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CHARACTERIZATION OF PROJECTIVE SPECIAL LINEAR GROUPS IN DIMENSION THREE BY THEIR ORDERS AND DEGREE PATTERNS

G. R. REZAEEZADEH*, M. BIBAK AND M. SAJJADI

(Communicated by Jamshid Moori)

Abstract. The prime graph $\Gamma(G)$ of a group G is a graph with vertex set $\pi(G)$, the set of primes dividing the order of G, and two distinct vertices p and q are adjacent by an edge written $p \sim q$ if there is an element in G of order pq. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$. For $p \in \pi(G)$, set $deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which is called the degree of p. We also set $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, where $p_1 < p_2 < ... < p_k$, which is called degree pattern of G. The group G is called k-fold OD-characterizable if there exists exactly k non-isomorphic groups M satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, as the main result, we prove that projective special linear group $L_3(2^n)$ where $n \in \{4, 5, 6, 7, 8, 10, 12\}$ is OD-characterizable.
Keywords: Prime graph, degree pattern, OD-characterizable.
MSC(2010): Primary: 20D05; Secondary: 20D06.

1. Introduction

For a finite group G, we denote by $\pi(G)$ the set of all prime divisors of G and the spectrum $\omega(G)$ of G is the set of element orders of G. Evidently $\omega(G)$ is partially ordered by the divisibility relation, hence, it is completely determined by the subset $\mu(G)$, of the maximal elements under the divisibility relation. The prime graph $\Gamma(G)$ of G is the graph with vertex set $\pi(G)$ where two distinct vertices p and q are adjacent by an edge if $pq \in \omega(G)$, in which case, we write $p \sim q$.
The degree $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident with p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define $D(G)$:=($\deg(p_1)$,$\deg(p_2)$,...,$\deg(p_k)$), which is called the degree pattern

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of G. A finite group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups M satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. In particular,a 1-fold OD-characterizable group is simply called OD-characterizable.
The interest in characterizing finite groups by degree pattern started in [4] by M.R. Darafsheh et al, in which the authors proved that if $|\pi(\frac{q^2+q+1}{d})| = 1$ where $d = (3, q-1)$ and $q > 5$, then the simple group $L_3(q)$ is OD-characterizable. In [10] it is proved that all finite simple groups whose orders are less than 10^8 except for A_{10} and $U_4(2)$ are OD-characterizable. In [11] and [14], the characterization by order and degree pattern of $L_2(q)$, where $q \geq 4$ is an odd prime power is proved. Finite groups with the same order and degree pattern as $U_3(5)$ and $U_6(2)$ are obtained in [12, 13]. Also in [7] proved that the automorphism groups of orthogonal groups $O^+_{10}(2)$ and $O^-_{10}(2)$ are OD-characterizable. The authors in [8] proved that the automorphism groups of simple K_3-groups except A_6 and $U_4(2)$ are OD-characterizable (we recall that a finite group possessing exactly n prime divisors is called K_n-group). In this paper our main aim is to show the recognizability of the group $L_3(2^n)$ where $|\pi(\frac{q^2+q+1}{d})| \neq 1$ and $d = (3, q-1)$ for certain n, by degree pattern in the prime graph and order of the group. In fact, we will prove the following Main Theorem.

Theorem 1.1 (Main Theorem)**.** *The simple group $L_3(2^n)$ where $n \in \{4, 5, 6, 7, 8, 10, 12\}$ is OD-characterizable.*

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. We denote the socle of G by $\mathrm{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively. All further unexplained notations are standard and can be found in [5].

2. **Preliminary results**

Let $p \geq 5$ be a prime. We denote by $\mathfrak{S}_p$ the set of all simple groups with prime divisors at most p. It is clear that $\mathfrak{S}_q \subseteq \mathfrak{S}_p$ where $q \leq p$. In this paper all simple groups in $\mathfrak{S}_p$ for $17 \leq p \leq 337$ are given in Table 1.

Lemma 2.1. *Let P be a simple group belonging to $\mathfrak{S}_{997}$, then $\pi(\mathrm{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$.*

Proof. All finite simple groups in $\mathfrak{S}_{997}$, are collected in [9]. So by computing the order of outer automorphism groups of them, we see that $\pi(\mathrm{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$ for every $P \in \mathfrak{S}_{997}$. In fact 11 divides only the order of outer automorphism group of $L_2(2^{11})$. □

To prove the propositions in the next section, we need degree patterns of the special linear groups under study. Since we obtain these degree patterns by a subset μ of these groups, we give following lemma.

Lemma 2.2 ([1]). *Let $L = L_3(q)$. Then* $\mu(L) = \{q-1, \frac{p(q-1)}{(3,q-1)}, \frac{q^2-1}{(3,q-1)}, \frac{q^2+q+1}{(3,q-1)}\}$.

Lemma 2.3 ([3]). *Let G be a Frobenius group with kernel K and complement H. Then:*
(a) K is a nilpotent group.
(b) $|K| \equiv 1(\mathrm{mod}|H|)$.

Definition 2.1. *A group G is called completely reducible if it is a direct product of simple group. A completely reducible group is simply called CR-group.*

Definition 2.2. *A CR-group is called centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups.*

The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.4 ([5]). *Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times ... \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then* $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times ... \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$, *where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group* $\mathbb{S}_{n_i}$ *in its natural permutation representation. Moreover, these isomorphisms induce the isomorphisms* $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times ... \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$.

3. **Proof of the main theorem**

This section is devoted to prove our main theorem. We break the proof into a number of separate propositions.

Proposition 3.1. *If G is a finite group such that $D(G) = D(L_3(2^4))$ and $|G| = |L_3(2^4)|$, then $G \cong L_3(2^4)$.*

Proof. By using Lemma 2.2, it follows that $D(L_3(2^4)) = (1,1,3,1,1,1)$. As $|G| = |L_3(2^4)| = 2^{12}.3^2.5^2.7.13.17$ and $D(G) = D(L_3(2^4))$, we conclude that the prime graph of G has the following form:

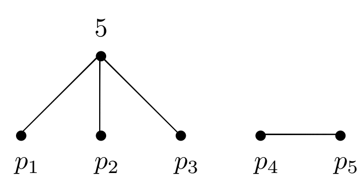

Figure 3-1

where $\{p_1, p_2, p_3, p_4, p_5\} = \{2, 3, 7, 13, 17\}$.
To simplify, we break the proof into several steps in every proposition.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{13, 17\}'$-group. In particular, G is non-solvable.

For proving Step 1, we consider two cases separately:
Case 1. $13.17 \notin \omega(G)$.
In this case, we show that K is a $\{13, 17\}'$-group. To prove this, assume first that $\{13, 17\} \subseteq \pi(K)$. Then K has a Hall $\{13, 17\}$-subgroup H. It is easy to see that H is an abelian subgroup of order 13.17, which implies that $13.17 \in \omega(K) \subseteq \omega(G)$, a contradiction. Next, we assume $13 \in \pi(K)$ and $17 \notin \pi(K)$. Then K is a $\{2, 3, 5, 7, 13\}$-group. Let $K_{13} \in \mathrm{Syl}_{13}(K)$. By Frattini argument, we deduce that $G = KN_G(K_{13})$. Therefore the normalizer $N_G(K_{13})$ contains an element of order 17, say x. Now $\langle x\rangle K_{13}$ is a subgroup of G of order 13.17, which is abelian. Hence, $13.17 \in \omega(G)$, a contradiction. Finally, we assume $17 \in \pi(K)$ and $13 \notin \pi(K)$. In this case, K is a $\{2, 3, 5, 7, 17\}$-group and we consider a Sylow 17-subgroup K_{17} of K. As before, we see that $G = KN_G(K_{17})$ and by similar argument, we get $13.17 \in \omega(G)$, which is a contradiction. Therefore K is $\{13, 17\}^{'}$-group. In addition since $G \neq K$, G is non-solvable.
Case 2. $13.17 \in \omega(G)$.
In this case according to $deg(13)$ and $deg(17)$, we conclude that that $13.7 \notin \omega(G)$ and $17.7 \notin \omega(G)$. Now, we show that K is a $p^{'}$-group where $p \in \{13, 17\}$. Assume to the contrary and let $p \in \pi(K)$. Then 7 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{p, 7\}$-subgroup of K. It is seen that T is an abelian subgroup of order $p.7$ and so $p.7 \in \omega(K) \subseteq \omega(G)$, a contradiction. Therefore $p \in \pi(K) \subseteq \pi(G) - \{7\}$. Let $K_p \in \mathrm{Syl}_p(K)$. By Frattini argument, $G = KN_G(K_p)$. Therefore, $N_G(K_p)$ contains an element x of order 7. Since G has no element of order $p.7$, $\langle x\rangle$ should act fixed point freely on K_p, that is implying $\langle x\rangle K_p$ is a Frobenius group. By using Lemma 2.3(b), we conclude that $|\langle x\rangle| | (|K_p| - 1)$, which is a contradiction. Therefore K is $\{13, 17\}^{'}$-group. In addition since $G \neq K$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, we have $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^4)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^4)$. Suppose that $m \geq 2$, we get a contradiction by considering two Cases 1 and 2.

