applied mathematics. In their studies, they obtained the order of accuracy $\mathrm{O}(\exp(-c\sqrt{n}))$ where c is a positive constant and n is the number of nodes or basis functions used in the method [6, 12, 14, 19, 20]. Takahasi and Mori introduced double exponential transformation (DE) to evaluate the integrals of an analytic function with end point singularity [23]. Then the researchers applied this transformation to Sinc method and developed it on quadrature, approximation, linear boundary value problems, etc. They found that the error of accuracy with Sinc method based on DE transformation is $\mathrm{O}(\exp(-kn/logn))$ with some positive k [16, 21, 22, 24, 25].

In this paper, we develop Sinc-Galerkin method based on double exponential transformation to a class of weakly singular nonlinear two-point boundary value problem (1).

The paper is organized into four sections. In Section 2, we recall some of the main properties of Sinc function, double exponential transformation, theorems and notations that we need in the sequel. In Section 3, we demonstrate how the Sinc-Galerkin based on DE transformation method can be used to replace problem (1) by an explicit system of nonlinear algebraic equations. In Section 4, the presented method is tested on several problems such as Bratu-type equation. Our numerical results are compared with numerical results of other existing methods.

2. Some properties of the Sinc function

In this section, we review some properties of the Sinc function, Sinc quadrature and notations from [12, 24], and [25] that we need later.

The Sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$\mathrm{Sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For any $h > 0$, the translated Sinc function with evenly spaced nodes are given by

$$\mathrm{S}(j,h)(x) = \mathrm{Sinc}(\frac{x - jh}{h}), \; j = 0, \pm 1, \pm 2, ... \tag{2.1}$$

The $S(j,h)$ is called the **j**th Sinc function with step size h at x.

Lemma 2.1 [12] Let $S(k,h)(x)$ be the kth Sinc function with step h. Thus,

$$\delta_{jk}^{(0)} = S(j,h)(kh) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{dz}[S(j,h)(z)](kh) = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{dz^2}[S(j,h)(z)](kh) = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

For the assembly of the discrete system, it is convenient to define the following matrices

$$I^{(l)} = [\delta_{jk}^{(l)}] \qquad l = 0, 1, 2, \tag{2.2}$$

where $\delta_{jk}^{(l)}$ denotes the (j,k)th entry of the matrix $I^{(l)}$. The matrix $I^{(0)}$ is the $m \times m$ identity matrix. The matrix $I^{(1)}$ is the skew symmetric Toeplitz matrix and $I^{(2)}$ is the symmetric Toeplitz matrix.

The following notation will be necessary for writing down the system. Let $D(g)$ be the $m \times m$ diagonal matrix

$$D\Big(g(x)\Big) = \begin{pmatrix} g(-Nh) & 0 & 0 & \dots & 0 \\ 0 & g((-N+1)h) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & & g((Nh)) \end{pmatrix}$$

If the function f is defied on the real line, then for $h > 0$ the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)\mathrm{Sinc}\Big(\frac{x-jh}{h}\Big),$$

is called the Whittaker cardinal expansion of f where this series converges [19]. These properties are derived in the infinite strip D_d of the complex plane, where for $d > 0$

$$D_d = \Big\{w = \xi + i\eta : |\eta| < d < \frac{\pi}{2}\Big\}.$$

To state the decay property of functions precisely, we introduce the following function space. Let $H^1(D_d)$ be a function space defined as

$$H^1(D_d) = \{f : D_d \to C | f \ is \ analytic \ on \ D_d \ and \ N^1(f, D_d) < \infty\}$$

where

$$N^1(f, D_d) \equiv \lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(t)| \ |dt|,$$

$$D_d(\varepsilon) = \{ \ t \in \mathbf{C}| \ |Re \ t| \leq 1/\varepsilon, \ |Im \ t| \leq d(1-\varepsilon)\}.$$

Theorem 2.2 [[24],Theorem 6.1] Assume that a function f satisfies

$$1) f \in H^1(D_d),$$
$$2) \forall x \in \mathbf{R} : | \ f(x) \ | \leq A \exp(-B \ \exp(\gamma \ \ | \ x \ |)),$$

for positive constants A, B, γ and d where $\gamma d \leq \frac{\pi}{2}$. Then there exists a constant C independent of N, such that:

$$\sup_{-\infty<x<\infty} \Big| \ f(x) - \sum_{k=-N}^{N} f(kh)S(k,h)(x)\Big| \leq C \exp\Big(- \frac{\pi d\gamma N}{log(\pi d\gamma N/B)}\Big),$$

where

$$h = \frac{\log(\pi d\gamma N/B)}{\gamma N}.$$

Theorem 2.3 [[25],Theorem 5.1] For $d > 0$, let f be a holomorphic function on D_d which satisfies

$$1) f \ \in H^1(D_d),$$
$$2) \forall x \ \in \mathbf{R} : | \ f(x) \ | \leq A \exp(-B \ \exp(\gamma \ \ | \ x \ |)),$$

for constants A, $B > 0$ and $\gamma > 0$ with $\gamma d \leq \frac{\pi}{2}$. Then there exists a constant C independent of N, such that:

$$(2.3) \quad | \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-N}^{N} f(kh) \ | \leq C \exp\Big(- \frac{2\pi d\gamma N}{log(2\pi d\gamma N/B)}\Big),$$

where

$$h = \frac{\log(2\pi d\gamma N/B)}{\gamma N}. \tag{2.4}$$

3. De-Sinc-Galerkin method for BVPs

Consider the following class of nonlinear two-point boundary value problems:

$$\begin{cases} L(u(x)) \equiv u''(x) + p(x)u'(x) + q(x)R(u(x)) = f(x), \ a < x < b, \\ \\ u(a) = \alpha, \ u(b) = \beta. \end{cases} \tag{3.1}$$

