Bulletin of the Iranian Mathematical Society Vol. 35 *No. 1* (*2009*), *pp 235-252.*

PROPERTIES OF GENERALIZED BERWALD CONNECTIONS

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Communicated by Karsten Grove

ABSTRACT. Recently we introduced a general class of Finsler connections which lead to a smart representation of connection theory in Finsler geometry and yielded a classification of Finsler connections into the three classes. Here, the properties of one of these classes, namely the Berwald-type connection which contains Berwald and Chern (Rund) connections as special cases are studied. It is proved, among other results, that the hv-curvature of these connections vanishes if and only if the Finsler space is a Berwald one. Some applications of this connection is discussed.

1. Introduction

Since there is always a hope to find a solution for some of unsolved problems by developing a connection theory, then it is useful to introduce new connections in Finsler geometry. As mentioned in [12], the study of hv-curvature of Finsler connections is of urgent necessity for the Finsler geometry as well as for theoretical physics. Similarly, as another application of Finsler connections in physics one can mention

MSC(2000): Primary: 53B40; Secondary: 53C60.
Keywords: General Finsler connection, Catran-type connection, Berwald-type connection, Shen connection.
Received: 24 April 2008, Accepted: 4 October 2008
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an example in Relativistic field theory. In this theory, different connections have been defined in Finsler geometry, where the connections, torsions, or curvatures can be related to fields which might be identified as electromagnetic or Yang-Mills fields. In this relation, see [2], [5] and [11].

Historically, in Riemannian geometry, the connection of choice was that constructed by Levi-Civita, which has two remarkable attributes; metric-compatibility and torsion-freeness. In 1926, L. Berwald [5] introduced a connection and two curvature tensors. The Berwald connection is torsion-free, but is not necessarily metric-compatible. It was Berwald who first successfully extended the notion of Riemann curvature to Finsler spaces. He also introduced a notion of non-Riemannian quantity called Berwald curvature. From this point of view, Berwald is the founder of differential geometry of Finsler spaces [14]. Next, Cartan in 1934 found locally the coefficients of a metric-compatible and h-torsion free connection, later called, Cartan connection. The global construction of this connection is given in a remarkable work of Akbar-Zadeh in 1988 [1]. Other progress came in 1943, when the Chern (Rund) connection was defined. In 1943, Chern studied the equivalence problem for Finsler spaces using the Cartan exterior differentiation method [8]. Chern came back to his connection in 1993, in a joint paper with Bao [3] and showed its usefulness in treating global problems in Finsler geometry. The Berwald and Chern connections also fail slightly, but expectedly, to be metric-compatible. The Chern connection has a simpler form, while the Berwald connection affects a leaner hh-curvature for spaces of constant flag curvature. Indeed the Berwald connection is particularly convenient when dealing with Finsler spaces of constant flag curvature. It is most directly related to the nonlinear connection coefficients and most amenable to the study of the geometry of paths. These connections (Berwald and Chern) coincide when the underlying Finsler structure is of Landsberg type. They further reduce to a linear connection on M, when the Finsler structure is of Berwald type [9], [10].

Recently, we defined a general class of Finsler connections which lead to a general representation of some Finsler connections in Finsler geometry and yielded a classification of Finsler connections into the three classes of Berwald-type, Cartan-type and Shen-type connections [6]. Here, we study the properties of a connection which contains Berwald and Chern (Rund) connections as special cases and is the most general connection of this kind. We prove in continuation of Berwald's and Chern's work that the hv-curvature of the Berwald-type connection characterizes the Berwald structure.

A distinguished property of the introduced connection is its adaptive form for different applications. In fact, one can use a suitable special case of this connection to find a geometric interpretation for solutions of certain differential equations formed by Cartan tensor and its derivatives. For example, in Section 4 we prove that if (M,F) is a complete Finsler manifold with bounded Landsberg tensor, then F is a Landsberg metric if and only if its hv-curvature P_{jkl} vanishes.

2. **Preliminaries**

Let M be a n-dimensional C^∞ manifold. For a point $x \in M$, denote by T_xM, the tangent space of M at x. The tangent bundle of M is the union of tangent spaces, $TM := \cup_{x\in M} T_xM$. We will denote elements in TM by (x, y) if $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$.
Throughout this paper, we use *Einstein summation convention* for expressions with indices.[1]

A *Finsler structure* on a manifold M is a function $F : TM \to [0,\infty)$ with the following properties:
(i)F is C^∞ on TM_0.
(ii)F is positively 1-homogeneous on the fibers of tangent bundle TM:

$$\forall\lambda > 0 \quad F(x, \lambda y) = \lambda F(x, y).$$

(iii) The Hessian of F^2 with elements $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^iy^j}$ is positively defined on TM_0.
The pair (M, F) is called a *Finsler manifold.* F is Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

Nonlinear connection.
Let us consider the tangent bundle (TM, π, M) of the manifold M. The tangent bundle of the manifold TM is (TTM, π_*, TM), where π_* is the tangent mapping of the projection π. A tangent vector field on TM can

[1]That is, when an index appears twice as a subscript as well as a superscript in a term, then the term is assumed to be summed over all values of that index.

be represented in the local natural frame $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ on TM by:

$$\bar{X} = X^i(x,y)\frac{\partial}{\partial x^i} + Y^i(x,y)\frac{\partial}{\partial y^i}.$$

It can be written in the form $\bar{X} = (x, y, X^i, Y^i)$ or shorter, $\bar{X} = (x, y, X, Y)$. The mapping $\pi_* : TTM \to TM$ has the local form,

$$\pi_*(x, y, X, Y) = (x, y).$$

Put $VTM := ker\pi_* = span\{\frac{\partial}{\partial y^i}\}_{i=1}^n$. VTM is an n-dimensional subbundle of $T(TM_0)$, whose fiber V_vTM at v is just the tangent space $T_v(T_xM) \subset T_v(TM_0)$. VTM is called the *vertical tangent bundle* of TM_0.
We can write the vertical subbundle as $(VTM, \pi_{_{VTM}}, TM)$. Its fibres are the linear vertical spaces V_uTM, $u \in TM$. The points of submanifold VTM are of the form $(x, y, 0, Y)$. Hence, the fibers V_uTM of the vertical bundle are isomorphic to the real vector space $\mathbb{R}^n$.

