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THE DISTRIBUTIONAL HENSTOCK-KURZWEIL INTEGRAL AND MEASURE DIFFERENTIAL EQUATIONS

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Abstract. In the present paper, measure differential equations involving the distributional Henstock-Kurzweil integral are investigated. Theorems on the existence and structure of the set of solutions are established by using Schauder$'s$ fixed point theorem and Vidossich theorem. Two examples of the main results paper are presented. The new results are generalizations of some previous results in the literatures.
Keywords: Distributional Henstock-Kurzweil integral, measure differential equation, Henstock-Stieltjes integral, distributional derivative.
MSC(2010): Primary: 26A39; Secondary: 28B05, 34A12, 46F05, 46G12.

1. **Introduction**

In this paper, we consider the following measure differential equation (MDE for short)

$$\begin{cases} Dx = f(x,t) + g(x,t)Du, \\ x(a) = x_0, \end{cases} \tag{1.1}$$

where $t \in [a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$, $x \in C([a,b])$, $u : [a,b] \to \mathbb{R}$ is a right continuous function of bounded variation, $f : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ is distributionally Henstock-Kurzweil integrable, $g : \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ is Henstock-Stieltjes (HS) integrable.

MDEs have been studied by many authors, we refer the readers to [5, 6, 11, 13, 15, 19] and the references therein. For instance, in [15], Tanwani, et al. considered the MDE (1.1) in a special case where $f(x,t) = f(x(t)), g(x,t) = g(x(t))$. Schmaedeke [11] considered the equation (1.1) in a complicated case where $g(x(t))$ is a continuous $n \times m$ matrix. Furthermore, Das and Sharma [6] generalized the results in [11] for a functional differential equation, i.e. $Dx = f(t, x_t) + g(t, x_t)Du$.

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All these papers use the ordinary derivative to discuss the MDEs. In this paper, we will use the distributional derivative to study the MDE (1.1). It is well known that the notion of a distributional derivative is a general concept, including ordinary derivatives and approximate derivatives. The distributional Henstock-Kurzweil integral is defined by the distributional derivative. In this case, we present some new results which are generalizations of the previous results in [5, 19].

The paper is organized as follows. In Section 2, we introduce some fundamental concepts and basic results of the distributional Henstock-Kurzweil integral. In Section 3, we apply the Schauder$'s$ fixed point theorem to verify the existence of solutions of the MDE (1.1). We also show that the set of solutions of the MDE (1.1) is an R_δ by using the Vidossich theorem stated in [16]. Here, an R_δ is the intersection of a decreasing sequence of compact absolute retracts (see [2] for details). In Section 4, we give two examples to show that Theorem 3.3 in Section 3 is more extensive.

2. **The distributional Henstock-Kurzweil integral**

In this section, we present the definition and some basic properties of the distributional Henstock-Kurzweil integral.

Define the space

$$C_c^\infty = \{\phi : \mathbb{R} \to \mathbb{R} \,|\phi \in C^\infty/\mathbb{R} \ \text{ and } \ \text{has compact support in } \ \mathbb{R}\},$$

where *support* of ϕ is denoted by $supp(\phi)$. A sequence $\{\phi_n\} \subset C_c^\infty$ converges to $\phi \in C_c^\infty$ if there is a compact set K such that all ϕ_n have supports in K and the sequence of derivatives $\phi_n^{(m)}$ converges to $\phi^{(m)}$ uniformly on K for every $m \in \mathbb{N} \cup \{0\}, \phi^{(0)} = \phi$. Denote C_c^∞ endowed with this convergence property by $\mathcal{D}$. Also, ϕ is called a $test\ function$ if $\phi \in \mathcal{D}$. The dual space to $\mathcal{D}$ is denoted by $\mathcal{D}'$ and its elements are called distributions. That is, if $f \in \mathcal{D}'$ then $f : \mathcal{D} \to \mathbb{R}$, and we write $\langle f, \phi\rangle \in \mathbb{R}$, for $\phi \in \mathcal{D}$.

For all $f \in \mathcal{D}'$, we define the distributional derivative Df of f to be a distribution satisfying $\langle Df, \phi\rangle = -\langle f, \phi'\rangle$, where ϕ is a test function and ϕ' is the ordinary derivative of ϕ. With this definition, all distributions have derivatives of all orders and each derivative is a distribution.