Before illustrating Sinc-Galerkin method based on DE transformation, we need to convert the nonhomogeneous boundary conditions to homogeneous ones. Therefore, we can define the following function:

$$\Theta(x) = Ax + B,$$

where the real constants A, B are $A = \frac{\beta-\alpha}{a-b}$ and $B = \frac{b\alpha - a\beta}{b-a}$. Now, by the following change of variable

$$v(x) = u(x) - \Theta(x), \tag{3.2}$$

problem (6) takes the following form:

$$\begin{cases} L(v(x)) \equiv v''(x) + p(x)v'(x) + q(x)R\Big(v(x) + \Theta(x)\Big) + \\ \qquad\qquad\qquad\qquad p(x)\Theta'(x) = f(x), \ a < x < b, \\ \\ v(a) = 0, \ v(b) = 0. \end{cases} \tag{3.3}$$

The suitable domain for Sinc function is $(-\infty, \infty)$. For problems with a different domain, there are two points of view. The first is to change the variables in the problem so that, in the new variables, the problem has domain $(-\infty, \infty)$. The second procedure is to move the numerical process and study it on the original domain of the problem. Here we take the first approach. For our problem with domain (a, b), the appropriate transformation is the following conformal map:

$$x = \psi(t) = \frac{b-a}{2}\tanh\Big(\frac{\pi}{2}\sinh(t)\Big) + \frac{a+b}{2}, \tag{3.4}$$

$$t = \phi(x) = \psi^{-1}(x) = \log\Big[\frac{1}{\pi}\log(\frac{x-a}{b-x}) + \sqrt{1 + \Big(\frac{1}{\pi}\log(\frac{x-a}{b-x})\Big)^2}\Big], \tag{3.5}$$

which is a double exponential (DE) transformation and the Sinc-Galerkin method based on this transformation is called DE-Sinc-Galerkin method. The DE transformation [23–25] maps $\mathbf{R}$ onto (a, b) and maps D_d onto the domain:

$$\psi(D_d) = \Big\{ z\epsilon\mathcal{C} : \Big| \arg\Big(\frac{1}{\pi}\log(\frac{z-a}{b-z}) + \sqrt{1 + \Big(\frac{1}{\pi}\log(\frac{z-a}{b-z})\Big)^2} \Big) \Big| < d \Big\}.$$

By applying ψ to problem (8), the problem is defined in the following form on $(-\infty, \infty)$

(3.6)

$$\begin{cases} L(v(\psi(t))) \equiv \frac{d^2}{dx^2}v(\psi(t)) + p(\psi(t))\frac{d}{dx}v(\psi(t)) + q(\psi(t))R(v(\psi(t)) + \Theta(\psi(t))) \\ \qquad + p(\psi(t))\Theta'(\psi(t)) = f(\psi(t)), \\ \lim_{t\to\pm\infty} v(\psi(t)) = 0. \end{cases}$$

Set $y(t) = v(\psi(t))$, so by the chain rule of differentiation we obtain

$$\frac{d}{dx}v(\psi(t)) = \frac{1}{\psi'(t)}y'(t), \tag{3.7}$$

$$\frac{d^2}{dx^2}(v(\psi(t))) = \Big(\frac{1}{\psi'(t)}\Big)^2 y''(t) - \frac{\psi''(t)}{(\psi'(t))^3}y'(t). \tag{3.8}$$

By using formulas (3.12) and (3.13) and multiplying by $\psi'(t)$ we have

(3.9)

$$\begin{cases} L(y(t)) \equiv \frac{1}{\psi'(t)}y''(t) + \Big(p(\psi(t)) - \frac{\psi''(t)}{(\psi'(t))^2}\Big)y'(t) + \psi'(t)q(\psi(t))R\Big(y(t) + \Theta(\psi(t))\Big) \\ \qquad + \psi'(t)p(\psi(t))\Theta'(\psi(t)) = \psi'(t)f(\psi(t)), \\ \lim_{t\to\pm\infty} y(t) = 0. \end{cases}$$

To approximate the solution of problem (3.14) we consider Sinc approximation by the formula

$$y_m(t) = \sum_{j=-N}^{N} c_j S_j(t), \ m = 2N+1, \tag{3.10}$$

where the bases Sinc functions $S_j(t) = S(j, h)(t)$ are defined in (2.1) and the unknown coefficients $\{c_j\}_{k=-N}^{N}$ need to be determined. Notice that y_m satisfies the boundary conditions because $\lim_{t\to\pm\infty} S_j(t) = 0$.

