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GENERALIZED DRAZIN INVERSE OF CERTAIN BLOCK MATRICES IN BANACH ALGEBRAS

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Abstract. Several representations of the generalized Drazin inverse of an anti-triangular block matrix in Banach algebra are given in terms of the generalized Banachiewicz–Schur form.
Keywords: Generalized Drazin inverse, Schur complement, block matrix.
MSC(2010): Primary: 46H05, 47A05; Secondary: 15A09.

1. Introduction

The Drazin inverse plays an important role in Markov chains, singular differential and difference equations, iterative methods in numerical linear algebra, etc. Representations for the Drazin inverse of block matrices under certain conditions where given in the literature [2–4, 10–12, 14, 16, 19]. Deng [7] investigated necessary and sufficient conditions for a partitioned operator matrix on a Hilbert space to have the Drazin inverse with the generalized Banachiewicz–Schur form. In [8], a representation for the Drazin inverse of an anti-triangular block matrix under some conditions was obtained, generalizing in different ways results from [6, 14]. Block anti-triangular matrices arise in many applications, ranging from constrained optimization problems to solution of differential equations, etc. Deng [9] presented some formulas for the generalized Drazin inverse of an anti-triangular operator matrix $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, acting on a Banach space, with the assumption that CA^dB is invertible.

In this paper, we study the generalized Drazin inverse of anti-triangular matrices in a Banach space, getting as particular cases recent results from [7–9].

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. For $a \in \mathcal{A}$, we use $\sigma(a)$, $r(a)$ and $\rho(a)$, respectively, to denote the spectrum, the spectral

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radius and the resolvent set of a. The sets of all invertible, nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively.

The generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a) is the element $b \in \mathcal{A}$ which satisfies

$$bab = b, \qquad ab = ba, \qquad a - a^2 b \in \mathcal{A}^{qnil}.$$

If the generalized Drazin inverse of a exists, it is unique and is denoted by a^d, and a is generalized Drazin invertible. The set of all generalized Drazin invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^d$. The Drazin inverse is a special case of the generalized Drazin inverse for which $a - a^2 b \in \mathcal{A}^{nil}$ instead of $a - a^2 b \in \mathcal{A}^{qnil}$, i.e., the Drazin inverse of a is the element b (denoted by a^D) which satisfies $bab = b$, $ab = ba$ and $a^{k+1} b = a^k$, for some nonnegative integer k. The least such k is called the Drazin index of a, and is denoted by $i(a)$. Obviously, if a is Drazin invertible, then it is generalized Drazin invertible. The group inverse is the Drazin inverse for which the conditon $a - a^2 b \in \mathcal{A}^{nil}$ is replaced with $a = aba$. We use $a^\#$ to denote the group inverse of a, and we use $\mathcal{A}^\#$ and $\mathcal{A}^D$ to denote the sets of all group invertible and Drazin invertible elements of $\mathcal{A}$, respectively.

Recall that $a \in \mathcal{A}$ is generalized Drazin invertible if and only if there exists an idempotent $p = p^2 \in \mathcal{A}$ such that

$$ap = pa \in \mathcal{A}^{qnil}, \qquad a + p \in \mathcal{A}^{-1}.$$

Then $p = 1 - aa^d$ is the *spectral idempotent* of a corresponding to the set $\{0\}$, and it will be denoted by a^π. The generalized Drazin inverse a^d double commutes with a, that is, $ax = xa$ implies $a^d x = x a^d$.

Let $p = p^2 \in \mathcal{A}$ be an idempotent. Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

We use the following lemmas.

Lemma 1.1. [5, Lemma 2.4] *Let $b, q \in \mathcal{A}^{qnil}$ and let $qb = 0$. Then $q + b \in \mathcal{A}^{qnil}$.*

Lemma 1.2. *Let $b \in \mathcal{A}^d$ and $a \in \mathcal{A}^{qnil}$.*

(i) [5, Corollary 3.4] *If $ab = 0$, then $a+b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n$.*

(ii) *If $ba = 0$, then $a + b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} a^n (b^d)^{n+1}$.*

Specializing [5, Corollary 3.4] (with multiplication reversed) to bounded linear operators N. Castro González and J. J. Koliha [5] recovered [13, Theorem 2.2] which is a spacial case of Lemma 1.2(ii).

Lemma 1.3. *Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1, and let p be an idempotent of $\mathcal{A}$. If $x \in p\mathcal{A}p$, then $\sigma_{p\mathcal{A}p}(x) \cup \{0\} = \sigma_{\mathcal{A}}(x)$, where $\sigma_{\mathcal{A}}(x)$ denotes the spectrum of x in the algebra $\mathcal{A}$, and $\sigma_{p\mathcal{A}p}(x)$ denotes the spectrum of x in the algebra $p\mathcal{A}p$.*

Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$. It is well known that if $a \in (p\mathcal{A}p)^{-1}$ and the Schur complement $s = d - ca^{-1}b \in ((1-p)\mathcal{A}(1-p))^{-1}$, then the inverse of x has Banachiewicz–Schur form

$$x^{-1} = \begin{bmatrix} a^{-1} + a^{-1}bs^{-1}ca^{-1} & -a^{-1}bs^{-1} \\ -s^{-1}ca^{-1} & s^{-1} \end{bmatrix}.$$

We investigate equivalent conditions under which x^d has the generalized Banachiewicz–Schur form in a Banach algebra. Also, we obtain several representations for the generalized Drazin inverse of an anti-triangular matrix $x = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ under different conditions. As particular cases, we get the corresponding results for the Drazin inverse in a Banach algebra. Thus, we extend some results from [7–9] to more general settings.