Let us consider the pullback tangent bundle π^*TM defined as follows [4]:

$$\pi^*TM = \{(u, v) \in TM \times TM | \pi(u) = \pi(v)\}.$$

Take a local coordinate system (x^i) in M. The local natural frame $\{\frac{\partial}{\partial x^i}\}$ for T_xM determines a local natural frame ∂_i for π^*TM, $\partial_i|_y := (y, \frac{\partial}{\partial x^i}|_x)$, $y \in T_xM$. This gives rise to a linear isomorphism between $\pi^*TM|_y$ and T_xM, for every $y \in T_xM$. There is a canonical section ℓ of π^*TM defined by $\ell = \ell^i\partial_i$, where $\ell^i = y^i/F(x,y)$.
The fibers of π^*TM, i.e., π_u^*TM are isomorphic to $T_{\pi(u)}M$. One can define the following morphism of vector bundle $\rho : TTM \to \pi^*TM$, $\rho(X_u) = (u, \pi_*(\bar{X}_u))$. It follows that

$$ker\rho = ker\pi_* = VTM.$$

By means of these considerations, one can see without any difficulties that the following sequence is exact:

$$0 \to VTM \xrightarrow{i} TTM \xrightarrow{\rho} \pi^*TM \to 0\ , \tag{2.1}$$

where i is natural inclusion map.

A *nonlinear connection* on the manifold TM is a left splitting of the exact sequence (2.1). Therefore, a nonlinear connection on TM is a vector bundle morphism $C : TTM \to VTM$, with the property $C \circ i =$

1_{VTM}. The kernel of the morphism C is a vector bundle of the tangent bundle (TTM, π_*, TM), denoted by (HTM, π_{HTM}, TM) and called the *horizontal subbundle*. Its fibres H_uTM determine a distribution $u \in TM \to H_uTM \subset T_uTM$, supplementary to the vertical distribution $u \in TM \to V_uTM \subset T_uTM$. Therefore, a nonlinear connection N induces the following Whitney sum:

$$TTM = HTM \oplus VTM. \tag{2.2}$$

Let

$$\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}\ , \tag{2.3}$$

where the above N^i_j are the components of N and are known in the trade as the *nonlinear connection coefficients* on TM_0.
Restriction of the morphism $\rho : TTM \to \pi^*TM$ to the HTM is an isomorphism of vector bundles, for which we have,

$$\rho(\frac{\partial}{\partial x^i}) = \partial_i \quad , \quad \rho(\frac{\partial}{\partial y^i}) = 0. \tag{2.4}$$

Let ∇ be a linear connection on π^*TM, $\nabla : \chi(TM_0) \times \pi^*TM \to \pi^*TM$ such that $\nabla : (\hat{X}, Y) \to \nabla_{\hat{X}}Y$. A *Finsler connection* is a pair of a linear connection ∇, and a nonlinear connection N.

Given a Finsler metric F on M, $F(y) = F(y^i \frac{\partial}{\partial x^i}|_x)$ is a function of $(y^i) \in \mathbb{R}^n$ at each point $x \in M$. Finsler metric F defines a fundamental tensor $g : \pi^*TM \otimes \pi^*TM \to [0, \infty)$ by the formula $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $v = y^i \frac{\partial}{\partial x^i}|_x$ and the g_{ij} are defined in the definition of Finsler structure. Then, (π^*TM, g) becomes a Riemannian vector bundle over TM_0. Let

$$A_{ijk}(x, y) = \frac{1}{2} F(x, y)[F^2(x, y)]_{y^i y^j y^k}.$$

Clearly, A_{ijk} is symmetric with respect to i, j, k. The Cartan tensor $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to \mathbb{R}$ is defined by $A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y)$. In some work, $C_{ijk} = A_{ijk}/F$ is called Cartan tensor. Riemannian manifolds are characterized by $A \equiv 0$. F is positively homogenous of degree 1 on M. Then, by the Euler's theorem we see that $y^i F_{y^i} = F$ and then $y^i F_{y^i y^j} = 0$. Using this the canonical section ℓ satisfies:

$$g(\ell, \ell) = 1 \quad , \quad A(X, Y, \ell) = 0,$$

where the second equation is equivalent to $A(X,Y,\frac{y^i}{F}\frac{\partial}{\partial x^i}) = \ell^i A(X,Y,\frac{\partial}{\partial x^i}) = 0$. Let $\bar{\ell}$ denote the unique vector field in HTM such that $\rho(\bar{\ell}) = \ell$. We call $\bar{\ell}$ a *geodesic* or *spray* field on TM_0.