To apply Galerkin method, first, we consider the following form of an inner product for arbitrary functions f and g:

$$\prec f, g \succ = \int_{-\infty}^{+\infty} f(t)g(t)w(t)dt, \tag{3.11}$$

in which $w(t)$ is a weight function. The weight function may be chosen based on several factors. Lund [12] used a suitable w for symmetrization of the system in a self-adjoint linear problem. However, we choose w so that the boundary terms occurring in the integration by past is zero. To do this, we choose

$w(t) = \psi'(t)$. For more information about different choices of the weight function see [12, 14, 16, 19].

Based on the Galerkin method, the coefficients $\{c_j\}_{j=-N}^{N}$ are determined by orthogonalizing the residual $Lv - \psi' f$ with respect to the functions $\Big\{S_k(x)\Big\}_{k=-N}^{N}$, in other word,

$$\prec Lv - \psi' f, S_k \succ = 0, \;\; k = -N, -N+1, ..., N.$$

So we have,

$$\begin{aligned}
&\prec \frac{1}{\psi'(t)} y''(t), S_k(t) \succ + \prec \Big(p(\psi(t)) - \frac{\psi''(t)}{(\psi'(t))^2}\Big) y'(t), S_k(t) \succ + \\
&\prec \psi'(t) q(\psi(t)) R\Big(y(t) + \Theta(\psi(t))\Big), S_k(t) \succ + \prec p(\psi(t)) \psi'(t) \Theta'(\psi(t)), S_k(t) \succ \\
&= \prec \psi'(t) f(\psi(t)), S_k(t) \succ, \;\; k = -N, -N+1, ..., N.
\end{aligned} \tag{3.12}$$

By applying the inner product formula defined in (2.16), the above inner products are calculated as follows:

$$\prec \frac{1}{\psi'(t)} y''(t), S_k(t) \succ = \int_{-\infty}^{+\infty} \frac{1}{\psi'(t)} y''(t) S_k \psi'(t)(t) dt. \tag{3.13}$$

Using the integration by part twice, we can write the above integration as follows,

$$\int_{-\infty}^{+\infty} y''(t) S_k(t) dt = y'(t) S_k(t) - y(t) S'_k(t) + \int_{-\infty}^{+\infty} y(t) S''_k(t) dt. \tag{3.14}$$

We know that $\lim_{t \to \pm\infty} S_k(t) = 0$, also by considering the boundary conditions, the first and the second terms in the right hand side of equation (3.19) vanish. Suppose that $y(t) S''_k(t)$ satisfies the conditions of Theorem 2.3, so we obtain

$$\begin{aligned}
&\Big| \int_{-\infty}^{+\infty} y(t) S''_k(t) dt - h \sum_{j=-N}^{N} y(jh) S''_k(jh) \Big| \\
&\qquad \le C_0 \exp\Big(-\frac{k' N}{log(k' N / B)}\Big),
\end{aligned} \tag{3.15}$$

where h is defined in (5) and $k' = 2\pi d\gamma$.

Another inner product is

$$\prec \Big(p(\psi(t) - \frac{\psi''(t)}{(\psi'(t))^2}\Big)y'(t), S_k(t) \succ = \int_{-\infty}^{+\infty} \psi'(t)\Big(p(\psi(t)) - \frac{\psi''(t)}{(\psi'(t))^2}\Big)y'(t)S_k(t)dt$$
$$= y(t)\frac{\psi''(t)}{\psi'(t)}S_k(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty} \Big\{y(t)\Big[\psi''(t)p(\psi(t)) + (\psi'(t))^2 p'(\psi(t))$$
$$- \Big(\frac{\psi''(t)}{\psi'(t)}\Big)'\Big]S_k(t) + \Big[\psi'(t)p(\psi(t)) - \frac{\psi''(t)}{\psi'(t)}\Big]S_k'(t)\Big\}dt.$$

If we suppose that $\Big((\psi'(t)p(\psi(t)) - \frac{\psi''(t)}{\psi'(t)})S_k\Big)'$ satisfies the conditions of Theorem 2.3, so by using the quadrature formula (4) we have

$$\Big| \prec \Big(p(\psi(t) - \frac{1}{\psi'(t)}\frac{\psi''(t)}{\psi'(t)}\Big)y'(t), S_k(t) \succ -h\sum_{j=-N}^{N} y(jh)\Big\{\Big[-\psi''(t)p(\psi(t)) -$$
$$(\psi'(t))^2 p'(\psi(t)) + \Big(\frac{\psi''(t)}{\psi'(t)}\Big)'\Big](jh)S_k(jh) + \Big[\frac{\psi''(t)}{\psi'(t)} - \psi'(t)p(\psi(t))\Big](jh)S_k'(jh)\Big\}\Big|$$
$$\le C_1 \exp\Big(-\frac{k'N}{log(k'N/B)}\Big). \tag{3.16}$$

The third term of the inner product in equation (17) can be written as

$$\prec \psi'(t)q(\psi(t))R\Big(y(t) + \Theta(\psi(t))\Big), S_k(t) \succ = \int_{-\infty}^{+\infty}(\psi'(t))^2 q(\psi(t))R\Big(y(t) + \Theta(\psi(t))\Big)S_k(t)dt.$$

By using the approximation of integral formula (4) we obtain

$$\Big|\int_{-\infty}^{+\infty}(\psi'(t))^2 q(\psi(t))R\Big(y(t) + \Theta(\psi(t))\Big)S_k(t)dt$$
$$- h\Big[(\psi'(kh))^2 q(\psi(kh))R\Big(y(kh) + \Theta(\psi(kh))\Big)\Big|$$
$$\le C_2 \exp\Big(-\frac{k'N}{log(k'N/B)}\Big). \tag{3.17}$$