2. Results

In the following lemma, we present necessary and sufficient conditions for an element $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of Banach algebra to have the generalized Drazin inverse with the generalized Banachiewicz–Schur form. We recover a new result concerning the Drazin inverse of Hilbert space operators (see [7, Corollary 3]).

Lemma 2.1. *Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\#}$, and let $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$ be the generalized Schur complement of a in x. Then the following statements are equivalent:*

(i) $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^{\#} + a^{\#}bs^{\#}ca^{\#} & -a^{\#}bs^{\#} \\ -s^{\#}ca^{\#} & s^{\#} \end{bmatrix}; \tag{2.1}$$

(ii) $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$, $\quad s^{\pi}ca^{\#} = s^{\#}ca^{\pi}$ $\quad$ *and* $\quad z = \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & 0 \end{bmatrix} \in \mathcal{A}^{qnil}$;

(iii) $a^{\pi}b = bs^{\pi}$, $\quad s^{\pi}c = ca^{\pi}$ $\quad$ *and* $\quad z = \begin{bmatrix} 0 & a^{\pi}b \\ s^{\pi}c & 0 \end{bmatrix} \in \mathcal{A}^{qnil}$.

Proof. (i) $\Leftrightarrow$ (ii): If the right hand side of (2.1) is denoted by y, then we obtain

$$xy = \begin{bmatrix} aa^{\#} - a^{\pi}bs^{\#}ca^{\#} & a^{\pi}bs^{\#} \\ s^{\pi}ca^{\#} & ss^{\#} \end{bmatrix},$$

$$yx = \begin{bmatrix} a^{\#}a - a^{\#}bs^{\#}ca^{\pi} & a^{\#}bs^{\pi} \\ s^{\#}ca^{\pi} & s^{\#}s \end{bmatrix}.$$

So, $xy = yx$ if and only if $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$ and $s^{\pi}ca^{\#} = s^{\#}ca^{\pi}$, because these equalities imply $(a^{\pi}bs^{\#})ca^{\#} = a^{\#}b(s^{\pi}ca^{\#}) = a^{\#}bs^{\#}ca^{\pi}$. Further, we can verify that $yxy = y$. Using $s = d - ca^{\#}b$, $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$ and $s^{\pi}ca^{\#} = s^{\#}ca^{\pi}$, we have

$$x - x^2y = \begin{bmatrix} -bs^{\#}ca^{\pi} & bs^{\pi} \\ ca^{\pi} & 0 \end{bmatrix}.$$

From $a^{\#}bs^{\pi} = a^{\pi}bs^{\#} = (p - aa^{\#})bs^{\#} = bs^{\#} - aa^{\#}bs^{\#}$, we obtain $bs^{\#} = a^{\#}bs^{\pi} + aa^{\#}bs^{\#}$ which gives $ca^{\pi}bs^{\#} = 0 = bs^{\#}ca^{\pi}bs^{\#}$ and

$$x - x^2y = \begin{bmatrix} p & -bs^{\#} \\ 0 & 1-p \end{bmatrix} z \begin{bmatrix} p & bs^{\#} \\ 0 & 1-p \end{bmatrix}.$$

Since $r(x - x^2y) = r(\begin{bmatrix} p & bs^{\#} \\ 0 & 1-p \end{bmatrix}\begin{bmatrix} p & -bs^{\#} \\ 0 & 1-p \end{bmatrix} z) = r(z)$, we deduce that $x - x^2y \in \mathcal{A}^{qnil}$ is equivalent to $z \in \mathcal{A}^{qnil}$.

(ii) $\Leftrightarrow$ (iii): We prove that $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$ is equivalent to $a^{\pi}b = bs^{\pi}$. Indeed, multiplying $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$ from the right by s and from the left by a, respectively, we obtain $a^{\pi}bs^{\#}s = 0$ and $aa^{\#}bs^{\pi} = 0$. Therefore, $bs^{\#}s = aa^{\#}bs^{\#}s = aa^{\#}b$ and

$$a^{\pi}b = b - aa^{\#}b = b - bs^{\#}s = bs^{\pi}.$$

On the other hand, if $a^{\pi}b = bs^{\pi}$, then $(a^{\pi}b)s^{\#} = bs^{\pi}s^{\#} = 0$ and $a^{\#}(bs^{\pi}) = a^{\#}a^{\pi}b = 0$, i.e. $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$.

Similarly, we can verify that $s^{\pi}ca^{\#} = s^{\#}ca^{\pi}$ is equivalent to $s^{\pi}c = ca^{\pi}$. Hence, the equivalence (ii) $\Leftrightarrow$ (iii) holds. □

Since the Drazin inverse is a particular but very important case of the generalized Drazin inverse, we give the next result which can be verified similar to Lemma 2.1.