Let ∇ be the Berwald (or Chern) connection. By means of ∇, the tensor $\dot{A}$ is defined by $\dot{A} : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to \mathbb{R}$,

$$\dot{A}(X,Y,Z) := \bar{\ell}A(X,Y,Z) - A(\nabla_{\bar{\ell}}X,Y,Z) - A(X,\nabla_{\bar{\ell}}Y,Z) - A(X,Y,\nabla_{\bar{\ell}}Z).$$

Putting $\overset{1}{A}_{ijk} = \dot{A}_{ijk}$, $\overset{2}{A}_{ijk} = \ddot{A}_{ijk}$, $\forall m \in \mathbb{N}$ we define $\overset{m+1}{A}$ as follows:

$$\overset{m+1}{A}(X,Y,Z) := \bar{\ell}\ \ \overset{m}{A}(X,Y,Z) - \overset{m}{A}(\nabla_{\bar{\ell}}X,Y,Z) - \overset{m}{A}(X,\nabla_{\bar{\ell}}Y,Z) - \overset{m}{A}(X,Y,\nabla_{\bar{\ell}}Z).$$

Obviously, $\forall m \in \mathbb{N}$, the tensors $\overset{m}{A}_{ijk}$ are symmetric with respect to the three indices. Moreover, using $\nabla_{\bar{\ell}}\ \ell = 0$, we have $\overset{m}{A}(X,Y,\ell) = 0$, $\forall m \in \mathbb{N}$. A and $\dot{A}$ are basic tensors in Finsler geometry. In the Riemannian case, both of them vanish. Therefore, by the above definition we know that for the Riemannian case, $\forall m \in \mathbb{N}$, $\overset{m}{A} = 0$.

A Finsler metric $F(x,y)$ on a manifold M is called *Berwald metric* if in any standard local coordinate system (x^i, y^i) in TM_0, the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ij}(x)$ are functions of $x \in M$ alone. In this case, $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y = y^i\frac{\partial}{\partial x^i}|_x$ and $F(x,y)$ is called a *Landsberg metric* if $L^i_{jk}(x,y) = 0$, that is,

$$L^i_{jk}(x,y) = \frac{\partial^2 G^i}{\partial y^j \partial y^k}(x,y) - \Gamma^i_{jk}(x,y).$$

Clearly, Minkowski and Riemannian metrics are trivial Berwald metrics. If $F(x,y)$ is a Berwald metric, then it is a Landsberg metric. But the converse might not be true, although no counter-example has been found yet [14]. A fundamental theorem in Finsler geometry says that a Finsler metric F is a Berwald metric if and only if the Cartan tensor is covariantly constant along all horizontal directions on the slit tangent bundle TM_0 (see [15] and [4] for a proof). Thus, in the Berwald case, the $\overset{m}{A}_{ijk}$ vanish, $\forall m \in \mathbb{N}$.

Flag curvature. A flag curvature is a geometrical invariant that generalizes the sectional curvature of Riemannian geometry. Let $x \in M$,

$0 \neq y \in T_xM$ and $V := V^i \frac{\partial}{\partial x^i}$. A flag curvature is obtained by carrying out the following computation at the point $(x, y) \in TM_0$, and viewing y, V as section of π^*TM:

$$K(y, V) := \frac{V^i(y^j\ R_{jikl}\ y^l)V^k}{g(y, y)g(V, V) - [g(y, V)]^2}\ ,$$

where g is a Riemannian metric on π^*TM. If K is independent of the transverse edge V, we say that the Finsler space has *a scalar flag curvature*. Denote this scalar by $\lambda = \lambda(x, y)$. When $\lambda(x, y)$ has no dependence on either x or y, then Finsler manifold is said to be of *constant flag curvature.*

3. Berwald-type connection on π^*TM

Here, we introduce a new family of Finsler connections which are torsion-free and almost compatible with the Finsler metric. In the sequel, we will refer to this connection by *"Berwald-type connection"*.

Definition 3.1. Let (M, F) be a Finsler manifold. Suppose that g and A denote the fundamental and the Cartan tensors in π^*TM, respectively. Let D be a Finsler connection on M.
(i) D *is* **torsion-free**, *if* $\forall \hat{X}, \hat{Y} \in \chi X(TM_0)$,

$$\mathfrak{T}_D(\hat{X}, \hat{Y}) := D_{\hat{X}}\rho(\hat{Y}) - D_{\hat{Y}}\rho(\hat{X}) - \rho([\hat{X}, \hat{Y}]) = 0. \tag{3.1}$$

(ii) D is **almost compatible** with the Finsler structure in the following sense: If for all $X, Y \in \pi^*TM$ *and* $\hat{Z} \in T_v(TM_0)$,

$$(D_{\hat{Z}}g)(X, Y) := \hat{Z}g(X, Y) - g(D_{\hat{Z}}X, Y) - g(X, D_{\hat{Z}}Y),$$

or equivalently,

$$\begin{aligned} (D_{\hat{Z}}g)(X, Y) &= -2k_1\dot{A}(\rho(\hat{Z}), X, Y) - \cdots - 2k_m\overset{m}{A}(\rho(\hat{Z}), X, Y) \\ &+ 2F^{-1}A(\mu(\hat{Z}), X, Y), \end{aligned} \tag{3.2}$$

where $\rho(\hat{Z}) := (v, \pi_*(\hat{Z}))$, $\mu(\hat{Z}) := D_{\hat{Z}}F\ell$, $m \in \mathbb{N}$ and $k_i \in \mathbb{R}$.