The fourth term of inner product in equation (17) is calculated as follows,

$$\prec \psi'(t)p(\psi(t))\Theta'(\psi(t)), S_k(t) \succ = \int_{-\infty}^{+\infty}(\psi'(t))^2 p(\psi(t))\Theta'(\psi(t))S_k(t)dt.$$

By using the approximation of integral formula (4) we get,

$$\Big|\int_{-\infty}^{+\infty}(\psi'(t))^2 p(\psi(t))\Theta'(\psi(t))S_k(t)dt - h(\psi'(kh))^2 p(\psi(kh))\Theta'(\psi(kh))\Big|$$
$$\le C_3 \exp\Big(-\frac{k'N}{log(k'N/B)}\Big). \tag{3.18}$$

Finally

$$\prec \psi'(t)f(\psi(t)), S_k(t) \succ = \int_{-\infty}^{+\infty} (\psi'(t))^2 f(\psi(t)) S_k(t) dt,$$

and we can approximate this integral by quadrature formula (3.4) as follows

$$\Big| \int_{-\infty}^{+\infty} (\psi'(t))^2 f(\psi(t)) S_k(t) dt - h(\psi'(kh))^2 f(\psi(kh)) \Big| \leq C_4 \exp\Big(-\frac{k'N}{log(k'N/B)}\Big). \tag{3.19}$$

Replacing each term in (3.17) by (3.20), (3.21), (3.22), (3.23) and (3.24) respectively, we have

$$\Big| \prec Ly - \psi' f, S_k \succ -h \sum_{j=-N}^{N} y(jh) \Big\{ S_k''(jh) + \Big[\frac{\psi''(jh)}{\psi'(jh)} - \psi'(jh) p(\psi(jh))\Big] S_k'(jh)$$
$$+ \Big[\Big(\frac{\psi''(jh)}{\psi'(jh)}\Big)' - \psi''(jh) p(\psi(jh)) - (\psi'(jh))^2 p'(\psi(jh)) + (\psi'(jh))^2 p(\psi(jh)) \Theta'(\psi(jh))\Big] S_k(jh) \Big\}$$
$$h(\psi'(kh))^2 q(\psi(kh)) R\Big(y(kh) + \Theta(\psi(kh))\Big) + h(\psi'(kh))^2 f(\psi(kh)) \Big|$$
$$\leq \Big| \prec \frac{1}{\psi'(t)} y''(t), S_k(t) \succ -h \sum_{j=-N}^{N} y(jh) S_k''(jh) \Big| + \Big| \prec \Big(p(\psi(t)) -$$
$$\frac{\psi''(t)}{(\psi'(t))^2}\Big) y'(t), S_k(t) \succ -h \sum_{j=-N}^{N} y(jh) \Big\{ \Big[-\psi''(jh) p(\psi(jh))$$
$$-(\psi'(jh))^2 p'(\psi(jh)) + \Big(\frac{\psi''(t)}{\psi'(t)}\Big)'(jh)\Big] S_k(jh) + h\Big[\frac{\psi''(jh)}{\psi'(jh)} - \psi'(jh) p(\psi(jh))\Big] S_k'(jh) \Big\} \Big|$$
$$+ \Big| \prec \psi'(t) q(\psi(t)) R\Big(y(t) + \Theta(\psi(t))\Big), S_k(t) \succ -h\Big(\psi'(kh)\Big)^2 q(\psi(kh)) R\Big(y(kh) + \Theta(\psi(kh))\Big) \Big|$$
$$+ \Big| \prec p(\psi(t)) \psi'(t) \Theta'(\psi(t)), S_k \succ -h\Big(\psi'(kh)\Big)^2 p(\psi(kh)) \Theta'(\psi(kh)) \Big|$$
$$+ \Big| \prec \psi'(t) f(\psi(t)), S_k \succ -h\Big(\psi'(kh)\Big)^2 f(\psi(kh)) \Big|$$

$$\leq (C_0 + C_1 + C_2 + C_3 + C_4) \exp\Big(-\frac{k'N}{log(k'N/B)}\Big). \tag{3.20}$$

By deleting the error term, replacing $y(jh)$ by c_j and dividing by h, we have the following nonlinear algebraic system

$$\sum_{j=-N}^{N} c_j \Big\{ S_k''(jh) + \Big[\frac{\psi''(jh)}{\psi'(jh)} - \psi'(jh)p(\psi(jh))\Big] S_k'(jh)$$

$$+ \Big[\Big(\frac{\psi''(jh)}{\psi'(jh)}\Big)' - \psi''(jh)p(\psi(jh)) - (\psi'(jh))^2 p'(\psi(jh))$$

$$+ (\psi'(jh))^2 p(\psi(jh))\Theta'(\psi(jh))\Big] S_k(jh)\Big\}$$

$$+ (\psi'(kh))^2 q(\psi(kh)) R(c_k + \Theta(\psi(kh))) = (\psi'(kh))^2 f(\psi'(kh)),$$

