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On convergence of certain nonlinear Durrmeyer operators at Lebesgue points

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ON CONVERGENCE OF CERTAIN NONLINEAR DURRMEYER OPERATORS AT LEBESGUE POINTS

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ABSTRACT. The aim of this paper is to study the behaviour of certain sequence of nonlinear Durrmeyer operators $ND_n f$ of the form

$$(ND_n f)(x) = \int\limits_0^1 K_n\left(x, t, f\left(t\right)\right) dt, \ 0 \le x \le 1, \quad n \in \mathbb{N},$$

acting on bounded functions on an interval $[0, 1]$, where $K_n\left(x, t, u\right)$ satisfies some suitable assumptions. Here we estimate the rate of convergence at a point x, which is a Lebesgue point of $f \in L_1\left([0, 1]\right)$ be such that $\psi o\left|f\right| \in BV\left([0, 1]\right)$, where $\psi o\left|f\right|$ denotes the composition of the functions ψ and $\left|f\right|$. The function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$, which appears from the $(L - \psi)$ Lipschitz conditions.

Keywords: Nonlinear Durrmeyer operators, bounded variation, Lipschitz condition, pointwise convergence.
MSC(2010): Primary: 41A35; Secondary: 41A25, 47G10.

1. Introduction

Let f be a Lebesgue integrable function defined on $[0, 1]$ and let $\mathbb{N} := \{1, 2, ...\}$. The classical Durrmeyer operators $D_n f$ applied to f are defined as

$$(1.1) \qquad (D_n f)(x) = \int\limits_0^1 f(t) K_n(x, t) \ dt, \ 0 \le x \le 1$$

where

$$K_n(x, t) = (n + 1) \sum_{k=0}^{n} p_{n,k}\left(x\right) p_{n,k}\left(t\right),$$

and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These operators were introduced by Durrmeyer [16] and independently by Lupas [29].

These operators are the integral modification of Bernstein polynomials [9] so as to approximate Lebesgue integrable functions defined on the interval [0,1]. Some remarkable approximation properties of these operators (1.1) are presented in [15, 17, 32] and [18].

The present paper concerns with pointwise convergence of certain sequence of nonlinear Durrmeyer operators $ND_n f$ of the form

$$(1.2) \qquad (ND_n f)(x) = \int_0^1 K_n(x, t, f(t)) \, dt \ , \ \ 0 \le x \le 1 \ , \ \ n \in \mathbb{N},$$

acting on Lebesgue integrable functions on the interval $[0, 1]$, where $K_n(x, t, u)$ satisfies some suitable assumptions. In particular, we obtain some pointwise convergence for a sequence of nonlinear Durrmeyer operators (1.2) to the point x, at the Lebesgue points of f, as $n \to \infty$.

We note that the approximation theory with nonlinear integral operators of convolution type was introduced by J. Musielak in [30] and widely developed in [4]. To the best of our knowledge, the approximation problem was limited to linear operators because the notion of singularity of an integral operator was closely connected with its linearity until the fundamental paper of Musielak [30]. In [30], the assumption of linearity of the singular integral operators was replaced by an assumption of a Lipschitz condition for the kernel function $K_\lambda(t, u)$ with respect to the second variable. After this approach, several mathematicians have undertaken the program of extending approximation by nonlinear operators in many ways, including pointwise and uniform convergence, Korovkin type theorems in abstract function spaces, sampling series and so on. Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t - x, f(t)) \, dt, \quad x \in (a, b),$$

and its special cases were studied by Bardaro, Karsli and Vinti [6, 7], Karsli [21, 22]- [25] and Karsli-Ibikli [24] in some Lebesgue spaces.

Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators.

For further reading, we also refer the reader to [1, 2], [5–8] as well as the monographs [13] and [4] where other kinds of convergence results of linear and nonlinear singular integral operators in the Lebesgue and Musielak-Orlicz spaces have been considered. Finally in the very recent paper due to Angeloni

and Vinti [3], some approximation properties with respect to the multidimensional $\varphi-$ variation for the linear cases of the operators of type (1.1) have been studied.

An outline of the paper is as follows: The next section contains basic definitions and notations. In Section 3 the main approximation result of this study are given. In Section 4 we give some certain results which are necessary to prove the main result. The final section, that is Section 5, deals with the proof of the main result presented in Section 3.

2. **Preliminaries**

In this section, we assemble the main definitions and notations which will be used throughout the paper.

Let X be the set of all Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that the function ψ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$.