The bundle map $\mu : T(TM_0) \to \pi^*TM$ defined in the above definition satisfies $\mu(\frac{\partial}{\partial y^i}) = \partial_i$. To prove this, take $\hat{\ell} = \ell^i \frac{\partial}{\partial x^i}$, where $\ell = \ell^i\partial_i$. Now,

$\rho(\hat{\ell}) = \ell$, and so from (3.1),

$$\mu(\frac{\partial}{\partial y^i}) = D_{\frac{\partial}{\partial y^i}} F\ell = \rho([\frac{\partial}{\partial y^i}, y^k \frac{\partial}{\partial x^k}]) = \partial_i. \tag{3.3}$$

Theorem 3.1. *Let (M, F) be a Finsler n-manifold. Then there is a unique linear torsion-free connection D in π^*TM, which is almost compatible with the Finsler structure in the sense of (2.2).*

Proof. In a standard local coordinate system (x^i, y^i) in TM_0, we write,

$$D_{\frac{\partial}{\partial x^i}} \partial_j = \Gamma^k_{ij} \partial_k \quad , \quad D_{\frac{\partial}{\partial y^i}} \partial_j = F^k_{ij} \partial_k \ .$$

By replacing $\hat{X}, \hat{Y}$ in (2.1) with the basis of $T_v(TM_0)$, i.e., $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$, we get,

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \tag{3.4}$$

$$F^k_{ij} = 0, \tag{3.5}$$

and by replacing X, Y (resp. $\hat{Z}$) in (3.2) with the basis of π^*TM, i.e., $\{\partial_i\}$, (resp. with the basis of $T_v(TM_0)$), we get,

$$\begin{aligned} \frac{\partial}{\partial x^k}(g_{ij}) = \Gamma^l{}_{ki} g_{lj} + \Gamma^l_{kj} g_{li} \quad &- \quad 2k_1 \dot{A}_{ijk} - \cdots - 2k_m \overset{m}{A}_{ijk} \\ &+ \quad 2A_{ijl}\Gamma^l_{km}\ell^m, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial}{\partial y^k}(g_{ij}) = F^l{}_{kj} g_{li} + F^l_{ik} g_{jl} \quad &- \quad 2\{k_1 \dot{A}_{ijk} + \cdots + k_m \overset{m}{A}_{ijk}\} F^l_{mk} \ell^m \\ &+ \quad 2F^{-1} A_{ijk}, \end{aligned} \tag{3.7}$$

where g_{ij}, A_{ijk} and $\overset{(m)}{A}_{ijk}$, $\forall m \in \mathbb{N}$, are all functions of (x, y). We shall compute Γ^k_{ij} by "Christoffel's trick" from (3.4) and (3.6). Then, making a permutation to i, j, k in (3.6), and using (3.4), we obtain,

$$\begin{aligned} \Gamma^k_{ij} \quad &= \quad \gamma^k_{ij} + k_1 \dot{A}^k{}_{ij} + \cdots + k_m \overset{m}{A}{}^k{}_{ij} \\ &+ \quad g^{kl} \left\{ A_{ijm}\Gamma^m_{lb} - A_{jlm}\Gamma^m_{ib} - A_{lim}\Gamma^m_{jb} \right\} \ell^b, \end{aligned} \tag{3.8}$$

where we let,

$$\gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\},$$

and $A^k_{ij} = g^{kl} A_{ijl}$. Multiplying (3.8) by ℓ^i, we obtain,

$$\Gamma^k_{ib}\ell^b = \gamma^k_{ib}\ell^b - A^k{}_{im}\Gamma^m_{lb}\ell^l\ell^b. \tag{3.9}$$

Multiplying (3.9) by ℓ^j, yields:

$$\Gamma^k_{ab}\ell^a\ell^b = \gamma^k_{ab}\ell^a\ell^b. \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain,

$$\Gamma^k_{ib}\ell^b = \gamma^k_{ib}\ell^b - A^k{}_{im}\gamma^m_{ab}\ell^a\ell^b. \tag{3.11}$$

Substituting (3.11) in (3.8), we obtain,

$$\begin{aligned}\Gamma^k_{ij} = \gamma^k_{ij} \quad + \quad & k_1\dot{A}^k{}_{ij} + \cdots + k_m\overset{m}{A}{}^k{}_{ij} \\ & +g^{kl}\left\{A_{ijm}\gamma^m_{lb} - A_{jlm}\gamma^m_{ib} - A_{lim}\gamma^m_{jb}\right\}\ell^b \\ & \qquad + \left\{A^k_{jm}A^m_{is} + A^k_{im}A^m_{js} - A^k_{sm}A^m_{ij}\right\}\gamma^s_{ab}\ell^b\ell^a.\end{aligned} \tag{3.12}$$

Then using (2.3), (3.12), we will have,

$$\Gamma^i_{jk} = \frac{g^{is}}{2}\left\{\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right\} + k_1\dot{A}^i{}_{jk} + \cdots + k_m\overset{m}{A}{}^i{}_{jk}. \tag{3.13}$$

This proves the uniqueness of D. The set $\{\Gamma^k_{ij}, F^k_{ij} = 0\}$, where the $\{\Gamma^k_{ij}\}$ are given by (3.13), defines a linear connection D on π^*TM satisfying (3.1) and (3.2). □

Definition 3.2. Let (M, F) be a Finsler manifold. A Finsler connection is called of *Berwald-type* (resp. *of Cartan-type* or *Shen-type*) if and only if vanishing of its hv-curvature, reduces the Finsler structure to the Berwaldian (resp. Landsbergian or Riemannian) one.

From this view point, one can compare some of the non-Riemannian Finsler connections according to the compatibility of the tensors S and T, as shown in Table 1.