Let (a, b) be an open interval in $\mathbb{R}$. We define

$$\begin{aligned}\mathcal{D}((a, b)) = \{\phi : (a, b) \to \mathbb{R} \mid \phi \in C_c^\infty \quad &\text{and} \\ &\text{has compact support in } (a, b)\}.\end{aligned}$$

The dual space of $\mathcal{D}((a, b))$ is denoted by $\mathcal{D}'((a, b))$.

Define

$$B_C = \{F \in C([a, b]) \mid F(a) = 0\}.$$

B_C is a Banach space with the uniform norm $\|F\|_\infty = \max_{[a,b]} |F|$.

Now we are able to introduce the definition of the D_{HK}-integral.

Definition 2.1. *A distribution f is distributionally Henstock-Kurzweil integrable or briefly D_{HK}-integrable on $[a, b]$ if f is the distributional derivative of a continuous function $F \in B_C$.*

The D_{HK}-integral of f on $[a, b]$ is denoted by $(D_{HK}) \int_a^b f = F(b)$, where F is called the primitive of f and "$(D_{HK}) \int$" denotes the D_{HK}-integral. Notice that if $f \in D_{HK}$ then f has many primitives in $C([a, b])$, but f has exactly one primitive in B_C.

The space of D_{HK}-integrable distributions is defined by

$$D_{HK} = \{f \in \mathcal{D}'((a, b)) \mid f = DF \text{ for some } F \in B_C\}.$$

With this definition, if $f \in D_{HK}$ then, for all $\phi \in \mathcal{D}((a, b))$,

$$\langle f, \phi \rangle = \langle DF, \phi \rangle = -\langle F, \phi' \rangle = -\int_a^b F\phi'.$$

Now we show that D_{HK}-integral includes the HK-integral, and hence includes Lebesgue and Riemann integrals (See [8–10] for details). So our results will extend the previous ones in [3, 4].

Remark 2.2. *In [9], Lee pointed out that if F is a continuous function and pointwise differentiable nearly everywhere on $[a, b]$, then F is ACG^*. So the primitive F of the HK-integrable function f is generalized absolutely continuous or briefly ACG^* (see [9, 10]). Furthermore, if F is a continuous function which is differentiable nowhere on $[a, b]$, then F is not ACG^*. Therefore, if $F \in C([a, b])$ but is differentiable nowhere on $[a, b]$, then DF exists and is D_{HK}-integrable but not HK-integrable. Conversely, if F is ACG^* then it belongs to $C([a, b])$. Hence F' is not only HK-integrable but also D_{HK}-integrable.*

Let us introduce some basic properties of the distributional Henstock-Kurzweil integral needed later.

Lemma 2.3 ([14, Theorem 4], Fundamental Theorem of Calculus)**.**
(a) Let $f \in D_{HK}$, and define $F(t) = (D_{HK}) \int_a^t f$. Then $F \in B_C$ and $DF = f$.
(b) Let $F \in C([a, b])$. Then $(D_{HK}) \int_a^t DF = F(t) - F(a)$ for all $t \in [a, b]$.

From definition of the positive measure, we impose a partial ordering on D_{HK}: for $f, g \in D_{HK}$, we say that $f \succeq g$ (or $g \preceq f$) if and only if $f - g$ is a positive measure on $[a, b]$. By this definition, if $f,\ g \in D_{HK}$ then $(D_{HK}) \int_a^b f \geq (D_{HK}) \int_a^b g$, whenever $f \succeq g$. See [1] for details.

It is shown that the following result holds.

Lemma 2.4 ([1, Corollary 1])**.** *If $f_1, f_2, f_3 \in \mathcal{D}'((a, b))$, $f_1 \preceq f_2 \preceq f_3$, and if f_1 and f_3 are D_{HK}-integrable, then f_2 is also D_{HK}-integrable.*

We now define the *Alexiewicz* norm by $\|f\| = \|F\|_\infty = \max_{[a,b]} |F|$ for $f \in D_{HK}$ and $F \in B_C$ with $DF = f$. Also, we say a sequence $\{f_n\} \subset D_{HK}$ converges strongly to $f \in D_{HK}$ if $\|f_n - f\| \to 0$ as $n \to \infty$.

Then the following holds.