(3.21)

$$k = -N, -N+1, ..., N.$$

We can use the notations of $\delta_{kj}^{(l)}$ in Lemma 2.1 and rewrite the system (3.26) as follow

$$\sum_{j=-N}^{N} c_j \Big\{ \frac{1}{h^2}\delta_{kj}^{(2)} + \Big[\frac{\psi''(jh)}{\psi'(jh)} - \psi'(jh)p(\psi(jh))\Big] \frac{1}{h}\delta_{kj}^{(1)}$$

$$+ \Big[\Big(\frac{\psi''(jh)}{\psi'(jh)}\Big)' - \psi''(jh)p(\psi(jh)) - (\psi'(jh))^2 p'(\psi(jh))$$

$$+ (\psi'(jh))^2 p(\psi(jh))\Theta'(\psi(jh))\Big] \delta_{kj}^{(0)}\Big\}$$

$$+ (\psi'(kh))^2 q(\psi(kh)) R(c_k + \Theta(\psi(kh))) = (\psi'(kh))^2 f(\psi'(kh)),$$
$$k = -N, -N+1, ..., N.$$

For simplicity, by recalling the notation in Section 2, we can write down the nonlinear algebraic system of (3.27) in a matrix-vector form. Let C be the m-vector with jth component given by c_j, and let $R(C+\Theta)$ be the m-vector with jth component given by $R(c_j + \Theta(\psi(jh)))$. Thus the nonlinear algebraic system is rewritten as follow

(3.22) $$AC + BR(C + \Theta) = Tf,$$

where

(3.23)

$$A = \frac{1}{h^2} I^{(2)} + \frac{1}{h} I^{(1)} D\Big(\frac{\psi''}{\psi'} - \psi' p(\psi)\Big) + I^{(0)} D\Big(\Big(\frac{\psi''(jh)}{\psi'(jh)}\Big)' - \psi'' p(\psi) - (\psi')^2 p'(\psi) + (\psi')^2 p(\psi)\Theta'(\psi)\Big),$$

(3.24)

$$B = D\Big((\psi')^2 q(\psi)\Big),$$

(3.25)

$$T = D\Big((\psi')^2\Big),$$

(3.26)

$$f = \Big(f(\psi(-Nh)), f(\psi((-N+1)h)), ..., f(\psi(Nh))\Big)^T.$$

Now, we have a nonlinear system of $m = 2N+1$ equations and m unknown coefficients $\{c_j\}_{j=-N}^{N}$. By solving this system, we can find the unknown coefficients $\{c_j\}_{j=-N}^{N}$ and calculate the Sinc approximation solution by (3.15).

For solving system (3.28), we use the Newton's method. In Newton's method, we start with an initial guess C_0 and then use the Newton iteration as follows

$$C_{k+1} = C_k - J^{-1}(C_k)\ \Big\{F(C_k)\Big\},$$

where

$$F(C) = AC\ +\ B\ R(C+\Theta) - Tf,$$

and

$$J(C) = A + B\ D(\frac{\partial}{\partial C} R(C+\Theta)).$$

Here, C_k is the current iteration and C_{k+1} is the new iteration. A common numerical rule is to stop the Newton iteration whenever the distance between two iterates is less than a given tolerance, i.e., where $\|C_{k+1} - C_k\| < \varepsilon$, where the Euclidean norm is used. By solving this system and obtaining $C = (c_{-N}, ..., c_N)^T$, we can calculate $y_m(t)$ in (15) as a numerical solution.

As we know, the Sinc-Galerkin method has two types of errors. The first error is due to the utility of the double exponential formula to numerical integration of the inner products whose order of magnitude is $O\Big(exp(-2\pi d/h)\Big)$ [23]. The second error is due to the application of Sinc expansion whose order of magnitude is $O\Big(exp(-\pi d/h)\Big)$ [22]. The fist error is much smaller than the second one and it can be ignored compared with the second error for small h, i.e., for large N. Thus we can conclude that the total error of the presented approximation method is of the order $O\Big(exp(-\pi d/h)\Big)$.

4. **Numerical results**

In this section, we consider six test examples which are chosen from literature and apply DE-Sinc-Galerkin (DESG) method to all of them. Also, in all examples, we consider the solution at 999 equally spaced gride points Ω where

$$\Omega = \{x_1, x_2, ..., x_{999}\},$$
$$x_k = a + \frac{k}{1000}(b-a), \qquad k = 1, 2, ..., 999,$$

and the maximum absolute error (MAE) in the solution is calculated as follows:

$$Maximum\ Absolute\ Error = \max_{1\le k\le 999} |y_{exatsolution}(x_k) - y_{m,DESG}(x_k)|.$$

Furthermore, to compare numerical results of the presented method with another methods, we determine the absolute error in the solution at some special points. All arising nonlinear systems are solved by Newton's method with the tolerance $\|C_{k+1} - C_k\| < 10^{-10}$ and the initial guess zero vector. All the computational programs developed on PC with 1 gigabyte memory by MATLAB.