Case 1. $2.17 \notin \omega(G)$.
In this case, we claim that 17 does not divide $|S|$. Assume the contrary and let $17 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 17$ which is a contradiction. Now, by Step 1, we observe that $17 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 17 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by 17, so 17 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4 we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore, $t \geq 17$ and so 2^{34} must divide the order of G, which is a contradiction. Therefore $m = 1$ and so $S = P_1$.
Case 2. $2.17 \in \omega(G)$.
In this case, we claim that 13 does not divide $|S|$, assume the contrary and let $13 \mid |S|$. Since $2 \in \pi(P_i)$ for every i, then we implies that $2 \sim 13$ which is a contradiction because $deg(2) = 1$ in $\Gamma(G)$. Now, by using a similar argument, as in Case 1, we can verify that $t \geq 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.
As $S \in \mathfrak{S}_{17}$, 13 and 17 do not divide $|\mathrm{Out}(S)|$ (by Lemma 2.1), so by Step 1 we conclude that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13.17$$

where $2 \leq \alpha_1 \leq 12, 0 \leq \alpha_2 \leq 2$, $0 \leq \alpha_3 \leq 2$ and $0 \leq \alpha_4 \leq 1$. Now by using Table 1, it follows that $S \cong L_3(2^4)$ and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_3(2^4)$.
By Step 2, $L_3(2^4) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_3(2^4))$. As $|G| = |L_3(2^4)|$, we deduce $K = 1$ and $G \cong L_3(2^4)$. □

Proposition 3.2. *If G is a finite group such that $D(G) = D(L_3(2^5))$ and $|G| = |L_3(2^5)|$, then $G \cong L_3(2^5)$.*

Proof. By using Lemma 2.2, we have $D(L_3(2^5)) = (1, 2, 1, 2, 3, 1)$. Since $|G| = |L_3(2^5)| = 2^{15}.3.7.11.31^2.151$ and $D(G) = D(L_3(2^5))$, we conclude that $\Gamma(G)$ has the following forms:

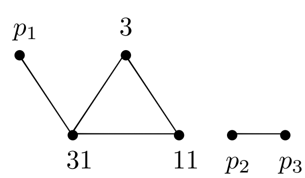

Figure 3-2

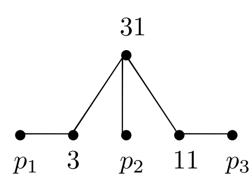

Figure 3-3

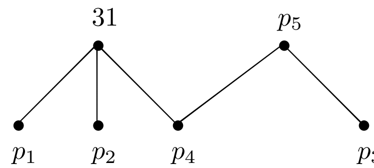

Figure 3-4

where $\{p_1, p_2, p_3\} = \{2, 7, 151\}$ and $\{p_4, p_5\} = \{3, 11\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{11, 151\}'$-group. In particular, G is non-solvable.

We prove this step by considering two cases 1 and 2:

Case 1. $11.151 \notin \omega(G)$.

First, we show that K is a $151'$-group. Assume to the contrary that $|K|$ is divisible by 151. Then 11 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{11, 151\}$-subgroup of K. It is seen that T is an abelian subgroup of order 11.151, hence $11.151 \in \omega(K) \subseteq \omega(G)$, a contradiction. Therefore $151 \in \pi(K) \subseteq \pi(G) - \{11\}$. Let $K_{151} \in \mathrm{Syl}_{151}(K)$. By Frattini argument, $G = KN_G(K_{151})$. Therefore, $N_G(K_{151})$ contains an element x of order 11. Since G has no element of order 11.151, $\langle x\rangle$ should act fixed point freely on K_{151}, which implies that $\langle x\rangle K_{151}$ is a Frobenius group. By using Lemma 2.3(b), we conclude that $|\langle x\rangle|\,|(|K_{151}| - 1)$, which is impossible. Therefore K is a $151^{'}$-group.

Next, we show that K is a $11^{'}$-group. Assume the contrary, $11 \in \pi(K)$. Let $K_{11} \in \mathrm{Syl}_{11}(K)$. By Frattini argument, $G = KN_G(K_{11})$. Therefore, $N_G(K_{11})$ has an element x of order 151. It is easy to see that $\langle x\rangle K_{11}$ is an abelian subgroup of G of order 11.151. Hence $11.151 \in \omega(G)$, which is impossible. Therefore K is a $\{11, 151\}'$-group. In addition since $G \neq K$, G is non-solvable.

Case 2. $11.151 \in \omega(G)$.

In this case, from the structure of degree pattern of G, it is easy to see that $7.11 \notin \omega(G)$ and $7.151 \notin \omega(G)$. Now, we show that K is a p'-group where $p \in \{11, 151\}$. Assume the contrary and let $p \in \pi(K)$. Then 7 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{p, 7\}$-subgroup of K. It is seen that T is an abelian subgroup of K of order $p.7$. Thus, $p.7 \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $p \in \pi(K) \subseteq \pi(G) - \{7\}$. Let $K_p \in \mathrm{Syl}_p(K)$. By Frattini argument, we deduce that $G = KN_G(K_p)$. Therefore, $N_G(K_p)$ contains an element of order 7, say x. Since G has no element of order $p.7$, $\langle\ x\rangle$ should act fixed point freely on K_p, that is implying $\langle x\rangle K_p$ is a Frobenius group. By using Lemma 2.3(b), we conclude that $|\langle x\rangle|\,|(|K_p| - 1)$, which is impossible. Therefore K is a $\{11, 151\}'$-group. In addition since $G \neq K$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^5)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^5)$. Assume to the contrary that $m \geq 2$. We get a contradiction by considering two cases 1 and 2:

Case 1. $2.11 \notin \omega(G)$.

In this case we claim that 11 does not divide $|S|$. Assume the contrary and let $11 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 11$, which is a contradiction. Now, by Step 1 we observe that $11 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 11 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{151}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by 11, so 11 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq 11$ and so 2^{22} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

Case 2. $2.11 \in \omega(G)$.

In this case, we claim that 151 does not divide $|S|$. Assume to the contrary and let $151 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 151$, which is a contradiction because $deg(2) = 1$ in $\Gamma(G)$. Thus, by Step 1 we observe that $151 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. Now, by using a similar argument as in the proof of Case 1, we can show that 2^{302} must divide $|G|$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

As $S \in \mathfrak{S}_{151}$, by using Lemma 2.1 we conclude that 11 and 151 don't divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.7^{\alpha_3}.11.31^{\alpha_4}.151$$

where $2 \leq \alpha_1 \leq 15$, $0 \leq \alpha_2 \leq 1$, $0 \leq \alpha_3 \leq 1$ and $0 \leq \alpha_4 \leq 2$. Now, by using Table 1 it follows that $S \cong L_3(2^5)$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_3(2^5)$.

By Step 2, $L_3(2^5) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_3(2^5))$. Since $|G| = |L_3(2^5)|$, we deduce $K = 1$, so $G \cong L_3(2^5)$ and the proof is complete. □

Proposition 3.3. *If G is a finite group such that $D(G) = D(L_3(2^6))$ and $|G| = |L_3(2^6)|$, then $G \cong L_3(2^6)$.*

Proof. By using lemma 2.2, we conclude that $D(G) = D(L_3(2^6)) = (2, 4, 3, 4, 3, 1, 1)$. Since $|G| = |L_3(2^6)| = 2^{18}.3^4.5.7^2.13.19.73$ and $D(G) = D(L_3(2^6))$, then we have the following forms for $\Gamma(G)$:

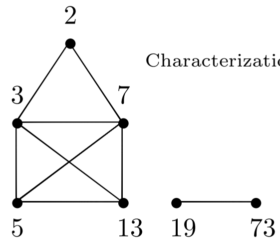

Figure 3-5

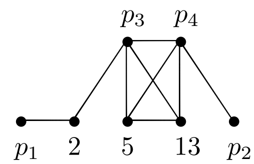

Figure 3-6

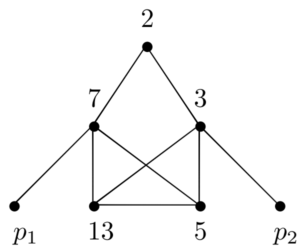

Figure 3-7

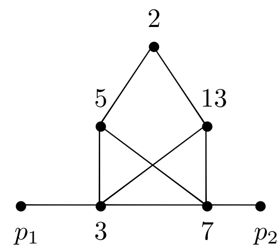

Figure 3-8

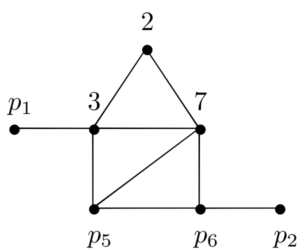

Figure 3-9

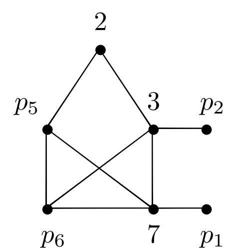

Figure 3-10

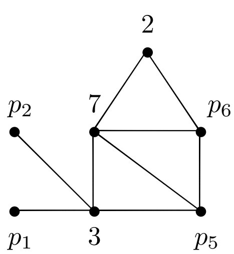

Figure 3-11

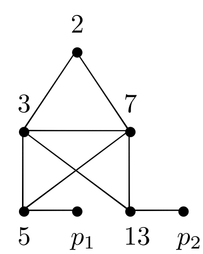

Figure 3-12

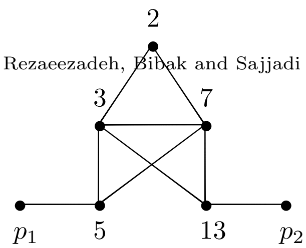

Figure 3-13

where $\{p_1, p_2\} = \{19, 73\}$, $\{p_3, p_4\} = \{3, 7\}$ and $\{p_5, p_6\} = \{5, 13\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{13, 19, 73\}^{'}$-group. In particular, G is non-solvable.

We prove this step by considering two Cases 1 and 2:

Case 1. $13.73 \notin \omega(G)$.

First, we show that K is a $\{13, 73\}'$-group. Assume that $\{13, 73\} \subseteq \pi(K)$. Then K has a Hall $\{13, 73\}$-subgroup H. It is easy to see that H is an abelian subgroup of order 13.73, which implies that $13.73 \in \omega(K) \subseteq \omega(G)$, a contradiction.

Now, we assume $13 \in \pi(K)$ and $73 \notin \pi(K)$. Then K is a $\{2, 3, 5, 7, 13\}$-group. Let $K_{13} \in \mathrm{Syl}_{13}(K)$. By Frattini argument, we deduce that $G = KN_G(K_{13})$. Therefore the normalizer $N_G(K_{13})$ contains an element of order 73, say x. Now $\langle x\rangle K_{13}$ is a subgroup of G of order 13.73, which is abelian. Hence, $13.73 \in \omega(G)$, a contradiction. Next, we assume $73 \in \pi(K)$ and $13 \notin \pi(K)$. In this case, K is a $\{2, 3, 5, 7, 73\}$-group and we consider a Sylow 73-subgroup K_{73} of K. As before, we see that $G = KN_G(K_{73})$ and by similar argument, we get $13.73 \in \omega(G)$, which is a contradiction.