Corollary 2.2. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\#}$, *and let* $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$ *be the generalized Schur complement of* a *in* x. *Then the following statements are equivalent:*

(i) $x \in \mathcal{A}^D$ *and*

$$x^D = \begin{bmatrix} a^{\#} + a^{\#}bs^{\#}ca^{\#} & -a^{\#}bs^{\#} \\ -s^{\#}ca^{\#} & s^{\#} \end{bmatrix};$$

(ii) $a^{\pi}bs^{\#} = a^{\#}bs^{\pi}$, $\quad s^{\pi}ca^{\#} = s^{\#}ca^{\pi}$ $\quad$ *and* $\quad z = \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & 0 \end{bmatrix} \in \mathcal{A}^{nil}$;

(iii) $a^{\pi}b = bs^{\pi}$, $\quad s^{\pi}c = ca^{\pi}$ $\quad$ *and* $\quad z = \begin{bmatrix} 0 & a^{\pi}b \\ s^{\pi}c & 0 \end{bmatrix} \in \mathcal{A}^{nil}$.

By Lemma 2.1, the following corollary recovers [1, Theorem 2].

Corollary 2.3. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\#}$, *and let* $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$ *be the generalized Schur complement of* a *in* x. *Then* $x \in \mathcal{A}^{\#}$ *and*

$$x^{\#} = \begin{bmatrix} a^{\#} + a^{\#}bs^{\#}ca^{\#} & -a^{\#}bs^{\#} \\ -s^{\#}ca^{\#} & s^{\#} \end{bmatrix}$$

if and only if

$$a^{\pi}b = 0 = bs^{\pi}, \quad s^{\pi}c = 0 = ca^{\pi}.$$

Now, we extend the well known result concerning the Drazin inverse of complex matrices to the generalized Drazin inverse of elements of a Banach algebra, see [8, Theorem 3.5].

Theorem 2.4. *Let*

$$x = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in \mathcal{A} \tag{2.2}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ *and let* $s = -ca^db \in ((1-p)\mathcal{A}(1-p))^d$. *If*

$$ss^dca^{\pi}b = 0, \qquad ss^dca^{\pi}a = 0, \qquad aa^dbs^{\pi}c = 0, \qquad bs^{\pi}ca^{\pi} = 0, \tag{2.3}$$

then $x \in \mathcal{A}^d$ *and*

$$\begin{aligned} x^d &= \left(r + \sum_{n=0}^{\infty} \begin{bmatrix} aa^{\pi} & a^{\pi}bs^{\pi} \\ s^{\pi}ca^{\pi} & 0 \end{bmatrix}^n \begin{bmatrix} 0 & a^{\pi}bss^d \\ s^{\pi}caa^d & 0 \end{bmatrix} r^{n+2} \right) \\ &\times \left(1 + r \begin{bmatrix} 0 & aa^dbs^{\pi} \\ ss^dca^{\pi} & 0 \end{bmatrix} \right), \end{aligned} \tag{2.4}$$

where

$$r = \begin{bmatrix} a^d + a^dbs^dca^d & -a^dbs^d \\ -s^dca^d & s^d \end{bmatrix}. \tag{2.5}$$

Proof. Applying $aa^d + a^{\pi} = p$ and $ss^d + s^{\pi} = 1 - p$, we have

$$x = \begin{bmatrix} a^2a^d & aa^db \\ ss^dc & 0 \end{bmatrix} + \begin{bmatrix} aa^{\pi} & a^{\pi}b \\ s^{\pi}c & 0 \end{bmatrix} := u + v.$$

The equalities $a^da^{\pi} = 0$ and (2.3) give $uv = 0$.

First, we show that $u \in \mathcal{A}^d$. If we write

$$u = \begin{bmatrix} a^2a^d & aa^dbss^d \\ ss^dcaa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^dbs^{\pi} \\ ss^dca^{\pi} & 0 \end{bmatrix} := u_1 + u_2,$$

we can get $u_2u_1 = 0$ and $u_2^2 = 0$. Let $A_{u_1} \equiv a^2a^d$, $B_{u_1} \equiv aa^dbss^d$, $C_{u_1} \equiv ss^dcaa^d$ and $D_{u_1} \equiv 0$. Then $u_1 = \begin{bmatrix} A_{u_1} & B_{u_1} \\ C_{u_1} & D_{u_1} \end{bmatrix}$ and, by $(a^2a^d)^{\#} = a^d$,