We now introduce a sequence of functions. Let $\{K_n(x, t, u)\}_{n \in \mathbb{N}}$ be a sequence of functions $K_n : [0, 1] \,\mathrm{x}\, [0, 1] \,\mathrm{x}\, \mathbb{R} \to \mathbb{R}$ defined by

$$(2.1) \qquad K_n(x, t, u) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \, p_{n,k}(t) \, H_n(u),$$

where $H_n : \mathbb{R} \to \mathbb{R}$ is a function such that $H_n(0) = 0$ and $p_{n,k}(x)$ is the Bernstein basis.

Throughout the paper we assume that $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n \to \infty} \mu(n) = \infty$.

First of all we assume that the following conditions hold:

a) $H_n : \mathbb{R} \to \mathbb{R}$ *is such that*

$$|H_n(u) - H_n(v)| \leq \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$ *and for every* $n \in \mathbb{N}$. *That is,* H_n *satisfies a* $(L - \psi)$ *Lipschitz condition.*

b) *We now set*

$$(2.2) \qquad F_n(x, t) := (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \, p_{n,k}(t).$$

and

$$A_n(x) := \int_{x - x/n^{\gamma/\beta}}^{x + (1-x)/n^{\gamma/\beta}} F_n(x, t)dt \quad \text{for any fixed } x \in (0, 1)$$

where $\beta > 0$, $\gamma \geq 1$.

c) denoting by $r_n(u) := H_n(u) - u$, $u \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *such that*

$$\lim_{n \to \infty} |r_n(u)| = 0$$

uniformly with respect to u.

We note that the use of the function $A_n(x)$ concerns the behavior of the approximation near the point x. Similar approach and some particular examples can be found in [8], [25], [20], [23] and [31].

Example 2.1. Here we give a concrete example of the nonlinear function $H_n(u)$ to show the validity of the definition of the operators. Let us consider the function $H_n : \mathbb{R} \to \mathbb{R}$ defined as

$$H_n(u) = \frac{nu\,|u|}{n\,|u| + 1}.$$

Some other examples can be found in [8]. The symbol $[a]$ will denote the greatest integer not exceeding a.

3. Convergence results

We take into consideration the following type nonlinear Durrmeyer operators, given by

$$(ND_n f)(x) = \int_0^1 K_n(x, t, f(t))\, dt,$$

with

$$
\begin{aligned}
K_n(x, t, f(t)) &= (n+1) \sum_{k=0}^n p_{n,k}(x)\, p_{n,k}(t) H_n(f(t)) \\
&= F_n(x, t) H_n(f(t)).
\end{aligned}
$$

We assume that this operator defined for every $f \in Dom\ ND_n f$, where $Dom\ ND_n f$ is the subset of X on which $ND_n f$ is well-defined.

We have the following.

Theorem 3.1. *Let the kernel function* $K_n(x, t, u)$ *satisfies a). If* $f \in L_1[0,1]$, *then* $ND_n f \in L_1[0,1]$ *and*

$$\|ND_n f\|_{L_1[0,1]} \leq \psi\left(\|f\|_{L_1[0,1]}\right)$$

for every $n \in \mathbb{N}$.

Proof. From the assumptions on K_n, we have

$$
\int_0^1 |(ND_nf)(x)|\, dx \;=\; \int_0^1 \left| \int_0^1 F_n(x,t) H_n(f(t)) dt \right| dx \leq \int_0^1 \int_0^1 F_n(x,t)\, |H_n(f(t))|\, dt dx
$$

$$
\leq \int_0^1 \int_0^1 F_n(x,t)\, |H_n(f(t))|\, dx dt
$$

$$
\leq \int_0^1 \psi\left(|f(t)|\right) \int_0^1 F_n(x,t) dx dt.
$$

Since

$$
\int_0^1 F_n(x,t) dt = \int_0^1 F_n(x,t) dx,
$$

one has

$$
\int_0^1 |(ND_nf)(x)|\, dx \leq \int_0^1 \psi\left(|f(t)|\right) dt.
$$

Using concavity of ψ, we obtain

$$
\int_0^1 |(ND_nf)(x)|\, dx \leq \psi\left(\int_0^1 |f(t)|\, dt \right) = \psi\left(\|f\|_{L_1[0,1]} \right).
$$

As a result, the nonlinear Durrmeyer operators acting on $L_1[0,1]$ into itself. This completes the proof. $\qquad\square$

We let

$$
(3.1) \qquad f_x(t) = \begin{cases} f(t) - f(x+) & , \; x < t \leq 1 \\ 0 & , \; t = x \\ f(t) - f(x-) & , \; 0 \leq t < x \end{cases} ,
$$

and $\bigvee_0^1 \psi(|f_x|)$ be the total variation of $\psi(|f_x|)$ on $[0,1]$.