Lemma 2.5 ([14, Theorem 2]). *With the Alexiewicz norm, D_{HK} is a Banach space.*

The Lebesgue integral has very important applications since the space L^1 of Lebesgue integrable functions is a Banach space and there are excellent convergence theorems. We have shown that D_{HK} is a Banach space, and by [14], $\mathcal{BV}$ is the dual space of D_{HK}. Next we will look at convergence theorems of D_{HK}-integral.

Lemma 2.6 ([1, Corollary 5, Dominated convergence theorem for the D_{HK}-integral]). *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in D_{HK} such that $f_n \to f$ in $\mathcal{D}'$. Suppose there exist $f_-, f_+ \in D_{HK}$ satisfying $f_- \preceq f_n \preceq f_+, \quad \forall\ n \in \mathbb{N}$. Then $f \in D_{HK}$ and $\lim_{n\to\infty}(D_{HK}) \int_a^b f_n = (D_{HK}) \int_a^b f$.*

If $g : [a, b] \to \mathbb{R}$, its variation is $Vg = \sup \sum_n |g(t_n) - g(s_n)|$, where the supremum is taken over every sequence $\{(t_n, s_n)\}$ of disjoint intervals in $[a, b]$. If $Vg < \infty$ then g is called a function with bounded variation. Denote the set of functions with bounded variation by $\mathcal{BV}$. As it is known that the dual space of D_{HK} is $\mathcal{BV}$ (see details in [14]), the next results have been presented.

Lemma 2.7 ([14, Definition 6, Integration by parts]). *Let $f \in D_{HK}$, and $g \in \mathcal{BV}$. Define $fg = DH$, where $H(t) = F(t)g(t) - \int_a^t F dg$. Then $fg \in D_{HK}$ and*

$$\int_a^b fg = F(b)g(b) - \int_a^b F dg.$$

Lemma 2.8 ([17, Theorem 2]). *Let $f(t)$ be a vector-valued function such that $f(t)$ is Henstock-Stieltjes integrable with respect to $g(t)$ on $[a, b]$. If $g(t)$ satisfies Lipschitz condition on $[a, b]$, then $F(t)$ is continuous on $[a, b]$, where*

$$F(t) = (HS) \int_a^t f(s) dg(s).$$

Lemma 2.9 ([18, Theorem 2]). *Let $g(t)$ be a function of bounded variation on $[a, b]$ and, $\{f_n(t)\}_{n=1}^{\infty}$ be a sequence of vector-valued functions such that $f_n(t)$ is Henstock-Stieltjes integrable with respect to $g(t)$ on $[a, b]$. If $f_n(t)$ converges uniformly to $f(t)$, then $f(t)$ is Henstock-Stieltjes integrable with respect to $g(t)$ on $[a, b]$ and*

$$\lim_{n\to\infty} (HS) \int_a^b f_n(t) dg(t) = (HS) \int_a^b f(t) dg(t).$$

3. Main results

In this section, we shall consider the existence of solutions and investigate the topological characterization of the set of solutions of the MDE (1.1). The main results are Theorems 3.3 and 3.5.

Define

$$B = \{x \in C([a,b]) : \|x - x_0\|_\infty \le r,\ r > 0\},$$

$$E = \{(x,t) : t \in [a,b],\ x \in B\}.$$

Now, we impose some assumptions on f and g in (1.1).

(C_1) $f(x(\cdot),\cdot)$ is D_{HK}-integrable for each $x \in B$;

(C_2) $f(\cdot,t)$ is continuous for all $t \in [a,b]$;

(C_3) There exist $f_-, f_+ \in D_{HK}$ such that $f_-(\cdot) \preceq f(x,\cdot) \preceq f_+(\cdot)$ for each $x \in B$;

(C_4) $g(x(\cdot),\cdot)$ is Henstock-Stieltjes integrable for each $x \in B$;

(C_5) $g(\cdot,t)$ is continuous for all $t \in [a,b]$;

(C_6) There exist g_-, g_+ such that

$$g_-(\cdot) \le g(x,\cdot) \le g_+(\cdot), \quad (x,t) \in E,$$

where g_-, g_+ are Henstock-Stieltjes (HS) integrable.

Here $f(x(\cdot),\cdot)$ is D_{HK}-integrable if each component of $f(x(\cdot),\cdot)$ is D_{HK}-integrable. In the space of D_{HK}, $f \preceq g$ if and only if $f^i \preceq g^i$, where f^i, g^i denote the ith component of f, g, $i = 1, 2, \cdots, n$.

The following two lemmas will play an important role to study the existence of solutions of MDE (1.1).