$A_{u_1} \in (p\mathcal{A}p)^{\#}$. Also, from $s = -ca^d b$, $S_{u_1} \equiv D_{u_1} - C_{u_1} A_{u_1}^{\#} B_{u_1} = s^2 s^d \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $(s^2 s^d)^{\#} = s^d$. Consequently, $A_{u_1}^{\pi} B_{u_1} S_{u_1}^{\#} = 0 = A_{u_1}^{\#} B_{u_1} S_{u_1}^{\pi}$, $S_{u_1}^{\pi} C_{u_1} A_{u_1}^{\#} = 0 = S_{u_1}^{\#} C_{u_1} A_{u_1}^{\pi}$ and $\begin{bmatrix} 0 & B_{u_1} S_{u_1}^{\pi} \\ C_{u_1} A_{u_1}^{\pi} & 0 \end{bmatrix} = 0 \in \mathcal{A}^{qnil}$. By Lemma 2.1, notice that $u_1 \in \mathcal{A}^d$ and

$$u_1^d = \begin{bmatrix} A_{u_1}^{\#} + A_{u_1}^{\#} B_{u_1} S_{u_1}^{\#} C_{u_1} A_{u_1}^{\#} & -A_{u_1}^{\#} B_{u_1} S_{u_1}^{\#} \\ -S_{u_1}^{\#} C_{u_1} A_{u_1}^{\#} & S_{u_1}^{\#} \end{bmatrix} = r.$$

Using Lemma 1.2(i), $u \in \mathcal{A}^d$ and $u^d = u_1^d + (u_1^d)^2 u_2 = r + r^2 u_2$.

To prove that $v \in \mathcal{A}^{qnil}$, observe that

$$\begin{aligned} v &= \begin{bmatrix} aa^{\pi} & a^{\pi} b s^{\pi} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^{\pi} c a^{\pi} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a^{\pi} b s s^d \\ s^{\pi} c a a^d & 0 \end{bmatrix} \\ &:= v_1 + v_2 + v_3. \end{aligned}$$

If $z = \begin{bmatrix} m & t \\ 0 & n \end{bmatrix}$, then $\lambda 1 - z = \begin{bmatrix} \lambda p - m & -t \\ 0 & \lambda(1-p) - n \end{bmatrix}$. Therefore,

$$\lambda \in \rho_{p\mathcal{A}p}(m) \cap \rho_{(1-p)\mathcal{A}(1-p)}(n) \ \Rightarrow \ \lambda \in \rho(z),$$

i.e.,

$$\sigma(z) \subseteq \sigma_{p\mathcal{A}p}(m) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(n).$$

Notice that, by $aa^{\pi} \in (p\mathcal{A}p)^{qnil}$, $v_1 \in \mathcal{A}^{qnil}$. It can be verified that $v_1 v_2 = 0$ and $v_2^2 = 0$, i.e., $v_2 \in \mathcal{A}^{nil}$. Now, by Lemma 1.1, $v_1 + v_2 \in \mathcal{A}^{qnil}$. Using Lemma 1.1 again, from $v_3^2 = 0$ and $v_3(v_1 + v_2) = 0$, we conclude that $v \in \mathcal{A}^{qnil}$.

Applying Lemma 1.2(ii), we deduce that $x \in \mathcal{A}^d$ and

$$x^d = \left(1 + \sum_{n=0}^{\infty} v^{n+1} (u^d)^{n+2}\right) u^d = \left(1 + \sum_{n=0}^{\infty} v^{n+1} (u^d)^{n+2}\right) r(1 + r u_2).$$

Since $u_2 r = u_2 u_1^d = (u_2 u_1)(u_1^d)^2 = 0$, then $(u^d)^{n+2} = (r + r^2 u_2)^{n+2} = r^{n+2}(1 + r u_2)$. From $r = \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r$, we obtain $vr = v \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r = \begin{bmatrix} 0 & a^{\pi} b s s^d \\ s^{\pi} c a a^d & 0 \end{bmatrix}$.

By $v^{n+1} = (v_1 + v_2)^n v$, we have $v^{n+1}(u^d)^{n+2} = (v_1 + v_2)^n \begin{bmatrix} 0 & a^{\pi} b s s^d \\ s^{\pi} c a a^d & 0 \end{bmatrix} r^{n+1}(1 + r u_2)$. Applying $u_2 r = 0$ again, we get (2.4). □

From Theorem 2.4, we get the following consequence.

Corollary 2.5. *Let x be defined as in* (2.2)*, $a \in (p\mathcal{A}p)^d$ and let r be defined as in* (2.5).

(1) *If* $ca^\pi = 0$ *and generalized Schur complement* $s = -ca^d b$ *is invertible, then* $x \in \mathcal{A}^d$ *and*

$$x^d = r + \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & 0 \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2}.$$

(2) *If* $ca^\pi = 0$, $a^\pi b = 0$ *and the generalized Schur complement* $s = -ca^d b$ *is invertible, then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d + a^d bs^{-1}ca^d & -a^d bs^{-1} \\ -s^{-1}ca^d & s^{-1} \end{bmatrix}.$$

(3) *If* $ca^\pi b = 0$, $ca^\pi a = 0$ *and the generalized Schur complement* $s = -ca^d b$ *is invertible, then* $x \in \mathcal{A}^d$ *and*

$$x^d = \left(r + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2} \right) \left(1 + r \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} \right).$$

The next result is a special case of Theorem 2.4.