For te sake of simplicity, first we determine the following functions:

(4.1)
$$\psi(t) = \frac{b-a}{2}\tanh\Big(\frac{\pi}{2}\sinh(t)\Big) + \frac{a+b}{2},$$
(4.2)
$$\psi'(t) = \frac{\pi}{b-a}\cosh(t)(\psi(t)-a)(b-\psi(t)),$$
(4.3)
$$\frac{\psi''(t)}{\psi'(t)} = \frac{\pi}{b-a}\cosh(t)(-2\psi(t)+a+b) + \tanh(t),$$
(4.4)
$$\Big(\frac{\psi''(t)}{\psi'(t)}\Big)' = \Big(\frac{1}{\cosh(t)}\Big)^2 - \pi\ \sinh(t)\tanh\Big(\frac{\pi}{2}\sinh(t)\Big) - \frac{\pi^2}{2}\Big(\frac{\cosh(t)}{\cosh\Big(\frac{\pi}{2}\sinh(t)\Big)}\Big)^2,$$

and substituting these functions in (39), (40), (41) and (42).

Example 4.1. Consider the following weakly singular nonlinear two-points boundary value problem

$$\begin{cases} u''(x) + \frac{1}{\sqrt{x}}u'(x) + \frac{1}{x}u(x) = f(x) + \sin(u^2(x)) + \exp(u^9(x)) - u^{11}(x), 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

where the exact solution is given by $u(x) = (x - x^2)\sin(x)$, and

$$f(x) = \frac{1}{\sqrt{x}}\Big[(2\sqrt{x} + x - 4x\sqrt{x} - x^2)\cos(x) + (1 - \sqrt{x} - 2x - 2x\sqrt{x} + x^2\sqrt{x})\sin(x)\Big] - \sin\Big(((x-x^2)\sin(x))^2\Big) - \exp\Big(((x-x^2)\sin(x))^9\Big) + \Big((x-x^2)\sin(x)\Big)^{11}.$$

Table 1. Absolute error in the solution for Example 1.

x	method in [26]	DESG $N = 10$	DESG $N = 20$	DESG $N = 30$
0.08	$2.5120E-7$	$6.76E-7$	$5.21E-11$	$1.51E-14$
0.16	$2.5039E-7$	$2.28E-7$	$1.29E-10$	$1.24E-14$
0.32	$2.2597E-7$	$4.76E-7$	$1.27E-10$	$1.37E-14$
0.48	$1.9226E-7$	$3.06E-7$	$3.14E-11$	$1.41E-14$
0.64	$1.5777E-7$	$4.81E-7$	$9.87E-11$	$1.65E-14$
0.80	$1.2554E-7$	$6.04E-7$	$4.11E-11$	$8.75E-15$
0.96	$9.6121E-8$	$6.50E-7$	$5.96E-11$	$1.41E-14$

Table 2. Maximum absolute error in the solution for Example 1

N	5	10	15	20	25	30	35
MAE	$4.40E-4$	$1.03E-6$	$9.63E-9$	$1.40E-10$	$2.11E-12$	$3.43E-14$	$1.19E-15$

This problem is singular at $x = 0$. We solve the problem by DE-Sinc-Galerkin (DESG) method which was developed in Section 3 with $h = log(\pi N/4)/N$ and compare our numerical results with reproducing kernel method [26]. In this example, the computed solutions are compared with the exact solution at the specified points in reference [26] and the results are shown in Table 1. This table shows that our method is considerably more accurate in comparison with the method in [26]. Also the maximum absolute error (MAE) in the solution in the set point Ω is presented in Table 2 for different values of N. The exponential convergence of our method can be seen in this table.

Example 4.2. Consider the following weakly singular nonlinear BVP with nonhomogeneous boundary conditions:

$$\begin{cases} u''(x) + \frac{1}{x}u'(x) + \frac{1}{x(1-x)}u(x) + x\sin(\sqrt{u(x)}) = f(x), & 0 < x < 1, \\ u(0) = 1, \quad u(1) = e, \end{cases}$$

where

$$f(x) = \frac{1}{x(1-x)}[-(\exp(x)(-2+x^2)) - (-1+x)x^2\sin\sqrt{\exp(x)}\ ],$$

with the exact solution $u(x) = e^x$. We know that this problem is weakly singular at $x = 0$ and $x = 1$. We have applied our method with $h = log(\pi N/3)/N$

Table 3. Absolute error in the solution for Example 2.

x	method in [7]	DESG $N = 10$	DESG $N = 20$	DESG $N = 30$
0.01	$2.5120E-7$	$9.68E-7$	$2.83E-11$	$1.60E-13$
0.08	$2.5039E-7$	$1.35E-6$	$1.11E-12$	$1.54E-13$
0.16	$2.2597E-7$	$1.17E-6$	$4.94E-11$	$1.39E-13$
0.32	$1.9226E-7$	$1.07E-6$	$4.95E-11$	$1.19E-13$
0.48	$1.5777E-7$	$8.02E-7$	$1.01E-11$	$8.85E-14$
0.64	$1.2554E-7$	$9.16E-7$	$2.45E-12$	$5.70E-14$
0.86	$1.2554E-7$	$1.98E-7$	$6.51E-12$	$3.09E-14$
0.96	$9.6121E-8$	$1.84E-8$	$9.62E-12$	$3.33E-15$

Table 4. Maximum absolute error in the solution for Example 2.