Finally, we show that K is a $19^{'}$-group. Assume the contrary and let $19 \in \pi(K)$. We claim that p does not divide the order of K, where $p \in \{13, 73\}$. Otherwise, we may suppose that T is a Hall $\{p, 19\}$-subgroup of K. It is seen that T is an abelian subgroup of order $p.19$. Thus, $p.19 \in \omega(G)$, a contradiction because from the structure of degree pattern of G, it is easy to see that if $19.13 \in \omega(G)$, then $19.73 \notin \omega(G)$. Also, if $19.73 \in \omega(G)$, then $13.73 \notin \omega(G)$. Thus, $19 \in \pi(K) \subseteq \pi(G) - \{p\}$. Let $K_{19} \in \mathrm{Syl}_{19}(K)$. By Frattini argument, $G = KN_G(K_{19})$. Therefore, $N_G(K_{19})$ contains an element x of order p. Since G has no element of order $19.p$, $\langle x\rangle$ should act fixed point freely on K_{19}, that is implying $\langle x\rangle K_{19}$ is a Frobenius group. By Lemma 2.3(b), $|\langle x\rangle|\,|(|K_{19}| - 1)$. It follows that $19|p - 1$, which is a contradiction. Therefore K is a $\{13, 19, 73\}^{'}$-group. In addition since $G \neq K$, G is non-solvable.

Case 2. $13.73 \in \omega(G)$.

First, we show that K is a $\{19, 73\}'$-group. To prove this, assume first that $\{19, 73\} \subseteq \pi(K)$. Then K has a Hall $\{19, 73\}$-subgroup H. It is easy to see that H is an abelian subgroup of order 19.73, which implies that $19.73 \in \omega(K) \subseteq \omega(G)$, a contradiction because $deg(73) = 1$ in $\Gamma(G)$.

Now, we assume $19 \in \pi(K)$ and $73 \notin \pi(K)$. Then K is a $\{2, 3, 5, 7, 13, 19\}$-group. Let $K_{19} \in \mathrm{Syl}_{19}(K)$. By Frattini argument, we deduce that $G = KN_G(K_{19})$. Therefore the normalizer $N_G(K_{19})$ contains an element of order 73, say x. Now $\langle x\rangle K_{19}$ is a subgroup of G of order 19.73, which is abelian. Hence, $19.73 \in \omega(G)$, a contradiction.
Next, we assume $73 \in \pi(K)$ and $19 \notin \pi(K)$. In this case, K is a $\{2, 3, 5, 7, 13, 73\}$-group and we consider a Sylow 73-subgroup K_{73} of K. As before, we see that $G = KN_G(K_{73})$ and by similar argument, we get $19.73 \in \omega(G)$, which is a contradiction.
Finally, we show that K is a $13'$-group. Assume the contrary and let $13 \in \pi(K)$. By the structure of degree pattern of G, it is easy to see that $13.19 \notin \omega(G)$. Now, we claim that 19 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{13, 19\}$-subgroup of K. It is seen that T is an abelian subgroup of K of order 13.19. Thus, $13.19 \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $13 \in \pi(K) \subseteq \pi(G) - \{19\}$. Let $K_{13} \in \mathrm{Syl}_{13}(K)$. By Frattini argument, we deduce that $G = KN_G(K_{13})$. Therefore the normalizer $N_G(K_{13})$ contains an element of order 19, say x. Since G has no element of order 13.19, $\langle x\rangle$ should act fixed point freely on K_{13}, that is implying $\langle x\rangle K_{13}$ is a Frobenius group. By using Lemma 2.3(b), we conclude that $|\langle x\rangle| \mid (|K_{13}| - 1)$, which is impossible. Therefore K is a $\{13, 19, 73\}'$-group. In addition since $G \neq K$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, we have $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^6)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^6)$. Assume to the contrary that $m \geq 2$. We get a contradiction by considering two cases 1 and 2:
Case 1. $2.19 \notin \omega(G)$.
In this case, we claim that 19 does not divide $|S|$. Assume the contrary and let $19 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 19$, which is a contradiction. Now, by Step 1, we observe that $19 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 19 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{73}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by 19, so 19 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4 we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq 19$ and so 2^{38} must divide the order of G, which is a contradiction. Therefore $m = 1$, and then $S = P_1$.
Case 2. $2.19 \in \omega(G)$.
In this case according to degree pattern of G we have $2.73 \notin \omega(G)$. Now, we

claim that 73 does not divide $|S|$. Assume the contrary and let $73 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 73$, which is a contradiction. Thus, by Step 1 we observe that $73 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. Now, by using a similar argument as in the proof of Case 1, we can show that 2^{146} must divide $|G|$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.
As $S \in \mathfrak{S}_{73}$, by using Lemma 2.1 we conclude that $13, 19$ and 73 do not divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13.19.73$$

where $2 \le \alpha_1 \le 18$, $0 \le \alpha_2 \le 4$, $0 \le \alpha_3 \le 1$ and $0 \le \alpha_4 \le 2$. Now, by using Table 1 it follows that $S \cong L_3(2^6)$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_3(2^6)$.
By Step 2, $L_3(2^6) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_3(2^6))$. As $|G| = |L_3(2^6)|$, we deduce $K = 1$, so $G \cong L_3(2^6)$ and the proof is complete. $\square$

Proposition 3.4. *If G is a finite group such that $D(G) = D(L_3(2^7))$ and $|G| = |L_3(2^7)|$, then $G \cong L_3(2^7)$.*

Proof. By using Lemma 2.2, we have $D(L_3(2^7)) = (1, 2, 1, 2, 3, 1)$. As $|G| = |L_3(2^7)| = 2^{21}.3.7^2$.
$43.127^2.337$ and $D(G) = D(L_3(2^7))$, then $\Gamma(G)$ has the following forms:

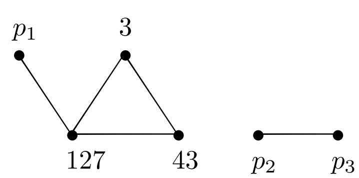

Figure 3-14

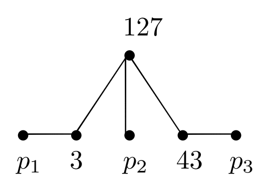

Figure 3-15

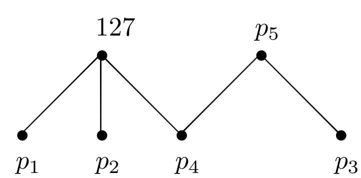

Figure 3-16

where $\{p_1, p_2, p_3\} = \{2, 7, 337\}$ and $\{p_4, p_5\} = \{3, 43\}$.

Let K be the maximal normal solvable subgroup of G, then by using a similar argument as in the proof of proposition 3.2, we can verify that K is a $\{43, 337\}'$-group and the factor group $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^7)$. Since $|G| = |L_3(2^7)|$, we deduce $K = 1$, so $G \cong L_3(2^7)$ and the proof is complete. $\square$

Proposition 3.5. *If G is a finite group such that $D(G) = D(L_3(2^8))$ and $|G| = |L_3(2^8)|$, then $G \cong L_3(2^8)$.*

Proof. By using Lemma 2.2, we conclude that $D(L_3(2^8)) = (2, 2, 4, 2, 2, 4, 2, 2)$. Since $|G| = |L_3(2^8)| = 2^{24}.3^2.5^2.7.13.17^2.241.257$ and $D(G) = D(L_3(2^8))$, then we have the following forms for $\Gamma(G)$:

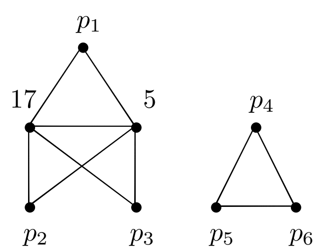

Figure 3-17

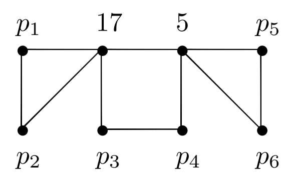

Figure 3-18

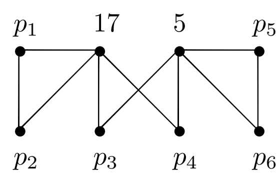

Figure 3-19

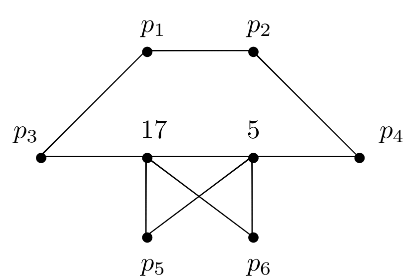

Figure 3-20

where $\{p_1, p_2, p_3, p_4, p_5, p_6\} = \{2, 3, 7, 13, 241, 257\}$.
We prove this proposition in two parts A and B:
Part A. $\Gamma(G)$ is a disconnected graph.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{13, 17, 257\}^{'}$-group. In particular, G is non-solvable.
For proving Step 1, we consider separate cases :
Case 1. $257 \in \{p_1, p_2, p_3\}$. First, we show that K is a $257'$-group. Without loss of generality, we can suppose that $p_1 = 257$. Assume to the contrary that $|K|$ is divisible by 257 and let x be an element of K of order 257. According to $\Gamma(G)$, $C_G(x)$ is a $\{5, 17, 257\}$-group. Since $\frac{N_G(\langle x\rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x\rangle) \cong \mathbb{Z}_{256}$, $\pi(N_G(\langle x\rangle)) \subseteq \{2, 5, 17, 257\}$. By Frattini argument, $G = KN_G(\langle x\rangle)$, which implies that $\{3, 7, 13, 241, 257\} \subseteq \pi(K)$. Since K is solvable, it follows that K has a Hall $\{13, 257\}$-subgroup H. It is seen that H is abelian subgroup of G of order 13.257. Thus $13 \sim 257$ in $\Gamma(G)$, which is a contradiction. Therefore K is a $257'$-group.