Corollary 2.6. *Let* x *be defined as in* (2.2), $a \in (p\mathcal{A}p)^D$, $i(a) = m$, *and let* $s = -ca^D b \in ((1-p)\mathcal{A}(1-p))^D$. *If* $ss^D ca^\pi b = 0$, $ss^D ca^\pi a = 0$, $aa^D bs^\pi c = 0$ *and* $bs^\pi ca^\pi = 0$, *then* $x \in \mathcal{A}^D$ *and*

$$\begin{aligned} x^D &= \left(r_1 + \sum_{n=0}^{m+1} \begin{bmatrix} aa^\pi & a^\pi bs^\pi \\ s^\pi ca^\pi & 0 \end{bmatrix}^n \begin{bmatrix} 0 & a^\pi bss^D \\ s^\pi caa^D & 0 \end{bmatrix} r_1^{n+2} \right) \\ &\times \left(1 + r_1 \begin{bmatrix} 0 & aa^D bs^\pi \\ ss^D ca^\pi & 0 \end{bmatrix} \right), \end{aligned}$$

where $r_1 = \begin{bmatrix} a^D + a^D bs^D ca^D & -a^D bs^D \\ -s^D ca^D & s^D \end{bmatrix}$.

Proof. Using the same notation as in the proof of Theorem 2.4, from $i(a) = m$, we have $v_1^{m+1} = 0$, $(v_1 + v_2)^{m+2} = 0$ and $v^{m+3} = (v_1 + v_2)^{m+2} v = 0$. Since v is nilpotent and u is Drazin invertible, we conclude that $x \in \mathcal{A}^D$ (see [15, 18]).

By $v^{n+1}(u^D)^{n+2} = (v_1+v_2)^n \begin{bmatrix} 0 & a^\pi bss^D \\ s^\pi caa^D & 0 \end{bmatrix} r_1^{n+1}(1+r_1 u_2)$, we obtain the representation for x^D. □

In the following theorems, we assume that $s = -ca^d b$ is the generalized Drazin invertible, and we prove representations of the generalized Drazin inverse of anti-triangular block matrices. Several results from [9] are extended.

Theorem 2.7. *Let* x *be defined as in* (2.2), $a \in (p\mathcal{A}p)^d$ *and let* $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. *If* $bca^\pi = 0$ *and* $aa^d bs^\pi = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n \left(1 + \begin{bmatrix} 0 & 0 \\ s^\pi c & 0 \end{bmatrix} r \right) r^{n+1}, \tag{2.6}$$

where r *is defined as in* (2.5).

Proof. We can write

$$x = \begin{bmatrix} a^2a^d & aa^db \\ caa^d & 0 \end{bmatrix} + \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix} := y + q.$$

Now, we get $yq = 0$, by the assumption $bca^\pi = 0$.

In order to prove that $y \in \mathcal{A}^d$, note that

$$\begin{aligned} y &= \begin{bmatrix} a^2a^d & aa^dbss^d \\ ss^dcaa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^dbs^\pi \\ s^\pi caa^d & 0 \end{bmatrix} \\ &= \begin{bmatrix} a^2a^d & aa^dbss^d \\ ss^dcaa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^\pi caa^d & 0 \end{bmatrix} := y_1 + y_2, \end{aligned}$$

$y_1y_2 = 0$ and $y_2^2 = 0$. Using Lemma 2.1, we have $y_1 \in \mathcal{A}^d$ and $y_1^d = r$. By Lemma 1.2(ii), $y \in \mathcal{A}^d$ and $y^d = y_1^d + y_2(y_1^d)^2 = r + y_2r^2$.

Further, we verify that $q \in \mathcal{A}^{qnil}$. Let

$$q = \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} := q_1 + q_2.$$

Thus, we deduce that $q_1 \in \mathcal{A}^{qnil}$ and $q_2 \in \mathcal{A}^{nil}$, because $aa^\pi \in (p\mathcal{A}p)^{qnil}$ and $q_2^2 = 0$. Since $q_1q_2 = 0$, by Lemma 1.1, $q \in \mathcal{A}^{qnil}$.

By Lemma 1.2(ii), $x \in \mathcal{A}^d$ and

$$x^d = \sum_{n=0}^{\infty} q^n (y^d)^{n+1} = \sum_{n=0}^{\infty} q^n (1 + y_2r) r^{n+1}.$$

The equality $r = \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r$ gives $y_2r = \begin{bmatrix} 0 & 0 \\ s^\pi c & 0 \end{bmatrix} r$, implying (2.6). □

Replacing the hypothesis $aa^dbs^\pi = 0$ with $s^\pi caa^d = 0$ in Theorem 2.7, we get the following theorem.

Theorem 2.8. *Let* x *be defined as in* (2.2), $a \in (p\mathcal{A}p)^d$ *and let* $s = -ca^db \in ((1-p)\mathcal{A}(1-p))^d$. *If* $bca^\pi = 0$ *and* $s^\pi caa^d = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n r^{n+1} \left(1 + r \begin{bmatrix} 0 & bs^\pi \\ 0 & 0 \end{bmatrix}\right), \tag{2.7}$$

where r *is defined in the same way as in* (2.5).