Definition 3.2. *A point $x_0 \in \mathbb{R}$ is called a Lebesgue point of the function f, if*

$$
(3.2) \qquad \lim_{h \to 0^+} \frac{1}{h} \int_0^h |f(x_0 + t) - f(x_0)|\, dt = 0,
$$

holds. (Butzer and Nessel [13]).

We are now ready to establish the main results of this study:

Theorem 3.3. *Let* $\psi \in \Psi$ *and* $f \in L_1\left([0,1]\right)$ *be such that* $\psi o\left|f\right| \in BV\left([0,1]\right).$ *Suppose that* $K_n\left(x,t,u\right)$ *satisfies conditions* $a) - c)$. *Then at each point* $x \in (0,1)$ *for which* (3.2) *holds we have for each* $\varepsilon > 0$ *and for sufficiently large* $n \in \mathbb{N}$,

$$\left|\left(ND_n f\right)(x) - f(x)\right| \quad \leq \quad \varepsilon B_n^*(x)\mu^{\beta-1}(n)$$

$$+\frac{B_n^*(x)}{\mu^{\beta}(n)}\left[\bigvee_0^1 \psi(|f_x|) + \sum_{k=1}^{\left[\mu^{\beta}(n)\right]} \bigvee_{x-x/k^{1/\beta}}^{x+(1-x)/k^{1/\beta}} \psi(|f_x|)\right] + \frac{1}{\mu(n)},$$

where $B_n^*(x) = B_n(x)\max\{x^{-\beta},(1-x)^{-\beta}\},\ (\beta > 0)$.

Now we are ready to establish a convergence result.

Theorem 3.4. *Let* $\psi \in \Psi$ *and* $f \in L_1\left([0,1]\right)$ *be such that* $\psi o\left|f\right| \in BV\left([0,1]\right).$ *Suppose that the kernel function* $K_n\left(x,t,u\right)$ *satisfies conditions* $a) - c)$. *Then at each point* $x \in (0,1)$ *for which* (3.2) *holds we have*

$$\lim_{n\to\infty}\left|\left(ND_n f\right)(x) - f(x)\right| = 0.$$

Proof. From Theorem 3.3 and $c)$ we reach the result, by the arbitrariness of $\varepsilon > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Corollary 3.5. *Let* $\psi \in \Psi$ *and* $f \in L_1\left([0,1]\right)$ *be such that* $\psi \circ \left|f\right| \in BV\left([0,1]\right).$ *Suppose that the function* $K_n\left(x,t,u\right)$ *satisfies conditions* $a) - c)$. *Then*

$$\lim_{n\to\infty}\left|\left(ND_n f\right)(x) - f(x)\right| = 0$$

holds almost everywhere in $(0,1)$.

Since almost all $x \in (0,1)$ are Lebesgue points of the function f, then the assertion follows by Theorem 3.4.

4. **Auxiliary result**

In this section we give certain results, which are necessary to prove our theorems.

Lemma 4.1. *For* $(D_n t^s)(x),\ s = 0,1,2,$ *one has*

$$(D_n 1)(x) \quad = \quad 1$$
$$(D_n t)(x) \quad = \quad x + \frac{1-2x}{n+2}$$
$$(D_n t^2)(x) \quad = \quad x^2 + \frac{[4n - 6(n+1)x]}{(n+1)(n+2)}x + \frac{2}{(n+2)(n+3)}.$$

For proof of this Lemma see [28].

By direct calculation, we find the following equalities:

$$(D_n\,(t-x)^2)(x) \le \frac{2nx(1-x)+2}{n^2} \quad,\quad (D_n\,(t-x))(x) = \frac{1-2x}{n+2}\,.$$

Lemma 4.2. *For all $x \in (0,1)$ and for each $n \in \mathbb{N}$, let*

$$(4.1) \qquad D_n((t-x)^\beta; x) = \int\limits_0^1 F_n(x,u)(u-x)^\beta du \le \frac{B_n(x)}{n^{\gamma/\beta}},\ \ (\beta > 0)$$

where $F_n(x,u)$ is defined as in Section 2. Then one has

$$(4.2) \qquad \lambda_n(x,z) =: \int\limits_0^z F_n(x,u)du \le \frac{B_n(x)}{(x-z)^\beta n^{\gamma/\beta}}\,,\quad 0 \le z < x,$$

and

$$(4.3) \qquad 1 - \lambda_n(x,z) = \int\limits_z^1 F_n(x,u)du \le \frac{B_n(x)}{(z-x)^\beta n^{\gamma/\beta}}\,,\quad x < z < 1.$$