Now, we show that K is a $17'$-group. Assume to the contrary and let $17 \in \pi(K)$. We know that one primes of $\{p_4, p_5, p_6\}$ is unequal to 2 and 3, we set it r. So r does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{r, 17\}$-subgroup of K. It is easy to see that T is a nilpotent subgroup of order $r.17^i$ for $i = 1$ or 2. Thus $r.17 \in \omega(K) \subseteq \omega(G)$, a contradiction. Hence, $17 \in \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$, by Frattini argument $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ has an element x of order r. Since G has no element of order $r.17$, $\langle x\rangle$ should act fixed point freely on K_{17}, implying $\langle x\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle x\rangle|||K_{17}| - 1$. It follows that $r||17^i| - 1$ for $i = 1$ or 2, which is a contradiction. Therefore K is a $17'$-group. Next, we prove that K is a $13'$-group. Assume to the contrary that $|K|$ is divisible by 13. Let $K_{13} \in \mathrm{Syl}_{13}(K)$, by Frattini argument $G = KN_G(K_{13})$. Therefore, $N_G(K_{13})$ contains an element of order 257, say x. It is easy to see that $\langle x\rangle K_{13}$ is an abelian subgroup of G of order 257.13. Thus $257.13 \in \omega(G)$, which is impossible. Therefore K is a $\{13, 17, 257\}'$-group. In addition since $G \neq K$, G is non-solvable.
Case 2. $257 \in \{p_4, p_5, p_6\}$. Without loss of generality, we can suppose that $p_4 = 257$. Now, we consider this part in different cases:
a. $13 \in \{p_1, p_2, p_3\}$. Without loss of generality, we can suppose that $p_1 = 13$. First, we show that K is a $257'$-group. Assume to the contrary that there exists an element x of K of order 257. By the structure of $\Gamma(G)$, we see that $C_G(x)$ is a $\{257, p_5, p_6\}$-group. Therefore $\pi(N_G(\langle x\rangle)) \subseteq \{2, 257, p_5, p_6\}$. Now, from Frattini argument, we deduce that $G = KN_G(\langle x\rangle)$, which implies that $\{5, 17, 257\} \subseteq \pi(K)$. Let T be a Hall $\{17, 257\}$-subgroup of K. It is easy to see that T is a nilpotent subgroup of G of order 257.17^i for $i = 1$ or 2. Hence $17 \sim 257$ in $\Gamma(G)$, which is impossible. Therefore K is a $257'$-group.

Next, we show that K is a $p^{'}$-group, where $p \in \{13, 17\}$. Assume the contrary and let $K_p \in \mathrm{Syl}_p(K)$. By Frattini argument $G = KN_G(K_p)$, hence $N_G(K_p)$ contains an element of order 257, say x. Since G has no element of order $p.257$, $\langle x \rangle$ should act fixed point freely on K_p, which implies that $\langle x \rangle K_p$ is a Frobenius group. By using Lemma 2.3(b) it follows that $|\langle x \rangle| | (|K_p| - 1)$, which is a contradiction. Therefore K is a $\{13, 17\}'$-group.

b. $13 \in \{p_5, p_6\}$. Without loss of generality, we can suppose that $p_5 = 13$. By using a similar argument as in the proof of Part **A**, we can verify that K is a $17^{'}$-group. Now, we show that K is a $p^{'}$-group, where $p \in \{13, 257\}$. Assume the contrary and let $K_p \in \mathrm{Syl}_p(K)$. By Frattini argument $G = KN_G(K_p)$, hence $N_G(K_p)$ contains an element of order 257, say x. It is easy to see that $\langle x \rangle K_p$ is an abelian subgroup of G of order $p.17$. Thus $p.17 \in \omega(G)$, which is a contradiction. Therefore K is a $\{13, 17, 257\}'$-group. In addition since $G \neq K$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^8)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^8)$. Assume to the contrary that $m \geq 2$. We get a contradiction by considering tow case 1 and case 2.

Case 1. $2 \in \{p_1, p_2, p_3\}$. Without loss of generality, we can assume that $p_1 = 2$. We claim that 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 13$, which is a contradiction. Now, by Step 1 we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times \ldots \times S_r$. Therefore for some j, 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{257}$, Lemma 2.1 follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13, so 13 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4 we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore $m = 1$, and then $S = P_1$.

Case 2. $2 \in \{p_4, p_5, p_6\}$. Without loss of generality, we can assume that $p_4 = 2$. By using similar argument, as in the proof of case 1 and replace 17 with 13 we conclude that 2^{34} must divide the order of G, which is impossible. Therefore $m = 1$ and then $S = P_1$.

As $S \in \mathfrak{S}_{257}$, by Lemma 2.1 we conclude that 13,17 and 257 don't divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13.17.241^{\alpha_5}.257$$

where $2 \leq \alpha_1 \leq 24, 0 \leq \alpha_2 \leq 2$, $0 \leq \alpha_3 \leq 2$, $0 \leq \alpha_4 \leq 1$ and $0 \leq \alpha_5 \leq 1$. Now, using Table 1 it follows that $S \cong L_3(2^8)$, and this completes the proof of Step

2.
Part B. $\Gamma(G)$ is a connected graph.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{241, 257\}^{'}$-group. In particular, G is non-solvable.
For proving this Step, we consider separate cases :
Case 1. $241 \sim 257$ and $\{p_1, p_2\} \neq \{241, 257\}$ in Figure 3-20.
In this case, we show that K is a p'-group where $p \in \{241, 257\}$. Assume the contrary and let $p \in \pi(K)$. Then 13 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{p, 13\}$-subgroup of K. It is seen that T is a nilpotent subgroup of K of order $p.13$. Thus, $p.13 \in \omega(K) \subseteq \omega(G)$, a contradiction. Therefore, $p \in \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_p \in \mathrm{Syl}_p(K)$. By Frattini argument, we deduce that $G = KN_G(K_p)$. Therefore the normalizer $N_G(K_p)$ contains an element of order 13, say x. Since G has no element of order $p.13$, $\langle x\rangle$ should act fixed point freely on K_p, implying $\langle x\rangle K_p$ is a Frobenius group. By Lemma 2.3(b), $|\langle x\rangle|||K_p| - 1$, which is a contradiction. Therefore K is a $\{241, 257\}'$-group.
Case 2. $\{p_1, p_2\} = \{241, 257\}$ in Figure 3-20.
In this case, we show that K is a p'-group where $p \in \{241, 257\}$. Assume the contrary and let $p \in \pi(K)$. Now, by using a similar argument as in the proof of case 1 and considering 17 instead of 13, we get a contradiction. Therefore K is a $\{241, 257\}'$-group.
Case 3. $241 \nsim 257$.
First assume that $\{241, 257\} \subseteq \pi(K)$. Then K has a Hall $\{241, 257\}$-subgroup H. It is easy to see that H is an abelian subgroup of order 241.257, which implies that $241.257 \in \omega(K) \subseteq \omega(G)$, a contradiction.
Next, we assume $241 \in \pi(K)$ and $257 \notin \pi(K)$. Then K is a $\{2, 3, 5, 7, 13, 17, 241\}$-group. Let $K_{241} \in \mathrm{Syl}_{241}(K)$. By Frattini argument, we deduce that $G = KN_G(K_{241})$. Therefore the normalizer $N_G(K_{241})$ contains an element of order 257, say x. Now $\langle x\rangle K_{241}$ is a subgroup of G of order 241.257, which is abelian. Hence, $241.257 \in \omega(G)$, a contradiction.
Finally, we assume $257 \in \pi(K)$ and $241 \notin \pi(K)$. In this case, K is a $\{2, 3, 5, 7, 13, 17, 257\}$-group and we consider a Sylow 257-subgroup K_{257} of K. As before, we see that $G = KN_G(K_{257})$ and by a similar argument, we get $241.257 \in \omega(G)$, which is a contradiction. Therefore K is a $\{241, 257\}^{'}$-group. In addition since $G \neq K$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^8)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^8)$.
Assume to the contrary that $m \geq 2$. By the structure of $\Gamma(G)$, we know

that there exists one prime number p in $\{241, 257\}$ such that $p \nsim 2$. Now, we claim that p does not divide $|S|$. Assume the contrary and let $p \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim p$, which is a contradiction. Now, by Step 1, we observe that $p \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, p divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{257}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by p, so p does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4 we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq p$ and so 2^{2p} must divide the order of G, which is a contradiction because $p \in \{241, 257\}$. Therefore $m = 1$ and $S = P_1$.
As $S \in \mathfrak{S}_{257}$, by Lemma 2.1 we conclude that 241 and 257 don't divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13^{\alpha_5}.17^{\alpha_6}.241.257$$

where $2 \leq \alpha_1 \leq 24, 0 \leq \alpha_2 \leq 2$, $0 \leq \alpha_3 \leq 2$, $0 \leq \alpha_4 \leq 1$, $0 \leq \alpha_5 \leq 1$ and $0 \leq \alpha_6 \leq 1$. Now, using Table 1 it follows that $S \cong L_3(2^8)$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_3(2^8)$.
By Step 2 in Parts A and B, we conclude that $L_3(2^8) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_3(2^8))$. As $|G| = |L_3(2^8)|$, we deduce that $K = 1$, so $G \cong L_3(2^8)$ and the proof is complete. $\square$

Proposition 3.6. *If G is a finite group such that $D(G) = D(L_3(2^{10}))$ and $|G| = |L_3(2^{10})|$, then $G \cong L_3(2^{10})$.*

Proof. By using Lemma 2.2, we conclude that $D(L_3(2^{10})) = (2, 2, 3, 2, 5, 5, 3, 2, 2)$. Since $|G| = |L_3(2^{10})| = 2^{45}.3^4.5.7.11^3.31^3.41.151.331$ and $D(G) = D(L_3(2^{10}))$, the prime graph of G has several possibilities shown in the following figures:

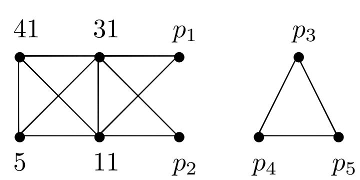

Figure 3-21

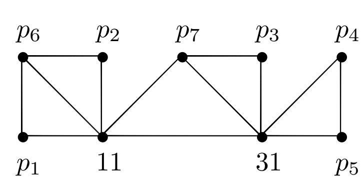

Figure 3-22

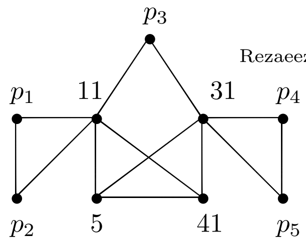

Figure 3-23

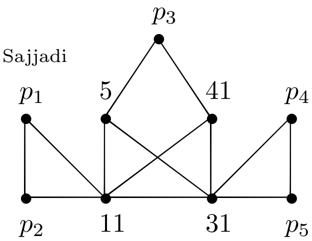

Figure 3-24

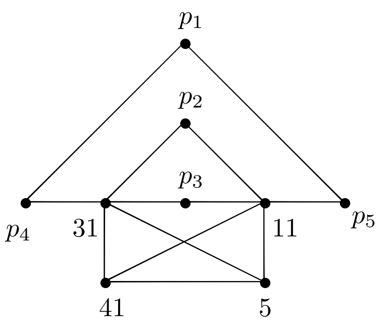

Figure 3-25

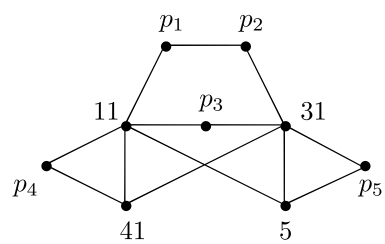

Figure 3-26

where $\{p_1, p_2, p_3, p_4, p_5\} = \{2, 3, 7, 151, 331\}$ and $\{p_6, p_7\} = \{5, 41\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{41, 151, 331\}^{'}$-group. In particular, G is non-solvable.