Proof. Similar to the proof of Theorem 2.7, by using

$$y = \begin{bmatrix} a^2a^d & aa^dbss^d \\ ss^dcaa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^dbs^\pi \\ 0 & 0 \end{bmatrix} := y_1 + y_2$$

and $y_2y_1 = 0$, we check this theorem. □

If $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$ and $s' = -s$, then $s^\pi = 0$ and $(s')^{-1} = -s^{-1}$. As a special case of Theorem 2.7 (or Theorem 2.8), we obtain the following result which recovers [9, Theorem 3.1] for bounded linear operators on a Banach space.

Corollary 2.9. *Let x be defined as in* (2.2)*, $a \in (p\mathcal{A}p)^d$ and let $s' = ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $bca^\pi = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n t^{n+1},$$

where $t = \begin{bmatrix} a^d - a^d b(s')^{-1} ca^d & a^d b(s')^{-1} \\ (s')^{-1} ca^d & -(s')^{-1} \end{bmatrix}$.

Sufficient conditions under which the generalized Drazin inverse x^d is represented by (2.6) or (2.7) are investigated in the following result.

Theorem 2.10. *Let x be defined as in* (2.2)*, $a \in (p\mathcal{A}p)^d$ and let $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. Suppose that $aa^d bca^\pi = 0$ and $ca^\pi b = 0$.*

(1) *If $aa^d bs^\pi = 0$ and ($aa^\pi b = 0$ or $caa^\pi = 0$), then $x \in \mathcal{A}^d$ and (2.6) is satisfied.*

(2) *If $s^\pi caa^d = 0$ and ($aa^\pi b = 0$ or $caa^\pi = 0$), then $x \in \mathcal{A}^d$ and (2.7) is satisfied.*

Proof. This result can be proved similarly as Theorem 2.7 and Theorem 2.8, applying $q_2 q_1 = 0$ when $caa^\pi = 0$, and the decomposition

$$q = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix}$$

when $aa^\pi b = 0$. □

Remark 2.11. *In the preceding theorem, if $ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$, then we obtain as a particular case [9, Theorem 3.2] for Banach space operator.*

We can easily show the next special cases of Theorems 2.7-2.10 for the Drazin inverse of x.

Corollary 2.12. *Let x be defined as in* (2.2)*, $a \in (p\mathcal{A}p)^D$, $i(a) = m$, $s = -ca^D b \in ((1-p)\mathcal{A}(1-p))^D$, and let r_1 be defined as in Corollary 2.6.*

(i) *If $bca^\pi = 0$ and $aa^D bs^\pi = 0$, then $x \in \mathcal{A}^D$ and*

$$x^D = \sum_{n=0}^{m+1} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n \left(1 + \begin{bmatrix} 0 & 0 \\ s^\pi c & 0 \end{bmatrix} r_1\right) r_1^{n+1}, \tag{2.8}$$

(ii) *If $bca^\pi = 0$ and $s^\pi caa^D = 0$, then $x \in \mathcal{A}^D$ and*

$$x^D = \sum_{n=0}^{m+1} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n r_1^{n+1} \left(1 + r_1 \begin{bmatrix} 0 & bs^\pi \\ 0 & 0 \end{bmatrix}\right), \tag{2.9}$$

(iii) *If* $aa^Dbca^\pi = 0$, $ca^\pi b = 0$, $aa^Dbs^\pi = 0$ *and* ($aa^\pi b = 0$ *or* $caa^\pi = 0$), *then* $x \in \mathcal{A}^D$ *and (2.8) is satisfied.*

(iv) *If* $aa^Dbca^\pi = 0$, $ca^\pi b = 0$, $s^\pi caa^D = 0$ *and* ($aa^\pi b = 0$ *or* $caa^\pi = 0$), *then* $x \in \mathcal{A}^D$ *and (2.9) is satisfied.*

The following result is well-known for complex matrices (see [17]).

Lemma 2.13. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ *and let* $\omega = aa^d + a^dbca^d$ *be such that* $a\omega \in (p\mathcal{A}p)^d$. *If* $ca^\pi = 0$, $a^\pi b = 0$ *and the generalized Schur complement* $s = d - ca^db$ *is equal to 0, then*

$$x^d = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} [(a\omega)^d]^2a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^db \\ 0 & 0 \end{bmatrix}. \tag{2.10}$$

Proof. Denote by y the right hand side of (2.10). Then we obtain

$$xy = \begin{bmatrix} (a+bca^d)[(a\omega)^d]^2a & (a+bca^d)[(a\omega)^d]^2b \\ (c+dca^d)[(a\omega)^d]^2a & (c+dca^d)[(a\omega)^d]^2b \end{bmatrix},$$

$$yx = \begin{bmatrix} [(a\omega)^d]^2(a^2+bc) & [(a\omega)^d]^2(ab+bd) \\ ca^d[(a\omega)^d]^2(a^2+bc) & ca^d[(a\omega)^d]^2(ab+bd) \end{bmatrix}.$$

By $ca^\pi = 0$ and $a^\pi b = 0$, we can conclude that $a + bca^d$ commutes with $a\omega$. Indeed,

$$\begin{aligned}(a+bca^d)(a\omega) &= (a^2+bca^da)(aa^d+a^dbca^d) = (a^2+aa^dbc)a^d(a+bca^d) \\ &= (a\omega)(a+bca^d).\end{aligned}$$