Proof. We have

$$\lambda_n(x,z) \quad = \quad : \int\limits_0^z F_n(x,u)du \le \int\limits_0^z F_n(x,u)\left(\frac{x-u}{x-z}\right)^\beta du$$

$$\le \quad \frac{1}{(x-z)^\beta}\int\limits_0^1 F_n(x,u)\,|u-x|^\beta\,du.$$

According to (4.1), we have

$$\lambda_n(x,z) \le \frac{B_n(x)}{(x-z)^\beta n^{\gamma/\beta}}.$$

Proof of (4.3) is analogous. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.3. *[32, Theorem 1] For all $x \in (0,1)$ and for all $n > \frac{256}{25x(1-x)}$, we have*

$$p_{n,k}\,(x) \le \frac{1}{\sqrt{2enx(1-x)}},$$

where $e = 2.71...$ is the Napierian constant.

Lemma 4.4. *Let $\psi \in \Psi$. Then if $x_0 \in \mathbb{R}$ is a Lebesgue point of the function f, we have*

$$(4.4) \qquad \left|\int\limits_0^h \psi\,(|f(x_0+t) - f(x_0)|)\,dt\right| = o(|h|) \quad as \quad h \to 0.$$

Proof. In order to prove our Lemma we will show the following two statements:

$$\left| \int_0^h \psi\left(|f(x_0+t) - f(x_0)|\right) dt \right| = o(h) \quad as \quad h \to 0^+,$$

$$\left| \int_h^0 \psi\left(|f(x_0+t) - f(x_0)|\right) dt \right| = o(-h) \quad as \quad h \to 0^-.$$

Since ψ is concave, one has for $h < 0$ and $h > 0$, respectively,

$$\frac{1}{-h} \int_h^0 \psi\left(|f(x_0+t) - f(x_0)|\right) dt \leq \psi\left(\frac{1}{-h} \int_h^0 |f(x_0+t) - f(x_0)| \, dt\right)$$

and

$$\frac{1}{h} \int_0^h \psi\left(|f(x_0+t) - f(x_0)|\right) dt \leq \psi\left(\frac{1}{h} \int_0^h |f(x_0+t) - f(x_0)| \, dt\right).$$

Hence, by continuity of ψ and $\psi(0) = 0$, we reach the desired result. $\square$

5. Proof of the theorems

Proof. **Proof of Theorem 3.3.** Suppose that

$$(5.1) \qquad\qquad x + \delta < 1, \;\; x - \delta > 0,$$

for any sufficiently small $0 < \delta$.

Let

$$|I_n(x)| = \left| \int_0^1 K_n\left(x, t, f(t)\right) dt - f(x) \right|.$$

From (1.2) and c), we can rewrite $|I_n(x)|$ as follows:

$$|I_n(x)| \leq \left| \int_0^1 K_n\left(x, t, f(t)\right) dt - \int_0^1 K_n\left(x, t, f(x)\right) dt \right| + \left| \int_0^1 K_n\left(x, t, f(x)\right) dt - f(x) \right|$$

$$\vdots \quad = I_{n,1}(x) + I_{n,2}(x)$$

From conditions $b)$ and $c)$ it is easy to see that the second term of the right-hand-side of the above inequality is less than or equal to $1/\mu(n)$. Indeed;

$$
\begin{aligned}
I_{n,2}(x) \;&=\; \left| \int_0^1 K_n\left(x,t,f\left(x\right)\right) dt - f(x) \right| = \left| \left[H_n(f\left(x\right)) - f(x)\right] \int_0^1 F_n\left(x,t\right) dt \right| \\
&=\; \left| H_n\left(f\left(x\right)\right) - f(x) \right| \int_0^1 F_n\left(x,t\right) dt \\
&\leq\; \frac{1}{\mu(n)}
\end{aligned}
$$

holds for n sufficiently large.