We consider this step in two parts A and B:

Part A. Consider Figures 3-21, 3-23 and 3-25.

First we show that K is a $p^{'}$-group, where $p \in \{151, 331\}$. Assume the contrary and let $p \in \pi(K)$. Then $41 \nmid |K|$, otherwise we may suppose that H is a Hall $\{41, p\}$-subgroup of K. It is easy to see that H is an abelian subgroup of G of order $41.p$. Hence $41 \sim p$, which is a contradiction. Therefore $p \in \pi(K) \subseteq \pi(G) - \{41\}$. Suppose that $K_p \in \mathrm{Syl}_p(K)$, then by Frattini argument $G = KN_G(K_p)$. Therefore $41 \in \pi(N_G(K_p))$. If x is an element of $N_G(K_p)$ of order 41, then $\langle x \rangle$ should act fixed point freely on K_p, since G has no element of order $41.p$. Hence by Lemma 2.3(b) we obtain that $|\langle x \rangle| \,|\, |K_p| - 1$, that is impossible. Therefore K is a $p^{'}$-group.

Next, we show that K is a $41^{'}$-group. Assume the contrary and let x be an element of order 41. According to $\Gamma(G)$, $C_G(x)$ is a $\{5, 11, 31, 41\}$-group. Since $\frac{N_G(\langle x \rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x \rangle) \cong \mathbb{Z}_{40}$, $\pi(N_G(\langle x \rangle)) \subseteq \{2, 5, 11, 31, 41\}$. By Frattini argument, $G = KN_G(\langle x \rangle)$, so 331 must divide the order of K, which is a contradiction. Therefore K is a $\{41, 151, 331\}^{'}$-group and this completes the proof of this part.

Part B. Consider Figures 3-22, 3-24 and 3-26.

Now, we consider this part in different cases:

Case 1. $151 \sim 331$. Then the proof is similar to Part A.

Case 2. $151 \nsim 331$. First, we show that K is a $\{151,331\}^{'}$-group. Assume that $\{151,331\} \subseteq \pi(K)$. Then K has a Hall $\{151,331\}$-subgroup H. It is easy to see that H is an abelian subgroup of order 151.331, which implies that $151.331 \in \omega(K) \subseteq \omega(G)$, a contradiction.
Now, we assume $151 \in \pi(K)$ and $331 \notin \pi(K)$. Then K is a $\{2,3,5,7,11,31,41,151\}$-group. Let $K_{151} \in \mathrm{Syl}_{151}(K)$. By Frattini argument, we deduce that $G = KN_G(K_{151})$. Therefore the normalizer $N_G(K_{151})$ contains an element of order 331, say x. Now $\langle x\rangle K_{151}$ is a subgroup of G of order 151.331, which is abelian. Hence, $151.331 \in \omega(G)$, a contradiction.
Next, we assume $331 \in \pi(K)$ and $151 \notin \pi(K)$. In this case, K is a $\{2,3,5,7,11,31,41,331\}$-group and we consider a Sylow 331-subgroup K_{331} of K. As before, we see that $G = KN_G(K_{331})$ and by a similar argument, we get $151.331 \in \omega(G)$, which is a contradiction. Therefore K is $\{151,331\}^{'}$-group.
Finally, we show that K is a $41^{'}$-group. Assume the contrary and let $41 \in \pi(K)$. By the structure of $\Gamma(G)$, we know that there exists one prime number p in $\{7,151,331\}$ such that $p \nsim 41$. Now, we claim that p does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{41,p\}$-subgroup of K. It is seen that T is an abelian subgroup of K of order $41,p$. Thus, $41.p \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $41 \in \pi(K) \subseteq \pi(G) - \{p\}$. Let $K_{41} \in \mathrm{Syl}_{41}(K)$. By Frattini argument, $G = KN_G(K_{41})$. Therefore, $N_G(K_{41})$ contains an element x of order p. Since G has no element of order $p.41$, $\langle\, x\rangle$ should act fixed point freely on K_{41}, which implies that $\langle x\rangle K_{41}$ is a Frobenius group. By using Lemma 2.3(b), we conclude that $|\langle x\rangle||(|K_{41}|-1)$, which is impossible. Therefore K is a $\{41,151,331\}^{'}$-group. In addition since $G \neq K$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^{10})$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^{10})$. Assume to the contrary that $m \geq 2$. We get a contradiction by considering three cases 1 and 2.

Case 1. Consider Figures 3-21, 3-23 and 3-25.
In this case, we claim that 41 does not divide $|S|$. Assume the contrary and let $41 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 41$, which is a contradiction. Now, by Step 1 we observe that $41 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times \ldots \times S_r$. Therefore, for some j, 41 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{331}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by 41, so 41 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq 41$ and so 2^{82} must divide the order of G, which is a contradiction. Therefore $m = 1$,

and then $S = P_1$.
Case 2. Consider Figures 3-22, 3-24 and 3-26.
In this case, by the structure of $\Gamma(G)$, we know that there exists one prime number p in $\{151, 331\}$ such that $p \nsim 2$. Now, we claim that p does not divide $|S|$. Assume the contrary and let $p \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim p$, which is a contradiction. Hence, by Step 1 we observe that $p \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. Now, by using a similar argument as in the proof of Case 1, we can verify that 2^{2p} must divide $|G|$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.
As $S \in \mathfrak{S}_{331}$, by using Lemma 2.1 we conclude that 41,151 and 331 don't divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.11^{\alpha_5}.31^{\alpha_6}.41.151.331$$

where $2 \leq \alpha_1 \leq 45$, $0 \leq \alpha_2 \leq 4$, $0 \leq \alpha_3 \leq 1$, $0 \leq \alpha_4 \leq 1$, $0 \leq \alpha_5 \leq 1$ and $0 \leq \alpha_6 \leq 1$. Now, by using Table 1 it follows that $S \cong L_3(2^{10})$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_3(2^{10})$.
By Step 2, $L_3(2^{10}) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_3(2^{10}))$. As $|G| = |L_3(2^{10})|$, we deduce $K = 1$, so $G \cong L_3(2^{10})$ and the proof is complete. $\square$

Proposition 3.7. *If G is a finite group such that $D(G) = D(L_3(2^{12}))$ and $|G| = |L_3(2^{12})|$, then $G \cong L_3(2^{12})$.*

Proof. By using Lemma 2.2, we conclude that $D(L_3(2^{12})) = (4, 6, 6, 6, 6, 5, 3, 3, 3, 3, 5)$. Since $|G| = |L_3(2^{12})| = 2^{36}.3^5.5^2.7^2.13^2.17.19.37.73.109.241$ and $D(G) = D(L_3(2^{12}))$, then we have the following forms for $\Gamma(G)$:

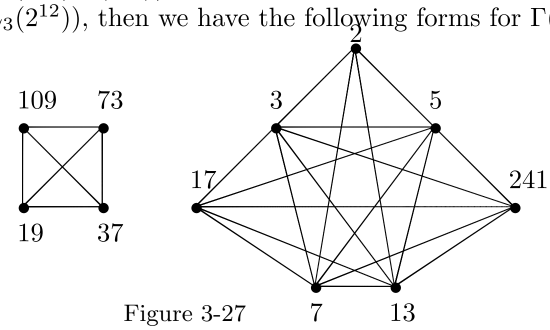

Figure 3-27

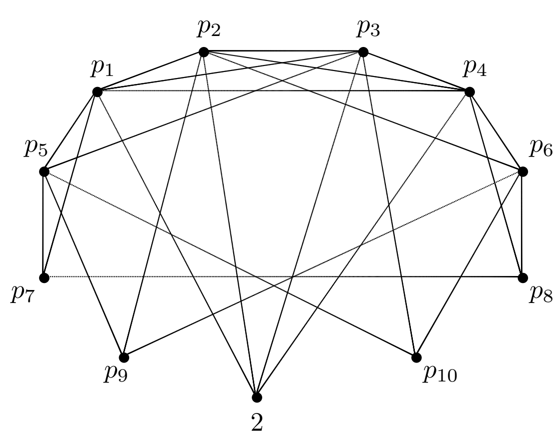

Figure 3-28

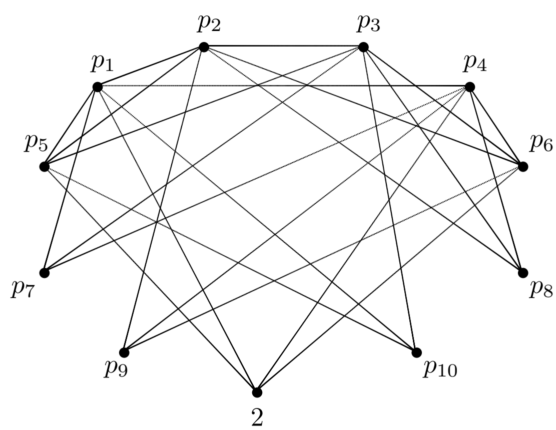

Figure 3-29

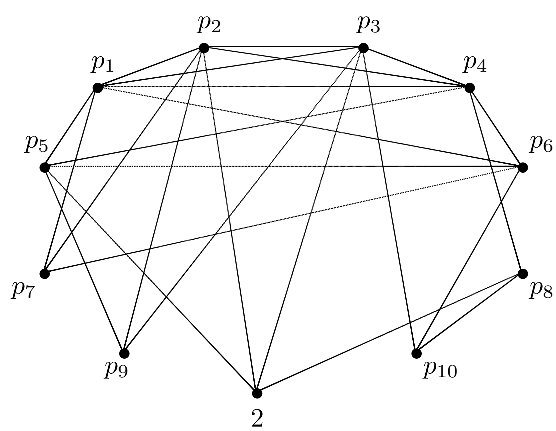

Figure 3-30

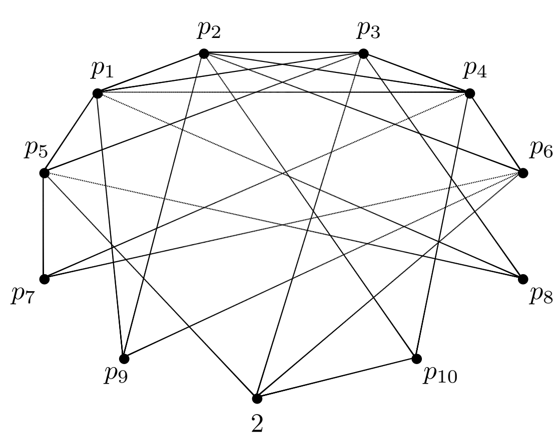

Figure 3-31

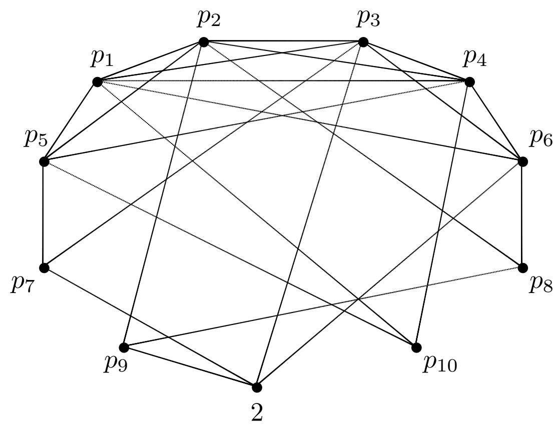

Figure 3-32

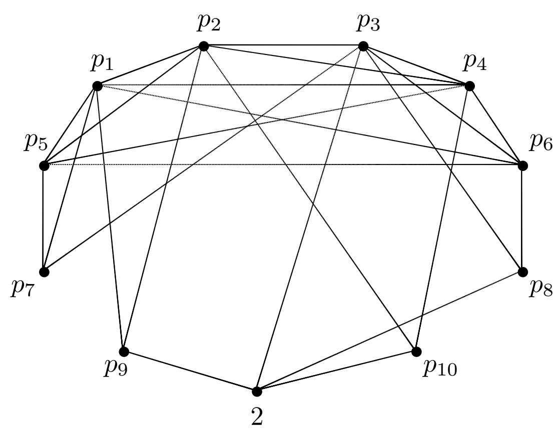

Figure 3-33

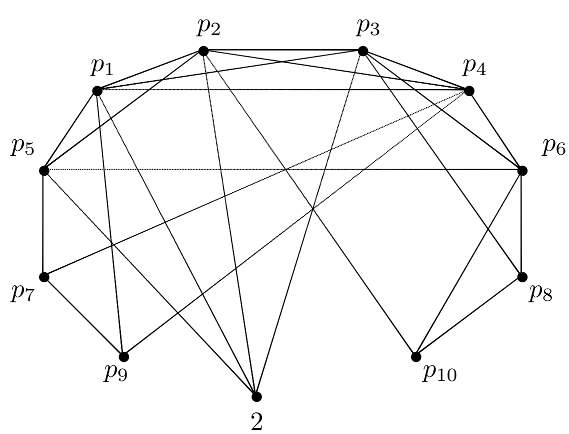

Figure 3-34

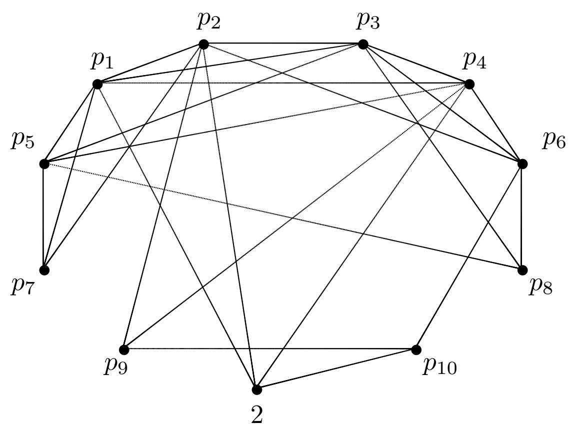

Figure 3-35

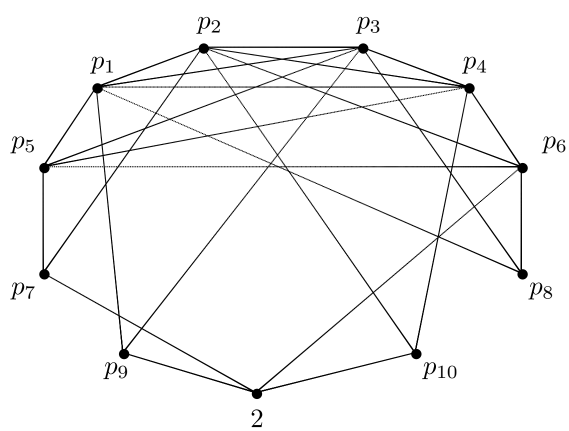

Figure 3-36

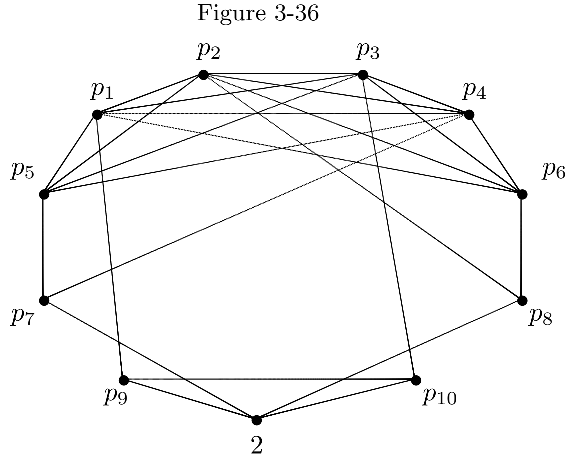

Figure 3-37

where $\{p_1, p_2, p_2, p_2\} = \{3, 5, 7, 13\}$, $\{p_5, p_6\} = \{17, 241\}$ and $\{p_7, p_8, p_9, p_{10}\} = \{19, 37, 73, 109\}$.
we prove this proposition in two parts A and B:
Part A. Consider Figure 3-22, 3-24, 3-25, 3-26, 3-27, 3-29 and 3-31.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{19, 37, 73, 109, 241\}^{'}$-group. In particular, G is non-solvable.
We consider this step in two cases:
Case 1. Consider Figure 3-27.
First we show that K is a $241^{'}$-group. Assume the contrary and let $241 \in \pi(K)$. We claim that 19 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{19, 241\}$-subgroup of K. It is easy to see that T is an abelian subgroup of order 19.241 and so $19.241 \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $241 \in \pi(K) \subsetneq \pi(G) - \{19\}$. Let $K_{241} \in \mathrm{Syl}_{241}(K)$. By Frattini argument, $G = KN_G(K_{241})$. Therefore, $N_G(K_{241})$ contains an element x of order 19. Since G has no element of order 19.241, $\langle x\rangle$ should act fixed point freely on K_{241}, which implies that $\langle x\rangle K_{241}$ is a Frobenius group. By using Lemma 2.3(b) it follows that $|\langle x\rangle| | (|K_{241}| - 1)$, which is a contradiction.
Next, we show that K is a $p^{'}$-group, where $p \in \{19, 37, 73, 109\}$. Assume the contrary and let x be an element of order p. According to $\Gamma(G)$, $C_G(x)$ is a $\{19, 37, 109, 73\}$-group. Since $\frac{N_G(\langle x\rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x\rangle) \cong \mathbb{Z}_{p-1}$, $\pi(N_G(\langle x\rangle)) \subseteq \{2, 3, 19, 37, 109, 73\}$. By Frattini argument, $G = KN_G(\langle x\rangle)$, so 241 must divide the order of K, which is impossible. Therefore K is a $\{19, 37, 73, 109, 241\}'$-group.
Case 2. Consider Figures 3-29, 3-30, 3-31, 3-32, 3-34 and 3-36.
First, we show that K is a $p^{'}$-group, where $p \in \{p_8, p_{10}\}$. Assume the contrary and let $p \in \pi(K)$. Then p_7 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{p, p_7\}$-subgroup of K. It is easy to see that T is an abelian subgroup of order $p.p_7$ and so $p.p_7 \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $p \in \pi(K) \subseteq \pi(G) - \{p_7\}$. Let $K_p \in \mathrm{Syl}_p(K)$. By Frattini argument, $G = KN_G(K_p)$. Therefore, $N_G(K_p)$ contains an element x of order p_7. Since G has no element of order $p.p_7$, $\langle x\rangle$ should act fixed point freely on K_p, which implies that $\langle x\rangle K_p$ is a Frobenius group. By using Lemma 2.3(b) it follows that $|\langle x\rangle| | (|K_p| - 1)$, which is a contradiction because $\{p_7, p\} \subseteq \{19, 37, 73, 109\}$. Therefore K is a $\{p_8, p_{10}\}'$-group.
Now, by using similar argument as above, we conclude that K is a $\{p_7, p_9\}'$-group, because p_{10} does not divide the order of K.
Next, we prove that K is a $241^{'}$-group. We assume to the contrary that $241 | |K|$ and $K_{241} \in \mathrm{Syl}_{241}(K)$. Then by Frattini argument $G = KN_G(K_{241})$. In Figures 3-32, 3-34 and 3-36, we set $r = p_9$ and in Figures 3-29 and 3-30, we

set $r = p_8$ and in Figure 3-31, we set $r = p_{10}$. Since r does not divide $|K|$, we conclude that r must divide the order of $N_G(K_{241})$. Let x be an element of $N_G(K_{241})$ of order r. As $\langle x\rangle$ normalizes K_{241}, then $\langle x\rangle K_{241}$ is a subgroup of G, which is abelian. Thus $r \sim 241$, which is impossible. Therefore K is a $\{19, 37, 73, 109, 241\}'$-group. In addition since $G \neq K$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^{12})$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^{12})$. Assume to the contrary that $m \geq 2$. We get a contradiction by considering two cases:

Case 1. Consider Figure 3-27.