Since $a + bca^d$ commutes with $a\omega$, it also commutes with $(a\omega)^d$ and we have

$$(a+bca^d)[(a\omega)^d]^2a = [(a\omega)^d]^2(a+bca^d)a = [(a\omega)^d]^2(a^2+bc).$$

From $s = 0$, we get $c + dca^d = ca^da + ca^dbca^d = ca^d(a + bca^d)$. Thus,

$$(c+dca^d)[(a\omega)^d]^2a = ca^d(a+bca^d)[(a\omega)^d]^2a = ca^d[(a\omega)^d]^2(a^2+bc).$$

Also, $ab + bd = ab + bca^db = (a + bca^d)b$ and we obtain

$$(a+bca^d)[(a\omega)^d]^2b = [(a\omega)^d]^2(ab+bd)$$

$$(c+dca^d)[(a\omega)^d]^2b = ca^d[(a\omega)^d]^2(ab+bd).$$

So, we proved that

$$xy = yx = \begin{bmatrix} [(a\omega)^d]^2(a+bca^d)a & [(a\omega)^d]^2(a+bca^d)b \\ ca^d[(a\omega)^d]^2(a+bca^d)a & ca^d[(a\omega)^d]^2(a+bca^d)b \end{bmatrix}.$$

Further, we can verify that $yxy = y$. Indeed, we have

$$\begin{aligned} yxy &= \begin{bmatrix} [(a\omega)^d]^2 a & [(a\omega)^d]^2 b \\ ca^d[(a\omega)^d]^2 a & ca^d[(a\omega)^d]^2 b \end{bmatrix} \\ &\times \begin{bmatrix} [(a\omega)^d]^2(a+bca^d)a & [(a\omega)^d]^2(a+bca^d)b \\ ca^d[(a\omega)^d]^2(a+bca^d)a & ca^d[(a\omega)^d]^2(a+bca^d)b \end{bmatrix} \\ &= \begin{bmatrix} [(a\omega)^d]^4(a+bca^d)^2 a & [(a\omega)^d]^4(a+bca^d)^2 b \\ ca^d[(a\omega)^d]^4(a+bca^d)^2 a & ca^d[(a\omega)^d]^4(a+bca^d)^2 b \end{bmatrix}. \end{aligned}$$

The equalities $a+bca^d = a-a^2a^d+a^2a^d+bca^d = aa^\pi + a\omega$ and $a^\pi\omega = 0 = \omega a^\pi$ give $(a+bca^d)^2 = a^2a^\pi + (a\omega)^2$. Therefore,

$$\begin{aligned} (a\omega)^d(a+bca^d)^2 &= (a\omega)^d(a^2a^\pi + (a\omega)^2) \\ &= [(a\omega)^d]^2(a\omega)a^\pi a^2 + (a\omega)^d(a\omega)^2 = (a\omega)^d(a\omega)^2 \end{aligned}$$

and $[(a\omega)^d]^4(a+bca^d)^2 = [(a\omega)^d]^4(a\omega)^2 = [(a\omega)^d]^2$ implying

$$yxy = \begin{bmatrix} [(a\omega)^d]^2 a & [(a\omega)^d]^2 b \\ ca^d[(a\omega)^d]^2 a & ca^d[(a\omega)^d]^2 b \end{bmatrix} = y.$$

We now obtain

$$x - x^2y = \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ ca^d(a\omega)^\pi a & ca^d(a\omega)^\pi b \end{bmatrix} = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix}.$$

Notice that, by $a+bca^d = aa^\pi + a\omega$, $(a\omega)^\pi(a+bca^d) = aa^\pi + (a\omega)(a\omega)^\pi$. Since $aa^\pi, (a\omega)(a\omega)^\pi \in (p\mathcal{A}p)^{qnil}$ and $aa^\pi(a\omega)(a\omega)^\pi = 0$, by Lemma 1.1, we have that $aa^\pi + (a\omega)(a\omega)^\pi \in (p\mathcal{A}p)^{qnil}$ and $r_{p\mathcal{A}p}((a\omega)^\pi(a+bca^d)) = 0$. From

$$\begin{aligned} r(x-x^2y) &= r\left(\begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \right) \\ &= r\left(\begin{bmatrix} (a\omega)^\pi(a+bca^d) & 0 \\ 0 & 0 \end{bmatrix} \right) = r_{p\mathcal{A}p}((a\omega)^\pi(a+bca^d)) = 0, \end{aligned}$$

we deduce that $x - x^2y \in \mathcal{A}^{qnil}$ which proves that $x^d = y$. $\square$

The following result is a special case of Lemma 2.13 holding for Drazin inverse.

Corollary 2.14. *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^D$ *and let* $\omega = aa^D + a^Dbca^D$ *be such that* $a\omega \in (p\mathcal{A}p)^D$. *If* $ca^\pi = 0$, $a^\pi b = 0$ *and the generalized Schur complement* $s = d - ca^Db$ *is equal to 0, then*

$$x^D = \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} [(a\omega)^D]^2 a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^Db \\ 0 & 0 \end{bmatrix}. \tag{2.11}$$

Proof. Using the notations as in the proof of Lemma 2.13, we prove in the same way the equations $xy = yx$ and $yxy = y$. The proof of nilpotency of $x - x^2y$ follows.