N	5	10	15	20	25	30	35
MAE	$1.07E-4$	$1.71E-6$	$9.43E-9$	$6.05E-11$	$2.83E-12$	$1.69E-13$	$6.41E-14$

to solve this problem. The numerical results are compared with numerical results of reproducing kernel Hilbert space method [7] at special points which are used in [7]. These results are tabulated in Table 3. Also the maximum absolute error (MAE) in the solution at set point Ω for different values of N are shown in Table 4. The results show that our method is accurate and the accuracy's is increased by increasing N.

Example 4.3. Consider the following nonlinear two-point BVP

$$\begin{cases} u''(x) - u^2 = 2\pi^2 \cos(2\pi x) - \sin^4(\pi x), & 0 < x < 1, \\ u(0) = 0 \quad , u(1) = 0, \end{cases}$$

in which, the exact solution is $u(x) = \sin^2(\pi x)$.

This problem has been solved by several methods such as extended Adomian decomposition method (EADM) [10], nonlinear shooting method (NSM) [8], and Homotopy perturbation method (HPM) [4]. We solve the problem by presented method with $h = log(\pi N/7)/N$ which is developed in Section 3. The obtained numerical results are compared with numerical results in [4,8] and [10] at special points $0.1, 0.2, 0.3, ..., 0.9$. The data are recorded in Table 5. Also the maximum absolute error in the solution for our method in the set Ω is listed in Table 6.

TABLE 5. Absolute error in the solution for Example 3

x	HPM [4]	EADM [10]	NSM [8]	DE-Sinc-Galerkin $M=20$	DE-Sinc-Galerkin $M=30$
0.1	$0.1E-09$	$6.9E-07$	$0.8E-06$	$3.70E-10$	$5.32E-13$
0.2	$0.5E-09$	$1.3E-06$	$2.8E-06$	$4.18E-10$	$1.16E-13$
0.3	$0.5E-09$	$1.9E-06$	$5.4E-06$	$3.12E-10$	$1.83E-14$
0.4	$0.1E-09$	$2.3E-06$	$7.5E-06$	$1.02E-10$	$1.65E-13$
0.5	$0.1E-09$	$2.5E-06$	$8.3E-06$	$2.08E-10$	$1.06E-13$
0.6	$0.6E-09$	$2.3E-06$	$7.5E-06$	$1.02E-10$	$1.65E-13$
0.7	$0.6E-09$	$1.9E-06$	$5.4E-06$	$3.12E-10$	$1.89E-14$
0.8	$0.7E-09$	$1.3E-06$	$2.7E-06$	$4.18E-10$	$1.15E-13$
0.9	$0.9E-09$	$6.9E-06$	$0.6E-06$	$3.70E-10$	$5.32E-13$

TABLE 6. Maximum absolute error in the solution for Example 3.

N	5	10	15	20	25	30	35
MAE	$9.19E-3$	$2.73E-6$	$3.41E-8$	$9.05E-10$	$2.20E-11$	$5.64E-13$	$1.64E-14$

Example 4.4. Consider the following singular nonlinear BVP arising in Astronomy:

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) + u^5 = 0, & 0 < x < 1, \\ u(0) = 1 \,, u(1) = \frac{\sqrt{3}}{2}, \end{cases}$$

with exact solution $u(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$.

The problem has been solved by cubic spline method, finite difference method (FDM), and collocation method [17]. We have applied the DE-Sinc-Galerkin method based on double exponential transformation with $h = log(\pi N/4)/N$ which was developed in Section 3. The numerical results obtained in our method and other methods are shown in Table 7. Also Table 8 illustrate the maximum absolute error in the solution for our method at set points Ω for different values of N.

TABLE 7. Absolute error in the solution for Example 4

x	Spline [17]	FDM [17]	Collocation [17]	DE-Sinc-Galerkin $N=20$	DE-Sinc-Galerkin $N=30$
0.125	$1.29E-06$	$9.59E-05$	$5.59E-05$	$2.41E-11$	$2.15E-14$
0.250	$1.10E-06$	$7.33E-05$	$5.33E-05$	$4.04E-11$	$1.37E-14$
0.375	$8.26E-07$	$5.55E-05$	$4.55E-05$	$1.29E-11$	$1.62E-14$
0.500	$5.28E-07$	$3.89E-05$	$2.89E-05$	$3.82E-12$	$5.83E-15$
0.625	$2.61E-07$	$3.41E-05$	$2.41E-05$	$1.71E-11$	$1.22E-14$
0.750	$7.06E-08$	$1.29E-06$	$1.29E-06$	$1.92E-11$	$2.49E-15$
0.875	$1.93E-08$	$9.60E-06$	$9.60E-06$	$6.08E-12$	$1.66E-16$

TABLE 8. Maximum absolute error in the solution for Example 4.

N	5	10	15	20	25	30	35
MAE	$9.58E-5$	$1.51E-7$	$1.65E-9$	$4.07E-11$	$1.05E-12$	$2.91E-14$	$1.36E-15$

Example 4.5. Consider the following Bratu-type model in one dimension polar coordinates:

$$\begin{cases} u''(x) + a\exp(u(x)) = 0, & 0 < x < 1, \\ u(0) = 0 \ , u(1) = 0, \end{cases} \tag{4.5}$$

Several numerical method were applied to Bratu problem [1–3, 5, 11, 13, 15, 27].