In this case, we claim that 19 does not divide $|S|$. Assume to the contrary and let $19 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 19$, which is a contradiction. Now, by Step 1 we observe that $19 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 19 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{241}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by 19, so 19 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq 19$ and so 2^{38} must divide the order of G, which is a contradiction. Therefore $m = 1$, and then $S = P_1$.

Case 2. Consider Figures 3-29, 3-30, 3-31, 3-32, 3-34 and 3-36.

In Figures 3-29, 3-31, 3-32, 3-34 and 3-36, we set $r = p_8$ and in Figure 3-30, we set $r = p_7$. Now, we claim that r does not divide $|S|$. Assume to the contrary and let $r \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim r$, which is a contradiction. Thus, by Step 1 we observe that $r \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. Now, by using a similar argument as in the proof of Part A, we can show that 2^{2r} must divide $|G|$, which is a contradiction, because $r \in \{19, 37, 73, 109\}$. Therefore $m = 1$ and $S = P_1$.

As $S \in \mathfrak{S}_{241}$, by Lemma 2.1 we conclude that 19,37,73,109 and 241 don't divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13^{\alpha_5}.17^{\alpha_6}.19.37.73.109.241$$

where $2 \leq \alpha_1 \leq 36$, $0 \leq \alpha_2 \leq 5$, $0 \leq \alpha_3 \leq 2$, $0 \leq \alpha_4 \leq 2$, $0 \leq \alpha_5 \leq 2$ and $0 \leq \alpha_6 \leq 1$. Now, by using Table 1 it follows that $S \cong L_3(2^{12})$, and this completes the proof of Step 2.

Part B. Consider Figure 3-28, 3-23, 3-35 and 3-37.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $241'$-group. In particular, G is non-solvable.

First, we show that K is a p_6'-group. Assume to the contrary and let $p_6 \in \pi(K)$. Then p_7 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{p_6, p_7\}$-subgroup of K. It is easy to see that T is an abelian

subgroup of order $p_6.p_7$ and so $p_6.p_7 \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $p_6 \in \pi(K) \subseteq \pi(G) - \{p_7\}$. Let $K_{p_6} \in \mathrm{Syl}_{p_6}(K)$. By Frattini argument, $G = KN_G(K_{p_6})$. Therefore, $N_G(K_{p_6})$ contains an element x of order p_7. Since G has no element of order $p_6.p_7$, $\langle x\rangle$ should act fixed point freely on K_p, which implies that $\langle x\rangle K_p$ is a Frobenius group. By using Lemma 2.3(b) it follows that $|\langle x\rangle|\,|(|K_{p_6}|-1)$, which is a contradiction because $\{p_6, p_7\} \subseteq \{19, 37, 73, 109\}$. Therefore K is a p_6'-group.
Next, we show that K is a p_5'-group. In Figures 3-33, 3-35 and 3-37, we set $r = p_{10}$ and in Figure 3-28, we set $r = p_8$. Now, by using a similar argument as above and replacing r with p_7 we conclude that K is a p_5'-group. Therefore K is a $\{p_5, p_6\}'$-group. In addition since $G \neq K$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_3(2^{12})$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We show that $m = 1$ and $S = P_1 \cong L_3(2^{12})$.
Assume to the contrary that $m \geq 2$. We claim that 241 does not divide $|S|$. Assume to the contrary and let $241 \mid |S|$, on the other hand, $2 \in \pi(P_i)$ for every i, hence $2 \sim 241$, which is a contradiction. Now, by Step 1 we observe that $241 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 241 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{241}$, Lemma 2.1 implies that $|\mathrm{Out}(P_i)|$ is not divisible by 241, so 241 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by using Lemma 2.4, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore $t \geq 241$ and so 2^{482} must divide the order of G, which is a contradiction. Therefore $m = 1$, and then $S = P_1$.
As $S \in \mathfrak{S}_{241}$, by Lemma 2.1 we conclude that 241 does not divide $|\mathrm{Out}(S)|$, so Step 1 implies that

$$|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13^{\alpha_5}.17^{\alpha_6}.19^{\alpha_7}.37^{\alpha_8}.73^{\alpha_9}.109^{\alpha_{10}}.241$$

where $2 \leq \alpha_1 \leq 36$, $0 \leq \alpha_2 \leq 5$, $0 \leq \alpha_3 \leq 2$, $0 \leq \alpha_4 \leq 2$, $0 \leq \alpha_5 \leq 2$, $0 \leq \alpha_6 \leq 1$, $0 \leq \alpha_7 \leq 1$, $0 \leq \alpha_8 \leq 1$, $0 \leq \alpha_9 \leq 1$ and $0 \leq \alpha_{10} \leq 1$. Now, by using Table 1 it follows that $S \cong L_3(2^{12})$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_3(2^{12})$.
By Step 2 in Parts A and B, we conclude that $L_3(2^{12}) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_3(2^{12}))$. As $|G| = |L_3(2^{12})|$, we deduce $K = 1$ and so $G \cong L_3(2^{12})$. □

The proof of our main Theorem is completed now.
As a consequence of the main theorem we will mention the following corollary, which is related to characterizable by prime graph.

Corollary 1. *Let G be a finite group satisfying $|G| = |L_3(2^n)|$, where $n \in \{4, 5, 6, 7, 8, 10, 12\}$. If $\Gamma(G) = \Gamma(L_3(2^n))$, then $G \cong L_3(2^n)$.*

Proof. Since $|G| = |L_3(2^n)|$ and $\Gamma(G) = \Gamma(L_3(2^n))$, we obtain $|G| = |L_3(2^n)|$ and $D(G) = D(L_3(2^n))$. By using the main theorem, we have $G \cong L_3(2^n)$. □

Remark 3.1. *Shi and Bi in [6] put forward the following conjecture:*
Conjecture. *Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if*
(i) $|G| = |M|$,
(ii) $\omega(G) = \omega(M)$.

A series of papers proved that this conjecture is true for most of finite simple groups. As the consequence of the main theorem, we can conclude that this conjecture is valid for the group under study.

Corollary 2. *Let G be a finite group satisfying $|G| = |L_3(2^n)|$, where $n \in \{4, 5, 6, 7, 8, 10, 12\}$. If $\omega(G) = \omega(L_3(2^n))$, then $G \cong L_3(2^n)$.*

Table 1. Finite simple groups $S \in \mathfrak{S}_p$ except alternating group

S	$\|S\|$
$p = 17$	
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$
$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$
$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$
$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$
$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$
$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$
$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$
$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$
$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$
$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$
$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$
$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$
$p = 73$	
$U_3(9)$	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$
$L_3(8)$	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$
$L_2(73)$	$2^3 \cdot 3^2 \cdot 37 \cdot 73$
$U_4(9)$	$2^9 \cdot 3^12 \cdot 5^3 \cdot 41 \cdot 73$
${}^3D_4(3)$	$2^6 \cdot 3^12 \cdot 7^2 \cdot 13^2 \cdot 73$
$L_2(2^9)$	$2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$
$G_2(8)$	$2^{18} \cdot 3^5 \cdot 7^2 \cdot 19 \cdot 73$

Table 1. (Continued)

S	$\|S\|$
$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 73$
$S_4(27)$	$2^6 \cdot 3^{12} \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 73$
$G_2(9)$	$2^8 \cdot 3^1 2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 73$
$L_4(8)$	$2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73$
$L_3(64)$	$2^{18} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 73$
$S_6(8)$	$2^{27} \cdot 3^7 \cdot 5 \cdot 7^3 \cdot 13 \cdot 19 \cdot 73$
$O_8^+(8)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^4 \cdot 13^2 \cdot 19 \cdot 73$
$L_3(3^4)$	$2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$
$S_6(9)$	$2^{12} \cdot 3^{18} \cdot 5^3 \cdot 7 \cdot 13 \cdot 41 \cdot 73$
$O_7(9)$	$2^{12} \cdot 3^{18} \cdot 5^3 \cdot 7 \cdot 13 \cdot 41 \cdot 73$
$F_4(3)$	$2^{15} \cdot 3^{24} \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 41 \cdot 73$
$O_8^+(9)$	$2^{16} \cdot 3^{24} \cdot 5^4 \cdot 7 \cdot 13 \cdot 41^2 \cdot 73$
$L_2(73^2)$	$2^4 \cdot 3^2 \cdot 5 \cdot 13 \cdot 37 \cdot 41 \cdot 73^2$
$S_4(73)$	$2^8 \cdot 3^4 \cdot 5 \cdot 13 \cdot 37^2 \cdot 41 \cdot 73^4$
$E_6(2)$	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 13 \cdot 17 \cdot 31 \cdot 73$
$U_4(27)$	$2^7 \cdot 3^{18} \cdot 5 \cdot 7^3 \cdot 13^2 \cdot 19 \cdot 37 \cdot 73$
$O_{12}^-(3)$	$2^{18} \cdot 3^{30} \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61 \cdot 73$
$L_6(9)$	$2^{18} \cdot 3^{30} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 41 \cdot 61 \cdot 73$
$O_{13}(3)$	$2^{21} \cdot 3^{36} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 41 \cdot 61 \cdot 73$
$S_{12}(3)$	$2^{21} \cdot 3^{36} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 41 \cdot 61 \cdot 73$
${}^2E_6(3)$	$2^{19} \cdot 3^{36} \cdot 5^2 \cdot 7^3 \cdot 13^2 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73$
$p = 151$	
$L_3(32)$	$2^{15}.3.7.11.31^2.151$
$L_4(32)$	$2^{30}.3^2.5^2.7.11^2.31^3.41.151$
$L_5(8)$	$2^{30}.3^4.5.7^4.13.31.73.151$
$L_6(8)$	$2^{45}.3^7.5.7^5.13.19.31.73^2.151$
$L_2(151)$	$2^3.3.5^2.19.151$
$p = 241$	
$L_2(241)$	$2^4.3.5.11^2.241$
$L_4(64)$	$2^{36}.3^7.5^2.7^3.13^2.17.19.73.241$
$S_8(8)$	$2^{48}.3^9.5^2.7^4.13^3.17.19.241$
$U_4(64)$	$2^{36}.3^4.5^3.7^2.13^3.17.37.109.241$
$O_{10}^+(8)$	$2^{60}.3^9.5^2.7^5.13^2.17^2.19.31.73.151.241$
$L_3(2^{12})$	$2^{36}.3^5.5^2.7^2.13^2.17.19.37.73.109.241$
$S_6(64)$	$2^{54}.3^6.5^3.7^3.13^3.17.19.37.109.241$
$O_8^+(64)$	$2^{72}.3^7.5^3.7^4.13^4.17^2.37.73.109.241^2$
$F_4(8)$	$2^{72}.3^{10}.5^2.7^4.13^2.17.37.73^2.109.241$