Lemma 3.1. *Under assumptions (C_1) − (C_6), the solutions of MDE (1.1) satisfy the integral equation for all $t \in [a,b]$,*

$$x(t) = x_0 + (D_{HK})\int_a^t f(x(s),s)ds + (HS)\int_a^t g(x(s),s)du(s). \tag{3.1}$$

The converse also holds.

Proof. It follows from $(C_3),(C_6)$ and Lemma 2.4 that the two integrals (D_{HK}) $\int_a^t f(x(s),s)ds$ and $(HS)\int_a^t g(x(s),s)du(s)$ exist. Denote $X^i = x^i(\cdot)$, where $x^i(\cdot)$ is the ith component of $x(\cdot)$.

Let $x(\cdot)$ satisfies (3.1). Obviously, $x(a) = x_0$. Since $\varphi \in C_c^\infty((a,b))$, define

$$\begin{aligned}\langle X^i, \varphi\rangle = \int_a^b \Big[x_0^i + \int_a^t f^i(x(s),s)ds \\ + \int_a^t g^i(x(s),s)du(s)\Big]\varphi(t)dt,\ i = 1,2,\cdots,n.\end{aligned} \tag{3.2}$$

Therefore,

$$
\begin{aligned}
\langle DX^i, \varphi\rangle &= -\langle X^i, \varphi^{'}\rangle \\
&= -\int_a^b \Big[x_0^i + \int_a^t f^i(x(s), s)ds \\
&\qquad + \int_a^t g^i(x(s), s)du(s)\Big] \varphi^{'}(t)dt, \\
&\qquad\qquad i = 1, 2, \cdots, n.
\end{aligned} \tag{3.3}
$$

Since $\varphi \in C_c^\infty((a, b))$, using Lemma 2.7, we have

$$
\begin{aligned}
-\int_a^b \Big[x_0^i + \int_a^t f^i(x(s), s)ds\Big] \varphi^{'}(t)dt = \int_a^b \varphi(t) f^i(x(t), t)dt, \\
i = 1, 2, \cdots, n.
\end{aligned} \tag{3.4}
$$

Moreover, $\int_a^t g^i(x(s), s)du(s)$ is right continuous on $[a, b]$,

$$
\begin{aligned}
&\int_a^b \Big[\int_a^t g^i(x(s), s)du(s)\Big] \varphi^{'}(t)dt \\
&= -\int_a^b \varphi(t) d\Big[\int_a^t g^i(x(s), s)du(s)\Big] \\
&= -\int_a^b g^i(x(t), t)\varphi(t)du(t), \quad i = 1, 2, \cdots, n.
\end{aligned} \tag{3.5}
$$

It follows from (3.4), (3.5) that

$$
\begin{aligned}
\langle DX^i, \varphi\rangle = \int_a^b \varphi(t) f^i(x(t), t)dt + \int_a^b g^i(x(t), t)\varphi(t)du(t), \\
i = 1, 2, \cdots, n.
\end{aligned} \tag{3.6}
$$

Thus, DX^i is identified with $f^i(x(t), t) + g^i(x(t), t)Du, i = 1, 2, \cdots, n$. Therefore, $x(\cdot)$ satisfies (1.1).

On the other side, suppose that $x(\cdot)$ satisfies (1.1). According to assumptions (C_1) and (C_4), by integrating equation (1.1) from a to t, we obtain that the integral equation (3.1) holds.

Hence the MDE (1.1) is equivalent to the integral equation (3.1).

This completes the proof. □

Lemma 3.2 ([12, Theorem 6.15]). *Let M be a convex, closed subset of a normed space X. Let T be a continuous map of M into a compact subset K of M. Then T has a fixed point.*

The first main result based on Lemma 3.2, is the existence of solutions of the MDE (1.1).