Table 1: A classification of Finsler connections according to their compatible tensors S and T

	Compatible tensors			
Connection	**S**	**T**	**Metric compatibility**	**Torsion**
1. Berwald	$A+\dot{A}$	0	almost compatible	free
2. Chern- Rund	A	0	almost compatible	free
3. Berwald-type	$A+\kappa_1\dot{A}+\cdots+\kappa_m\overset{m}{A}$	0	almost compatible	free
4. Cartan	A	A	metric compatible	not free
5. Hashiguchi	$A+\dot{A}$	A	almost compatible	not free
6. Cartan-type	$A+\kappa_1\dot{A}+\cdots+\kappa_m\overset{m}{A}$	A	depends on κ_i	not free
7. Shen	0	0	almost compatible	free
8. Shen-type	$\kappa_1\dot{A}+\cdots+\kappa_m\overset{m}{A}$	0	almost compatible	free
9. General-type	$\kappa_0 A+\kappa_1\dot{A}+\cdots+\kappa_m\overset{m}{A}$	rA	depends on κ_i and r	depends on r

In Table 1, A, $\dot{A}$, $\ddot{A}$,..., $\overset{m}{A}$ are Cartan tensors and their covariant derivatives κ_i and r are arbitrary real constants. The connections 1, 2, and 3 belong to the Berwald-type category. The connections 4, 5, and 6 are Cartan-type connections. The connections 7 and 8 belong to the Shen-type category. The connection 9 contains all other connections.

Remark 3.1. The Berwald and Chern connections are special cases of Berwald-type connection in the following way:
Putting $k_1=\cdots=k_m=0$ yields the **Chern connection**.
Putting $k_2=\cdots=k_m=0$ and $k_1=1$ yield the **Berwald connection**.

The bundle map $\mu: T(TM_0)\to\pi^*TM$ defined in Definition 1 can be expressed in the following form:

$$\mu(\frac{\partial}{\partial x^i})=N_i^k\partial_k,\qquad \mu(\frac{\partial}{\partial y^i})=\partial_i, \tag{3.14}$$

where $N_i^k=F\Gamma_{ij}^k\ell^j=F\{\gamma_{ij}^k\ell^j-A^k{}_{il}\gamma_{ab}^l\ell^a\ell^b\}$.
Using the nonlinear connection coefficients, for Berwald-type connection we have,
(3.15)

$$\Gamma_{jk}^i=\gamma_{jk}^i+k_1\dot{A}^i{}_{jk}+\cdots+k_m\overset{m}{A}{}^i{}_{jk}-g^{il}\{C_{ljs}N_k^s-C_{jks}N_l^s+C_{kls}N_j^s\}.$$

We summarize that $Ker\rho = VTM$, $Ker\mu = HTM$, ρ restricted to HTM is an isomorphism onto π^*TM, and μ restricted to VTM is the bundle isomorphism onto π^*TM.

4. **Curvature tensors**

Here, we study the curvature tensors of the Berwald-type connection. This connection is torsion-free and almost compatible with Finsler metric in the sense of (3.2). As a torsion-free connection, it defines two curvatures R and P. The R-term is the so-called Riemannian curvature tensor which is a natural extension of the usual Riemannian curvature tensor of Riemannian metrics, while the P-term is a purely non-Riemannian quantity. We prove also that the hv-curvature P of this connection vanishes if and only if the Finsler structure is a Berwald structure. The curvature tensor Ω of D is defined by:

$$\Omega(\hat{X},\hat{Y})Z = D_{\hat{X}}D_{\hat{Y}}Z - D_{\hat{Y}}D_{\hat{X}}Z - D_{[\hat{X},\hat{Y}]}Z, \tag{4.1}$$

where $\hat{X},\hat{Y} \in \chi(TM_0)$ and $Z \in \pi^*TM$.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal (with respect to g) frame field for the vector bundle π^*TM such that $g(e_i, e_n) = 0, i = 1,...,n-1$, and $e_n := \ell$. Put $\ell_i := g_{ij}\ell^j = F_{y^i}$. Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. The ω^i are local sections of the dual bundle π^*TM. One readily finds that $\omega^n : \frac{\partial F}{\partial y^i}dx^i = \omega$, which is the *Hilbert form*. It is obvious that $\omega(\ell) = 1$.

Put

$$\rho = \omega^i \otimes e_i, \quad De_i = \omega_i^{\ j} \otimes e_j, \quad \Omega e_i = 2\Omega_i^{\ j} \otimes e_j.$$

$\{\Omega_i^{\ j}\}$ and $\{\omega_i^{\ j}\}$ are called the *curvature forms* and *connection forms* of D with respect to $\{e_i\}$. We have $\mu := DF\ell = F\{\omega_n^{\ i} + d(logF)\delta_n^i\} \otimes e_i$. Put $\omega^{n+i} := \omega_n^{\ i} + d(logF)\delta_n^i$. It is easy to see that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. By definition,

$$\rho = \omega^i \otimes e_i, \quad \mu = F\omega^{n+i} \otimes e_i.$$

According to Theorem 3.1, there exits a connection 1-forms $\{\omega_j^i\}$ which satisfies the following torsion-freeness and almost compatibility as follows:

$$d\omega^i = \omega^j \wedge \omega_j^{\ i}, \tag{4.2}$$

$$dg_{ij} = g_{kj}\omega_i^{\ k} + g_{ik}\omega_j^{\ k} - 2\{k_1 \dot{A}_{ijk} + \cdots + k_m \overset{m}{A}_{ijk}\}\omega^k + 2A_{ijk}\omega^{n+k}. \tag{4.3}$$