Table 1. (Continued)

S	$\lvert S\rvert$
$p = 257$	
$L_2(257)$	$2^8.3.43.257$
$L_2(2^8)$	$2^8.3.5.17.257$
$S_4(16)$	$2^{16}.3^2.5^2.17^2.257$
$U_4(16)$	$2^{24}.3^2.5^2.17^3.241.257$
$O_8^-(4)$	$2^{24}.3^4.5^3.7.13.17.257$
$S_8(4)$	$2^{32}.3^5.5^4.7.13.17^2.257$
$L_2(241^2)$	$2^5.3.5.7^3.11^2.113.241^2.257$
$S_4(241)$	$2^{10}.3^2.5^2.11^4.113.241^2.257$
$U_3(257)$	$2^{11}.3^2.7.13.43.241.257^3$
$O_{10}^-(4)$	$2^{40}.3^5.5^6.7.13.17^2.41.257$
$L_3(2^8)$	$2^{24}.3^2.5^2.7.13.17^2.241.257$
$S_6(16)$	$2^{36}.3^4.5^3.7.13.17^3.241.257$
$O_8^+(16)$	$2^{48}.3^5.5^4.7.13.17^4.241.257$
$F_4(4)$	$2^{48}.3^6.5^4.7^2.13^2.17^2.241.257$
$O_{10}^+(4)$	$2^{40}.3^6.5^4.7.11.13.17^2.31.257$
$L_5(16)$	$2^{40}.3^5.5^4.7.11.13.17^2.31.41.257$
$S_{10}(4)$	$2^{50}.3^6.5^6.7.11.13.17^2.31.41.257$
$O_{12}^+(4))$	$2^{60}.3^8.5^7.7^2.11.13^2.17^2.31.41.257$
$U_8(4)$	$2^{56}.3^6.5^7.7.13^2.17^2.41.43.127.257$
$O_{12}^-(4)$	$2^{60}.3^6.5^6.7.11.13.17^3.31.41.241.257$
$L_6(16)$	$2^{60}.3^6.5^6.7^2.11.13^2.17^3.31.41.241.257$
$S_{12}(4)$	$2^{72}.3^8.5^7.7^2.11.13^2.17^3.31.41.241.257$
$O_{16}^-(2)$	$2^{56}.3^9.5^3.7^2.11.13.17.31.43.127.257$
$L_8(4)$	$2^{56}.3^9.5^4.7^2.11.13.17^2.31.43.127.257$
$S_{16}(2)$	$2^{64}.3^{10}.5^4.7^2.11.13.17^2.31.43.127.257$
${}^2E_6(4)$	$2^{72}.3^6.5^7.7^2.13^3.17^2.37.41.109.241.257$
$O_{18}^-(2)$	$2^{72}.3^{13}.5^4.7^2.11.13.17^2.19.31.43.127.257$
$E_6(4)$	$2^{72}.3^9.5^4.7^3.11.13^2.17^2.19.31.73.241.257$
$O_{18}^+(2)$	$2^{72}.3^{10}.5^4.7^3.11.13.17^2.31.43.73.127.257$
$U_9(4)$	$2^{72}.3^5.5^9.7.13^3.17^2.29.37.41.109.113.257$
$L_9(4)$	$2^{72}.3^{11}.5^4.7^3.11.13.17^2.19.31.43.73.127.257$
$S_{18}(2)$	$2^{81}.3^{13}.5^4.7^3.11.13.17^2.19.31.43.73.127.257$
$O_{20}^+(2)$	$2^{90}.3^{14}.5^4.7^3.11^2.13.17^2.19.31^2.43.73.127.257$
$O_{14}^-(4)$	$2^{84}.3^8.5^8.7^3.11.13^2.17^2.29.31.41.113.241.257$
$L_{10}(4)$	$2^{90}.3^{13}.5^6.7^3.11^2.13.17^2.19.31^2.41.43.73.127.257$

Table 1. (Continued)

S	$\|S\|$
$S_{20}(2)$	$2^{100}.3^{14}.5^{6}.7^{3}.11^{2}.13.17^{2}.19.31^{2}.41.43.73.127.257$
$U_{10}(4)$	$2^{90}.3^{6}.5^{10}.7.11.13^{3}.17^{2}.29.31.37.41^{2}.109.113.257$
$L_7(16)$	$2^{84}.3^{8}.5^{7}.7^{2}.11.13^{2}.17^{3}.29.31.41.43.113.127.241.257$
$S_{14}(4)$	$2^{98}.3^{7}.5^{6}.7^{2}.11.13^{2}.17^{3}.29.31.41.43.113.127.241.257$
$O^+_{16}(4)$	$2^{112}.3^{8}.5^{7}.7^{2}.11.13^{2}.17^{4}.29.31.41.43.113.127.241.257^{2}$
$O^+_{22}(2)$	$2^{110}.3^{14}.5^{6}.7^{3}.11^{2}.13.17^{2}.19.23.31^{2}.41.43.73.89.127.257$
$E_7(4)$	$2^{126}.3^{11}.5^{8}.7^{3}.11.13^{3}.17^{2}.19.29.31.37.41.43.73.109.113.127.241.257$
$p = 331$	
$L_2(331)$	$2^{2}.3.5.11.83.331$
$L_3(331)$	$2^{7}.3^{2}.5^{2}.31^{3}.331$
$U_3(32)$	$2^{15}.3^{2}.11^{2}.31.331$
$L_2(31^3)$	$2^{5}.3^{2}.5.7^{2}.19.31^{3}.331$
$G_2(31)$	$2^{12}.3^{3}.5^{2}.7^{3}.19.31^{6}.331$
$U_4(32)$	$2^{30}.3^{4}.5^{2}.11^{3}.31^{2}.41.331$
$L_4(31)$	$2^{13}.3^{4}.5^{3}.19.31^{6}.331$
$L_2(2^{15})$	$2^{15}.3^{2}.7.11.31.151.331$
$G_2(32)$	$2^{30}.3^{3}.7.11^{2}.31^{2}.151.331$
$U_5(8)$	$2^{30}.3^{9}.5.7^{2}.11.13.19.331$
$U_6(8)$	$2^{45}.3^{11}.5.7^{3}.11.13.19^{2}.73.331$
$L_3(2^{10})$	$2^{45}.3^{4}.5.7.11^{3}.31^{3}.41.151.331$
$S_6(32)$	$2^{30}.3^{2}.5^{2}.7.11^{2}.31^{2}.41.151.331$
$O^+_8(32)$	$2^{60}.3^{5}.5^{4}.7.11^{4}.31^{4}.41^{2}.151.331$
$L_3(31^2)$	$2^{13}.3^{2}.5^{2}.7^{2}.13.19.37.331$
$O_7(31)$	$2^{18}.3^{4}.5^{3}.7^{2}.13.19.31^{9}.37.331$
$S_6(31)$	$2^{18}.3^{4}.5^{3}.7^{2}.13.19.31^{9}.37.331$
$O^+_8(31)$	$2^{25}.3^{5}.5^{4}.7^{2}.13^{2}.19.31^{2}.37^{2}.41.331$
$O^-_{10}(8)$	$2^{60}.3^{11}.5^{2}.7^{4}.11.13^{2}.17.19.73.241.331$
$L_5(64)$	$2^{60}.3^{9}.5^{2}.7^{4}.11.13.17.19.31.73.151.241.331$
$S_{10}(8)$	$2^{75}.3^{11}.5^{2}.7^{4}.11.13^{2}.17.19.73.151.241.331$
$O^+_{12}(8)$	$2^{90}.3^{14}.5^{2}.7^{6}.11.13^{2}.17.19^{2}.31.73^{2}.151.241.331$
$O^-_{12}(8)$	$2^{90}.3^{11}.5^{3}.7^{5}.11.13^{3}.17.19.31.37.73^{2}.109.151.241.331$
$L_6(64)$	$2^{90}.3^{12}.5^{3}.7^{5}.11.13^{3}.17.19^{2}.31.37.73^{2}.109.151.241.331$
$S_{12}(8)$	$2^{108}.3^{14}.5^{3}.7^{6}.11.13^{3}.17.19^{2}.31.37.73^{2}.109.151.241.331$
$E_8(2)$	$2^{120}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.17^{2}.19.31^{2}.41.43.73.127.151.241.331$
$p = 337$	
$L_3(2^7)$	$2^{21}.3.7^{2}.43.127^{2}.337$

Table 1. (Continued)

S	$\|S\|$
$L_2(337^2)$	$2^5.3.5.7.13^2.41.277.337$
$S_4(337)$	$2^{10}.3^2.5.7^2.13^4.41.277.337^4$
$L_4(2^7)$	$2^{42}.3^2.5.7^2.29.43^2.113.127^3.337$
$L_7(8)$	$2^{63}.3^7.5.7^6.13.19.31.73.127.151.337$
$L_8(8)$	$2^{84}.3^9.5^2.7^8.13^2.17.19.31.73.127.151.241.337$
$O^+_{14}(8)$	$2^{126}.3^{14}.5^3.7^9.11.13^3.19^2.31.37.73^2.109.127.151.241.331.337$
$L_2(337)$	$2^4.3.7.13^2.337$

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