Theorem 3.3. *Assume that the functions f, g in (1.1) satisfy assumptions $(C_1) - (C_6)$. Then there exists at least one solution of the MDE (1.1).*

Proof. We first assume that

$$h_1 = \max_{t\in[a,b]} \left\{ \left|(D_{HK}) \int_a^t f_-\right|, \left|(D_{HK}) \int_a^t f_+\right| \right\}.$$

Then for each $t \in [a, b]$, we have

$$-h_1 \leq (D_{HK}) \int_a^t f_-, \quad (D_{HK}) \int_a^t f_+ \leq h_1. \tag{3.7}$$

According to Lemma 2.8, let

$$h_2 = \max_{t\in[a,b]} \left\{ \left|(HS) \int_a^t g_- du\right|, \left|(HS) \int_a^t g_+ du\right| \right\}.$$

It follows from Lemma 2.4 and assumptions (C_3) and (C_6) that

$$\left|(D_{HK}) \int_a^t f(x(s), s)ds\right| \leq h_1, \tag{3.8}$$

$$\left|(HS) \int_a^t g(x(s), s)du(s)\right| \leq h_2. \tag{3.9}$$

Let $B = \{x \in C([a, b]) : \|x - x_0\|_\infty \leq r,\ r = h_1 + h_2 > 0\}$. For each $x \in B$, we define an operator $T : B \to C([a, b])$, satisfying

$$\begin{aligned} T(x)(t) = x_0 &+ (D_{HK}) \int_a^t f(x(s), s)ds \\ &+ (HS) \int_a^t g(x(s), s)du(s), t \in [a, b]. \end{aligned} \tag{3.10}$$

Now we prove this theorem in three steps.

Step 1: $T : B \to B$.

By (3.10),

$$\begin{aligned} \|T(x) - x_0\|_\infty = \max_{t\in[a,b]} \bigg|(D_{HK}) &\int_a^t f(x(s), s)ds \\ &+(HS) \int_a^t g(x(s), s)du(s)\bigg|. \end{aligned} \tag{3.11}$$

Furthermore, by using (3.7)-(3.11), we have

$$\begin{aligned} \|T(x) - x_0\|_\infty \leq & \max_{t\in[a,b]} \left|(D_{HK}) \int_a^t f(x(s), s)ds\right| \\ & + \max_{t\in[a,b]} \left|(HS) \int_a^t g(x(s), s)du(s)\right| \\ \leq & \ h_1 + h_2 = r. \end{aligned} \tag{3.12}$$

This implies that $T(x) \in B$. Hence, $T(B) \subseteq B$.

Step 2: $T(B)$ is equi-continuous.

Let $t_1, t_2 \in [a,b], t_1 < t_2, x \in B$. By (C_3) and (C_6), one has

$$\begin{aligned} |Tx(t_1) - Tx(t_2)| &\leq \left|(D_{HK}) \int_{t_1}^{t_2} f(x(s), s) ds\right| \\ &\quad + \left|(HS) \int_{t_1}^{t_2} g(x(s), s) du(s)\right| \\ &\leq \left|(HS) \int_{t_1}^{t_2} g_-(s) du(s)\right| \\ &\quad + \left|(HS) \int_{t_1}^{t_2} g_+(s) du(s)\right| \\ &\quad + \left|(D_{HK}) \int_{t_1}^{t_2} f_-\right| + \left|(D_{HK}) \int_{t_1}^{t_2} f_+\right|. \end{aligned} \tag{3.13}$$

Since $g_-, g_+ \in HS$ and $u(t)$ is a right continuous function of bounded variation on $[a,b]$, by Lemma 2.8, we know that $(HS) \int_{t_1}^{t_2} g_- du(s)$ and $(HS) \int_{t_1}^{t_2} g_+ du(s)$ are continuous on $[a,b]$ and, hence, uniformly continuous on $[a,b]$. Besides, $f_-, f_+ \in D_{HK}$, the primitives of them are also uniformly continuous on $[a,b]$. Thus, $T(B)$ is equiuniformly continuous on $[a,b]$ for all $x \in B$.

In view of Ascoli-Arzelà theorem, $T(B)$ is relatively compact.

Step 3: T is a continuous mapping.

Let $x \in B$, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in B and $x_n \to x$ as $n \to \infty$.

By (C_2) and (C_5), one has

$$f(x_n, \cdot) \to f(x, \cdot) \quad \text{as} \quad n \to \infty,$$

$$g(x_n, \cdot) \to g(x, \cdot) \quad \text{as} \quad n \to \infty.$$

According to assumption (C_1) and Lemma 2.6, we have

$$\lim_{n\to\infty} (D_{HK}) \int_a^t f(x_n(s), s) ds = (D_{HK}) \int_a^t f(x(s), s) ds, \quad t \in [a,b].$$

It follows from Lemma 2.9 that

$$\lim_{n\to\infty} (HS) \int_a^t g(x_n(s), s) du(s) = (HS) \int_a^t g(x(s), s) du(s).$$

Then $\lim_{n\to\infty} T(x_n) = T(x)$, so T is continuous.