In fact, using the local orthonormal frame field $\{e_i\}_{i=1}^n$ for the vector bundle π^*TM and its dual co-frame field $\{\omega^i\}_{i=1}^n$, (3.1) and (3.2), respectively, after a straightforward calculation analogous to the proof of Theorem 3.1, we will have (4.2) and (4.3).
Let,

$$dg_{ij} - g_{kj}\omega_i^{\ k} - g_{ik}\omega_j^{\ k} = g_{ij|k}\omega^k + g_{ij.k}\omega^{n+k}, \tag{4.4}$$

where $g_{ij.k}$ and $g_{ij|k}$ are respectively the vertical and horizontal covariant derivatives of g_{ij}. This gives:

$$g_{ij|k} = -2\{k_1 \dot{A}_{ijk} + \cdots + k_m \overset{m}{A}_{ijk}\}, \tag{4.5}$$

and

$$g_{ij.k} = 2A_{ijk}. \tag{4.6}$$

Moreover, the torsion freeness is equivalent to:

$$\omega_j^{\ i} = \Gamma^i_{jk} dx^k. \tag{4.7}$$

Clearly, (4.1) is equivalent to:

$$d\omega_i^{\ j} - \omega_i^{\ k} \wedge \omega_k^{\ j} = \Omega_i^{\ j}. \tag{4.8}$$

Since the $\Omega_j^{\ i}$ are 2-forms on the manifold TM_0, they can be generally expanded as:

$$\Omega_i^{\ j} = \frac{1}{2} R_i^{\ j}{}_{kl}\omega^k \wedge \omega^l + P_i^{\ j}{}_{kl}\omega^k \wedge \omega^{n+l} + \frac{1}{2} Q_i^{\ j}{}_{kl}\omega^{n+k} \wedge \omega^{n+l}. \tag{4.9}$$

Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM, \dot{e}_i \in VTM$ such that $\rho(\bar{e}_i) = e_i, \mu(\dot{e}_i) = Fe_i$. The objects R, P and Q are respectively the hh-, hv- and vv-curvature tensors of the connection D and with $R(\bar{e}_k, \bar{e}_l)e_i = R_i^{\ j}{}_{kl}e_j$, $\quad P(\bar{e}_k, \dot{e}_l)e_i = P_i^{\ j}{}_{kl}e_j$, and $Q(\dot{e}_k, \dot{e}_l)e_i = Q_i^{\ j}{}_{kl}e_j$. From (4.9) we see that

$$R_i^{\ j}{}_{kl} = -R_i^{\ j}{}_{lk} \quad \textit{and} \quad Q_i^{\ j}{}_{lk} = -Q_i^{\ j}{}_{kl}. \tag{4.10}$$

If D is a torsion-free, then $Q = 0$. Differentiating (4.2), then we have the first Bianchi identity,

$$\omega^i \wedge \Omega_i^j = 0, \tag{4.11}$$

which implies the first *Bianchi identity* for R:

$$R_i^{\ j}{}_{kl} + R_k^{\ j}{}_{li} + R_l^{\ j}{}_{ik} = 0, \tag{4.12}$$

and

$$P_i{}^j{}_{kl} = P_k{}^j{}_{il}. \tag{4.13}$$

The Exterior differentiation of (4.8) gives rise to the *Second Bianchi identity*:

$$d\Omega_i{}^j - \omega_i{}^k \wedge \Omega_k{}^j + \omega_k{}^j \wedge \Omega_i{}^k = 0. \tag{4.14}$$

We decompose the covariant derivatives of the Cartan tensor on TM as:

$$dA_{ijk} - A_{ljk}\omega_i{}^l - A_{ilk}\omega_j{}^l - A_{ijl}\omega_k{}^l = A_{ijk|l}\omega^l + A_{ijk.l}\omega^{n+l}, \tag{4.15}$$

and in a similar way, $\forall m \in \mathbb{N}$, for $\overset{m}{A}_{ijk}$ we have,

$$d\overset{m}{A}_{ijk} - \overset{m}{A}_{ljk}\omega_i{}^l - \overset{m}{A}_{ilk}\omega_j{}^l - \overset{m}{A}_{ijl}\omega_k{}^l = \overset{m}{A}_{ijk|l}\omega^l + \overset{m}{A}_{ijk.l}\omega^{n+l}. \tag{4.16}$$

Clearly, from (4.15) and (4.16), we find that for each l and $\forall m \in \mathbb{N}$,

$$A_{ijk|l}, A_{ijk.l},\ \overset{m}{A}_{ijk|l} \quad \text{and} \quad \overset{m}{A}_{ijk.l}, \tag{4.17}$$

are symmetric in i, j, k. Put $\overset{m}{A}_{ijk} = \overset{m}{A}(e_i, e_j, e_k)$ and $\overset{m}{A}{}^l{}_{ij} = g^{kl}\overset{m}{A}_{ijk}$, $\forall m \in \mathbb{N}$. By definition of $\dot{A}$ and $\overset{m}{A}$, one has,

$$A_{ijk|n} = \dot{A}_{ijk}, \tag{4.18}$$

where we use the notation $\overset{m}{A}_{ijk|n} = \overset{m}{A}_{ijk|s}\ell^s$ for all $m \in \mathbb{N}$ and

$$\overset{m}{A}_{ijk|n} = \overset{m+1}{A}{}_{ijk}. \tag{4.19}$$

It follows from (4.15),

$$A_{njk|l} = 0 \quad , \quad A_{njk.l} = -A_{jkl}, \tag{4.20}$$

and from (4.16) we have,

$$\forall m \in \mathbb{N}, \quad \overset{m}{A}_{njk|l} = 0 \quad , \quad \overset{m}{A}_{njk.l} = -\overset{m}{A}_{jkl}. \tag{4.21}$$

In this relation the following results are well known.