Let $z = x - x^2y$. It holds

$$\begin{aligned} z^2 &= \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi (a + bca^D) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

By induction, we have

$$\begin{aligned} z^n &= \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi (a + bca^D) & 0 \\ 0 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} \left((a\omega)^\pi (a + bca^D)\right)^{n-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Since $a + bca^D = aa^\pi + a\omega$, we get $(a\omega)^\pi(a + bca^D) = aa^\pi + (a\omega)(a\omega)^\pi$. Also, aa^π and $(a\omega)(a\omega)^\pi$ commute and $aa^\pi(a\omega)(a\omega)^\pi = (a\omega)(a\omega)^\pi aa^\pi = 0$. So, we have $\left((a\omega)^\pi(a + bca^D)\right)^n = (aa^\pi)^n + ((a\omega)(a\omega)^\pi)^n$, for all $n \in \mathbb{N}$. Let $k = \max\{i(a), i(a\omega)\} + 1$. Since a and $(a\omega)$ are Drazin invertible, it holds that $(aa^\pi)^{k-1} = ((a\omega)(a\omega)^\pi)^{k-1} = 0$. Now, it follows that

$$\begin{aligned} z^k &= \begin{bmatrix} p & 0 \\ ca^D & 0 \end{bmatrix} \begin{bmatrix} (aa^\pi)^{k-1} + ((a\omega)(a\omega)^\pi)^{k-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which proves $z = x - x^2y \in \mathcal{A}^{nil}$. Therefore, $x \in \mathcal{A}^D$ and x^D is equal to the right hand side of f (2.11). □

In the following theorem, we extend [9, Theorem 3.3 and Theorem 3.4] for Banach space operators to elements of a Banach algebra.

Theorem 2.15. *Let x be defined as in* (2.2), $a \in (p\mathcal{A}p)^d$ *and let* $k = a^2a^d + aa^dbca^d \in (p\mathcal{A}p)^d$. *If* $ca^db = 0$ *and if one of the following conditions holds:*

(1) $bca^\pi = 0$;
(2) $aa^dbca^\pi = 0$, $aa^\pi b = 0$ *and* $ca^\pi b = 0$;
(3) $aa^dbca^\pi = 0$, $caa^\pi = 0$ *and* $ca^\pi b = 0$;

then $x \in \mathcal{A}^d$ *and*

$$x^d = \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n \begin{bmatrix} (k^d)^2 a & (k^d)^2 b \\ ca^d (k^d)^2 a & ca^d (k^d)^2 b \end{bmatrix}^{n+1}. \tag{2.12}$$

Proof. To prove (1) suppose that $x = y + q$, where s and y are defined as in the proof of Theorem 2.7. It follows that $yq = 0$ and $q \in \mathcal{A}^{qnil}$. Applying Lemma

2.13, we conclude that $y \in \mathcal{A}^d$ and

$$y^d = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} (k^d)^2 a^2 a^d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^d b \\ 0 & 0 \end{bmatrix}.$$

Since $kaa^d = k$, then $k^d aa^d = k^d$ and

$$y^d = \begin{bmatrix} (k^d)^2 a & (k^d)^2 b \\ ca^d (k^d)^2 a & ca^d (k^d)^2 b \end{bmatrix}.$$

Using Lemma 1.2(ii), we conclude that $x \in \mathcal{A}^d$ and $x^d = \sum_{n=0}^{\infty} q^n (y^d)^{n+1}$. Thus, (2.12) holds.

Parts (2) and (3) can be checked similar to part (1) and the proof of Theorem 2.10. □

If $c = 0$ or $b = 0$ in Theorem 2.15, we have $k = a^2 a^d \in (p\mathcal{A}p)^d$ and $k^d = a^d$. As a consequence of Theorem 2.15, we obtain the following result.

Corollary 2.16. *Let x be defined as in (2.2) and let $a \in (p\mathcal{A}p)^d$.*

(1) *If $c = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ 0 & 0 \end{bmatrix}.$$

(2) *If $b = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & 0 \\ c(a^d)^2 & 0 \end{bmatrix}.$$

The next corollary can be proved similar to Theorem 2.15.

Corollary 2.17. *Let x be defined as in (2.2), $a \in (p\mathcal{A}p)^D$, $i(a) = m$, and let $k = a^2 a^D + aa^D bca^D \in (p\mathcal{A}p)^D$. If $ca^D b = 0$ and if one of the following conditions holds:*

(1) *$bca^\pi = 0$;*
(2) *$aa^D bca^\pi = 0$, $aa^\pi b = 0$ and $ca^\pi b = 0$;*
(3) *$aa^D bca^\pi = 0$, $caa^\pi = 0$ and $ca^\pi b = 0$;*

then $x \in \mathcal{A}^D$ and

$$x^D = \sum_{n=0}^{m+1} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n \begin{bmatrix} (k^D)^2 a & (k^D)^2 b \\ ca^D (k^D)^2 a & ca^D (k^D)^2 b \end{bmatrix}^{n+1}. \tag{2.13}$$

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