Hence T satisfies the hypotheses of Lemma 3.2, then there exists a fixed point of T which is a solution of the MDE (1.1). □

We now consider the topological characterization of the set of solutions of the MDE (1.1).

Let $C_u(K, Y)$ be the space of all continuous mappings $x : K \to Y$, where K is a compact convex subset of a normed space and Y is a metric space equipped with the topology of uniform convergence. Denote by $B(t_0, \varepsilon)$ the closed ball with center t_0 and radius ε. Denote by $x|_A$ the restriction of the map x to A.

Now we present the well-known Vidossich theorem.

Lemma 3.4 ([16, Corollary 1.2, Vidossich theorem]). *Let K be a compact convex subset of a normed space, Y a closed convex subset of a Banach space Y_0, F a compact mapping $C_u(K, Y) \to C_u(K, Y)$. Suppose that there exist $t_0 \in K$ and $y_0 \in Y$ such that the following two conditions hold.*
(i) $F(x)(t_0) = y_0 \quad (x \in C(K, Y))$.
(ii) For every $\varepsilon > 0$,

$$x|_{K_\varepsilon} = y|_{K_\varepsilon} \Rightarrow F(x)|_{K_\varepsilon} = F(y)|_{K_\varepsilon} \quad (x, y \in C(K, Y)),$$

where $K_\varepsilon = B(t_0, \varepsilon) \cap K$. Then the set of fixed points of F is an R_δ.

Recall that if a set is an R_δ, it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Furthermore, G. Vidossich pointed out that R_δ is a nonempty, compact and connected set in [16].

The second main result concerned with the structure of the set of solutions of the MDE (1.1) can be stated as follow.

Theorem 3.5. *Under the above assumptions $(C_1) - (C_6)$, the solution set of MDE (1.1) on $[a, b]$ is an R_δ.*

Proof. In Theorem 3.3, we have verified that the mapping $T : B \to B$ is compact. Hence, conditions (i) and (ii) of Lemma 3.4 hold. Therefore, the solution set of the MDE (1.1) is an R_δ. □

4. **Examples**

In this section, we first show a lemma which is obtained by Y. T. Xu in literature [19]. And then, we will give two examples to explain that Theorem 3.3 in Section 3 is more generalized than Theorem 6.2 in [19] and Theorem 1 in [5].

Lemma 4.1 ([19, Theorem 6.2]). *Let f and g in (1.1) satisfy the following conditions:*

(1) $f(x, t)$ is measurable in t for each x and continuous in x for all t;
(2) there exists a Lebesgue integrable function $f(t)$ such that

$$|f(x, t)| \le f(t), \quad (x, t) \in E;$$

(3) $g(x, t)$ is continuous in x for all t and $du-$integrable for each $x \in \mathcal{BV}([a, b], B)$;
(4) there exists a dV_u-integrable function $g(t)$ such that

$$|g(x, t)| \le g(t), \quad (x, t) \in E,$$

where V_u denotes the total variation function of u.

Then there exists at least one solution of the MDE (1.1).

We can see that if f and g in (1.1) satisfy assumptions $(1)-(4)$ in Lemma 4.1, then f and g satisfy $(C_1)-(C_6)$. According to Theorem 3.3, the MDE (1.1) has at least one solution. However, the following example shows that Lemma 4.1 is only a special case of Theorem 3.3.

Example 1. *Consider the following MDE*

$$Dx(t) = r(t) + x(t) + \frac{2}{t+1}x(t)DH(t), \quad x(0) = 0, \quad t \in [0,2], \tag{4.1}$$

where $x(t)$ is continuous and $H(t)$ is the Heaviside function, i.e., $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$, then $DH = \delta$ is the Dirac measure. Let $r(t)$ be the distributional derivative of Riemann function $R(t) = \sum_{n=1}^{\infty} \frac{\sin n^2 \pi t}{n^2}$. Then equation (4.1) has at least one solution. Moreover, the solution set is an R_δ.

We can assume that $f(x,t) = r(t) + x(t)$, $g(x,t) = \frac{2}{t+1}x(t)$ and $u(t) = H(t)$, then (4.1) can be regarded as the MDE (1.1).