Theorem A. ([7], [10]) *Let (M, F) be a Finsler manifold. Then, for the Cartan connection (or Hashiguchi connection), hv-curvature $P_j{}^i{}_{kl} = 0$ if and only if F is a Landsberg metric.*

Theorem B. ([4])*Let (M, F) be a Finsler manifold. Then, for the Chern connection (or Berwald connection), hv-curvature $P_j{}^i{}_{kl} = 0$ if and only if F is a Berwald metric.*

Theorem C. ([13]) *Let (M,F) be a Finsler manifold. Then, for the Shen connection, hv-curvature $P_j{}^i{}_{kl}=0$ if and only if F is Riemannian.*

Analogously we have the following result.

Theorem 4.1. *Let (M,F) be a Finsler manifold. Then, for the Berwald-type connection, hv-curvature $P_j{}^i{}_{kl}=0$ if and only if F is a Berwald metric.*

proof. Let (M,F) be a Finsler manifold. Differentiating (4.3), and using (4.2), (4.3), (4.8), (4.15), (4.16), (4.17), (4.18) , (4.19), (4.20) and (4.21) lead to:

$$\begin{aligned} g_{kj}\Omega_i{}^k + g_{ik}\Omega_j{}^k &= -2A_{ijk}\Omega_n^k - 2A_{ijk|l}\omega^l\wedge\omega^{n+k} + 2A_{ijk.l}\omega^{n+k}\wedge\omega^{n+l} \\ &+ k_1(\dot{A}_{ijk|l}\omega^l + \dot{A}_{ijk.l}\omega^{n+l})\wedge\omega^k + \cdots \\ &+ k_m(\overset{m}{A}_{ijk|l}\omega^l + \overset{m}{A}_{ijk.l}\omega^{n+l})\wedge\omega^k. \end{aligned} \tag{4.22}$$

By using (4.9) and (4.22), we have the followings:

$$\begin{aligned} R_{ijkl} + R_{jikl} &= 2k_1\left\{\dot{A}_{ijl|k} - \dot{A}_{ijk|l}\right\} + \cdots + 2k_m\left\{\overset{m}{A}_{ijl|k} - \overset{m}{A}_{ijk|l}\right\} \\ &- 2A_{ijs}R_n{}^s{}_{kl}, \end{aligned} \tag{4.23}$$

$$P_{ijkl}+P_{jikl} = -2\{k_1\dot{A}_{ijk.l}+\cdots+k_m\overset{m}{A}_{ijk.l}\} - 2A_{ijl|k} - 2A_{ijs}P_n{}^s{}_{kl}, \tag{4.24}$$

$$A_{ijk.l} = A_{ijl.k}. \tag{4.25}$$

Permuting i,j,k in (4.24) yields:

$$\begin{aligned} P_{ijkl} &= -\{k_1\dot{A}_{ijk.l} + \cdots + k_m\overset{m}{A}_{ijk.l}\} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) \\ &+A_{kis}P_n{}^s{}_{jl} - A_{jks}P_n{}^s{}_{il} - A_{ijs}P_n{}^s{}_{kl}, \end{aligned} \tag{4.26}$$

and

$$P_{njkl} = \{k_1\dot{A}_{jkl} + \cdots + k_m\overset{m}{A}_{jkl}\} - \dot{A}_{jkl}, \tag{4.27}$$

Because $P_{njnl}=0$. Now, if F is a Berwald metric, then from (4.26) and (4.27) we conclude $P=0$.
Conversely, let $P=0$. It follows from (4.27),

$$k_1\dot{A}_{jkl} + \cdots + k_m\overset{m}{A}_{jkl} = \dot{A}_{jkl}, \tag{4.28}$$

By means of (4.26) we have,

$$k_1\dot{A}_{ijk.l} + \cdots + k_m\overset{m}{A}_{ijk.l} = A_{kil|j} - A_{ijl|k} - A_{jkl|i}.$$

Permuting i, j, k in the above identity yields:

$$k_1 \dot{A}_{ijk.l} + \cdots + k_m \overset{m}{A}_{ijk.l} = A_{jkl|i} - A_{kil|j} - A_{ijl|k},$$

and then,

$$A_{ijl|k} = A_{jkl|i}.$$

Letting $k = n$ in the above relation, we conclude:

$$\dot{A}_{ijk} = 0. \tag{4.29}$$

It is obvious that

$$\forall m \in \mathbb{N}, \quad \overset{m}{A}_{ijk} = 0. \tag{4.30}$$

Therefore, from (4.24), (4.26), (4.27) and (4.30), we conclude that $A_{ijk|l} = 0$, and thus F is a Berwald metric. $\square$

5. Some applications

5.1. **Preliminaries on geodesics and completeness.** Here, we explore the notion of geodesics to introduce the concept of completeness for Finsler manifolds. Let $c : [a, b] \to M$ be a unit speed C^∞ curve in (M, F). The canonical lift of c to TM_0 is defined by:

$$\hat{c} := \frac{dc}{dt} \in TM_0.$$

It is easy to see that $\rho(\frac{d\hat{c}}{dt}) = \ell_{\hat{c}}$, where c is called a *geodesic* if its canonical lift $\hat{c}$ satisfies:

$$\frac{d\hat{c}}{dt} = \bar{\ell}_{\hat{c}},$$

where $\bar{\ell}$ is the geodesic field on TM_0, defined for $\ell \in HTM$ by $\rho(\bar{\ell}) = \ell$. Let $I_xM = \{v \in T_xM, F(v) = 1\}$ and $IM = \bigcup_{p \in M} I_xM$, where I_xM is called the *indicatrix*, and it is a compact set. We can show that the projection of integral curve $\varphi(t)$ of $\bar{\ell}$ with $\varphi(0) \in IM$ is a unit speed geodesic c whose canonical lift is $\hat{c}(t) = \varphi(t)$.