The exact solution of problem (37) is given by:

$$u(x) = -2\ln\Big[\frac{\cosh(\frac{\xi}{2}(x-\frac{1}{2}))}{\cosh(\frac{\xi}{4})}\Big]$$

where ξ satisfies

$$\xi = \sqrt{2a}\ \cosh(\frac{\xi}{4}).$$

We have applied DE-Sinc-Galerkin (DESG) method to this problem with $h = log(\pi N/4)/N$ for $a = 1, 2$ and 3.513830719. Table 9 displays the numerical results of the absolute error in the solution for presented method, Lie-group shooting method (LGSM) [1], B-Spline method (BSM) [5], optimal spline method(OSM) [2], Laplace method (LAM) [11], parametric spline method (PSM) [27] at special points $0.1, 0.2, 0.3, ..., 0.9$ for $a = 1$. Table 10

TABLE 9. Absolute error in the solution for Example 5 with $a = 1$

x	LGSM [1]	LAM [11]	BSM [5]	OSM [2]	PSM [27]	DESG N=40
0.1	$7.50E-07$	$1.97E-06$	$2.98E-06$	$4.63E-08$	$5.87E-10$	$9.02E-17$
0.2	$1.01E-06$	$3.93E-06$	$5.46E-06$	$1.02E-07$	$2.58E-10$	$1.38E-16$
0.3	$9.04E-07$	$5.85E-06$	$7.33E-06$	$1.44E-07$	$5.59E-11$	$4.16E-17$
0.4	$5.23E-07$	$7.70E-06$	$8.50E-06$	$1.71E-07$	$8.77E-11$	$2.77E-17$
0.5	$5.06E-09$	$9.46E-06$	$8.78E-06$	$1.81E-07$	$1.38E-10$	$1.66E-16$
0.6	$5.13E-07$	$1.11E-05$	$8.50E-06$	$1.71E-07$	$8.77E-11$	0.00
0.7	$8.94E-07$	$1.25E-05$	$7.33E-06$	$1.44E-07$	$5.59E-11$	$1.38E-17$
0.8	$1.00E-06$	$1.34E-05$	$5.46E-06$	$1.02E-07$	$2.58E-10$	$1.38E-17$
0.9	$7.41E-07$	$1.19E-05$	$2.98E-06$	$4.63E-08$	$5.87E-10$	$1.39E-16$

TABLE 10. Absolute error in the solution for Example 5 with $a = 2$

x	LGSM [1]	LAM [11]	BSM [5]	PSM [27]	DESG N=40
0.1	$4.03E-06$	$2.13E-03$	$1.72E-05$	$1.25E-08$	$1.33E-15$
0.2	$5.70E-06$	$4.21E-03$	$3.25E-05$	$1.95E-08$	$1.02E-15$
0.3	$5.22E-06$	$6.19E-03$	$4.49E-05$	$2.73E-08$	$1.44E-15$
0.4	$3.07E-06$	$8.00E-03$	$5.28E-05$	$3.31E-08$	$9.43E-16$
0.5	$1.45E-08$	$9.60E-03$	$5.56E-05$	$3.53E-08$	$1.22E-15$
0.6	$3.04E-06$	$1.09E-03$	$5.28E-05$	$3.31E-08$	$9.43E-16$
0.7	$5.19E-06$	$1.19E-02$	$4.49E-05$	$2.73E-08$	$1.22E-15$
0.8	$5.67E-06$	$1.24E-02$	$3.25E-05$	$1.95E-08$	$9.71E-16$
0.9	$4.01E-06$	$1.09E-02$	$1.72E-05$	$1.25E-08$	$1.29E-15$

shows the numerical results of the absolute error in the solution for DE-Sinc-Galerkin method, Lie-group shooting method [1], B-Spline method [5], Laplace method [11] and parametric spline [27] for $a = 2$. Also the numerical results of the absolute error in the solution for DE-Sinc-Galerkin, B-spline method [5], Lie-group shooting method [1] for $a = 3.513830719$ are tabulated in Table 11. Furthermore, the Figure 1 shows the graph of approximation solution for $a = 1, 2$ and 3.513830719.

TABLE 11. Absolute error in the solution for Example 5 with $a = 3.15$

x	BSM [5]	LGSM [1]	DESG N=40
0.1	$3.84E-02$	$4.45E-05$	$2.31E-07$
0.2	$7.48E-02$	$7.12E-05$	$4.51E-07$
0.3	$1.06E-01$	$7.30E-05$	$6.43E-07$
0.4	$1.27E-01$	$4.46E-05$	$7.67E-07$
0.5	$1.35E-01$	$6.75E-07$	$8.14E-07$
0.6	$1.27E-01$	$4.56E-05$	$7.67E-07$
0.7	$1.06E-01$	$7.20E-05$	$6.43E-07$
0.8	$7.48E-02$	$7.05E-05$	$4.51E-07$
0.9	$3.84E-02$	$4.41E-05$	$2.31E-07$

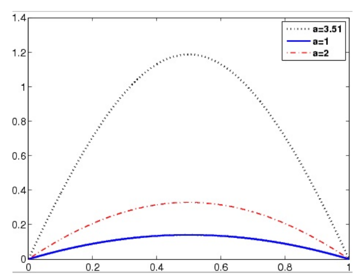

FIGURE 1. Graph of approximation solution with $a = 1,\ 2\ and\ 3.513830719$ for Bratu problem.

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