It is easy to see that $f(x,t)$ is continuous in x and $D_{HK}-$integrable on $[0,2]$. Since $x(t)$ is continuous, there exists $M_0 > 0$ such that $-M_0 \leq x(t) \leq M_0$. Note that $r(t)$ is $D_{HK}-$integrable, we have $r(t) \pm M_0 \in D_{HK}$. Moreover,

$$r(t) - M_0 \preceq f(x,t) \preceq r(t) + M_0, \quad t \in [0,2].$$

Therefore, $f(x,t)$ satisfies $(C_1)-(C_3)$.

Obviously, $\frac{2}{t+1}x(t)$ is continuous in x. Because $H(t)$ is of bounded variation and right continuous on $[a,b]$, so $(HS)\int_a^t \frac{2}{s+1}x(s)dH(s)$ exists. Hence, we obtain that $(C_4)-(C_6)$ hold.

Applying Theorem 3.3, we can obtain that equation (4.1) has at least one solution. Furthermore, by using Theorem 3.5, one can see that the solution set is an R_δ.

Remark 4.2. *Let F be the primitive of f. Since F is continuous but not ordinarily differentiable in [7], f is neither Henstock-Kurzweil integrable nor Lebesgue integrable in the above example. So f does not satisfy the hypotheses* $(1)-(2)$ *in Lemma 4.1. Hence Theorem 6.2 in [19] is not applicable for the case of Example 1.*

Moreover, we can use a similar method to verify that Theorem 3.3 is a generalization of Theorem 1 in [5], where $g(x,t)$ is an $n \times n$ matrix and $u \in \mathbb{R}^n$ is a vector.

Let us give another example to illustrate this.

Example 2. *Consider the following MDE*

$$Dx = r(t) + x(t) + A(t)x(t)Du, \quad x(a) = 0, \quad t \in [a,b], \tag{4.2}$$

where

$$x(t)=\begin{pmatrix} x_1(t)\\ x_2(t)\\ \cdots\\ x_n(t)\end{pmatrix}, r(t)=\begin{pmatrix} r_1(t)\\ r_2(t)\\ \cdots\\ r_n(t),\end{pmatrix}, \tag{4.3}$$

$$A(t)=\begin{pmatrix} e^t & 0 & 0 & 0\\ 0 & e^t & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & e^t\end{pmatrix}, u=\begin{pmatrix} u_1\\ u_2\\ \cdots\\ u_n\end{pmatrix} \tag{4.4}$$

and $x_i(t)$ is continuous, $r_i(t)$ is the distributional derivative of the Riemann function $R_i(t)=\sum_{n=1}^{\infty}\frac{\sin n^2\pi t}{n^2}$, $Du_i=\delta$ is the Dirac measure, $i=1,2,\cdots,n$. Then (4.2) has at least one solution.

Proof. According to (4.3) and (4.4), let

$$f(x,t)=r(t)+x(t)=\begin{pmatrix} x_1(t)+r_1(t)\\ x_2(t)+r_2(t)\\ \cdots\\ x_n(t)+r_n(t)\end{pmatrix}, \tag{4.5}$$

$$g(x,t)=A(t)x(t)=\begin{pmatrix} e^t & 0 & 0 & 0\\ 0 & e^t & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & e^t\end{pmatrix}\begin{pmatrix} x_1(t)\\ x_2(t)\\ \cdots\\ x_n(t)\end{pmatrix}. \tag{4.6}$$

Since $x_i(t)$ is continuous, there exists $N_i>0$ such that $-N_i\le x_i(t)\le N_i$. Note that $r_i(t)$ is $D_{HK}-$integrable, thus $r_i(t)\pm N_i\in D_{HK}$. Let $N=\max_{1\le i\le n}N_i$, one has

$$r(t)-N\preceq f(x,t)\preceq r(t)+N,\quad t\in[a,b].$$

It is clear that $g(x,t)$ is continuous in x, $e^t x_i(t)$ is Henstock-Kurzweil integrable on $[a,b]$ and u is bounded variation on $[a,b]$, $f(x,t)$ and $g(x,t)$ satisfy the assumptions of Theorem 3.3. So equation (4.2) has at least one solution. However, equation (4.2) does not satisfy the conditions of Theorem 1 in [5]. Hence, Theorem 1 in [5] is not applicable. □

It follows from Examples 1 and 2 that Theorem 3.3 in Section 3 generalizes not only Theorem 6.2 in [19] but also Theorem 1 in [5].

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