A Finsler manifold (M, F) is said to be *backward geodesically complete* (or *forward geodesically complete*) if every geodesic $c(t)$, $a \leq t < b$ ($a < t \leq b$), parameterized to have a constant Finslerian speed, can be extended to a geodesic defined on $a \leq t < \infty$ ($-\infty < t \leq b$). A Finsler manifold (M, F) is said to be *complete* if it is both forward and backward geodesically complete.

Let c be a unit speed geodesic in M. A section $X = X(t)$ of π^*TM

along $\hat{c}$ is said to be parallel if $D_{\frac{d\hat{c}}{dt}}X = 0$. For $v \in TM_0$, define $\|A\|_v = supA(X,Y,Z)$ and $\|\dot{A}\|_v = sup\dot{A}(X,Y,Z)$, where the supremum is taken over all unit vectors of π_v^*TM. Put $\|A\|_v = sup_{v\in IM}\|A\|_v$ and $\|\dot{A}\|_v = sup_{v\in IM}\|\dot{A}\|_v$.

5.2. **Application of Berwald-type connections.** In this subsection we are going to use two special cases of Berwald-type connections introduced in Section 3. A useful property of this connection is its adaptive form for different applications. In fact, one can use a suitable special case of this connection to find a geometric interpretation for solutions of some differential equations formed by Cartan tensor and its derivatives in Finsler spaces. For example, we prove the following theorem.

Theorem 5.1. *Let (M,F) be a complete Finsler manifold with bounded Landsberg tensor. Then, F is a Landsberg metric if and only if* $P_{jkl} = 0$.

proof. To prove the theorem, we introduce a connection for which we have put $k_1 = k_3 = \cdots = k_m = 0$ and $k_2 \neq 0$ in (4.27). Let F be a Landsberg metric, then from (4.27) we find that $P_{jkl} = 0$. Conversely, if $P_{jkl} = 0$, then we have following differential equation:

$$k_m A^{(m)} + \cdots + k_2 A^{(2)} + (k_1 - 1)\dot{A} = 0. \tag{5.1}$$

If $k_1 = k_3 = \cdots = k_m = 0$ and $k_2 \neq 0$, then we find an special Berwald-type connection for which we have,

$$k_2\ddot{A} - \dot{A} = 0. \tag{5.2}$$

On the other hand,

$$\frac{d\dot{A}}{dt} = \ddot{A}. \tag{5.3}$$

We have $\dot{A} = e^{k_2 t}\dot{A}(0)$. Using $\|\dot{A}\| < \infty$, and letting $t \to +\infty$, we then have $\dot{A}(0) = \dot{A}(X,Y,Z) = 0$, or $\dot{A} = 0$, i.e., F is a Landsberg metric. $\Box$

By means of the Theorem 5.1, every compact Finsler manifold is a Landsberg space if and only if P_{jkl} vanishes. Next, we consider an special Berwald-type connection and give another proof for the following well-known result.

Corollary 5.1. *Let (M,F) be a complete Finsler manifold with a negative constant flag curvature and a bounded Cartan tensor. Then, F is Riemannian.*

proof. Let (M, F) be a complete Finsler manifold with the constant flag curvature λ. If $\lambda \neq 0$ then we put in (4.27) $k_2 = k_4 = \cdots = k_m = 0$, $k_1 = 2$ and $k_3 = \frac{1}{\lambda} \neq 0$. We obtain a connection for which the hv-curvature P becomes:

$$\begin{aligned} P_{ijkl} &= -\{2\dot{A}_{ijk.l} + \frac{1}{\lambda}\dddot{A}_{ijk.l}\} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) \\ &\quad + A_{kis}P_{n\ jl}^{\ s} - A_{jks}P_{n\ il}^{\ s} - A_{ijs}P_{n\ kl}^{\ s}, \end{aligned} \tag{5.4}$$

and

$$P_{njkl} = \frac{1}{\lambda}\dddot{A} + \dot{A}. \tag{5.5}$$

As M has a constant flag curvature, we have,

$$\ddot{A} + \lambda A = 0, \tag{5.6}$$

From which we have $P_{njkl} = \frac{1}{\lambda}\dddot{A} + \dot{A} = 0$. By solving this differential equation, we find,

$$A(t) = c_1 + c_2 e^{\sqrt{-\lambda}t} + c_3 e^{-\sqrt{-\lambda}t}. \tag{5.7}$$

By the assumption that the Cartan tensor is bounded, and letting $t \to \infty$ and $t \to -\infty$, we see that $c_2 = c_3 = 0$. Then, $A = c_1$, and therefore $\dot{A} = 0$ and F is a Landsberg metric. From (5.6), it is easy to see that $A = 0$. □

Acknowledgments

The authors express their sincere gratitude to Professor Z. Shen for his remarks and encouragements.

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