As to the first term, by $a)$, we have the following inequality,

$$
\left| I_{n,1}(x) \right| \leq \int_0^1 F_n\left(x,t\right) \psi\left(\left|f\left(t\right) - f(x)\right|\right) dt.
$$

According to $b)$, we can split the last integral in three terms as follows:

$$
\left| I_{n,1}(x) \right| \;\leq\; \left(\int_0^{x-x/\mu(n)} + \int_{x-x/\mu(n)}^{x+(1-x)/\mu(n)} + \int_{x+(1-x)/\mu(n)}^1 \right) \psi\left(\left|f(t) - f(x)\right|\right) d_t\left(\lambda_n(x,t)\right)
$$

$$
: \quad = I_1(n,x) + I_2(n,x) + I_3(n,x)
$$

We estimate $I_2(n,x)$. We have for $t \in \left[x - x/\mu(n), x + \left(1 - x\right)/\mu(n)\right]$

$$
\left| I_2(n,x) \right| = \int_{x-x/\mu(n)}^{x+(1-x)/\mu(n)} \psi\left(\left|f(t) - f(x)\right|\right) d_t\left(\lambda_n(x,t)\right)
$$

$$
\leq \int_{x-x/\mu(n)}^{x} \psi\left(\left|f(t) - f(x)\right|\right) d_t\left(\lambda_n(x,t)\right) + \int_{x}^{x+(1-x)/\mu(n)} \psi\left(\left|f(t) - f(x)\right|\right) d_t\left(\lambda_n(x,t)\right)
$$

$$
= I_{2,1}(n,x) + I_{2,2}(n,x).
$$

Setting

$$
G(t) := \int_t^x \psi\left(\left|f(y) - f(x)\right|\right) dy,
$$

then, according to (4.4), to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

(5.2) $G(t) \leq \varepsilon\left(x - t\right)$

for all $0 < x - t \leq \delta$.

We now fix this δ and estimate $I_{2,1}(n,x)$ and $I_{2,2}(n,x)$ respectively.

Now, we recall the Lebesgue-Stieltjes integral representation, so we can write $I_{2,1}(n,x)$ as

$$(5.3) \qquad I_{2,1}(n,x) = \int\limits_{x-x/\mu(n)}^{x} d_t \left(\lambda_n(x,t)\right) d\left(G(t)\right).$$

Applying partial Lebesgue-Stieltjes integration (5.3) and using (5.2) we obtain,

$$\begin{aligned}
I_{2,1}(n,x) &= -G(x - x/\mu(n))\, d_t \left(\lambda_n(x, x - x/\mu(n))\right) + \int\limits_{x-x/\mu(n)}^{x} G(t) \frac{\partial}{\partial t} d_t \left(\lambda_n(x,t)\right)\, dt \\[2mm]
&\leq -G(x - x/\mu(n))\, d_t \left(\lambda_n(x, x - x/\mu(n))\right) + \int\limits_{x-x/\mu(n)}^{x} G(t) \frac{\partial}{\partial t} d_t \left(\lambda_n(x,t)\right)\, dt \\[2mm]
&\leq \varepsilon\, x/\mu(n)\, d_t \left(\lambda_n(x, x - x/\mu(n))\right) + \varepsilon \int\limits_{x-x/\mu(n)}^{x} (x-t) \frac{\partial}{\partial t} d_t \left(\lambda_n(x,t)\right)\, dt.
\end{aligned}$$

Integration by parts again gives

$$\begin{aligned}
I_{2,1}(n,x) &\leq \varepsilon\, x/\mu(n)\, d_t \left(\lambda_n(x, x - x/\mu(n))\right) \\[2mm]
&+ \varepsilon \left\{ -x/\mu(n) d_t \left(\lambda_n(x, x - x/\mu(n))\right) + \int\limits_{x-x/\mu(n)}^{x} d_t \left(\lambda_n(x,t)\right) \right\} \\[2mm]
&= \varepsilon \int\limits_{x-x/\mu(n)}^{x} d_t \left(\lambda_n(x,t)\right).
\end{aligned}$$

Setting

$$I_{2,1,1}(n,x) := \int\limits_{-x/\mu(n)}^{0} d_t \left(\lambda_n(x, t+x)\right),$$

according to (4.2) we can now obtain the following estimate:

$$\begin{aligned}
I_{2,1}(n,x) &= \varepsilon I_{2,1,1}(n,x) = \varepsilon \left(\lambda_n \left(x, x - \frac{x}{\mu(n)} \right) \right) \int\limits_{-x/\mu(n)}^{0} dt \\[2mm]
&= \varepsilon B_n(x) x^{-\beta} \mu^{\beta-1}(n).
\end{aligned}$$

We can use a similar method for $I_{2,2}(n,x)$. Then, in view of (4.3) we find the following inequality,

$$I_{2,2}(n,x) \leq \varepsilon B_n(x) \left(1 - x\right)^{-\beta} \mu^{\beta-1}(n).$$

Next, we estimate $I_1(n,x)$. Using partial Lebesgue-Stieltjes integration, we obtain

$$|I_1(n,x)| = \int_0^{x-x/\mu(n)} \psi\left(|f_x(t)|\right) d_t\left(\lambda_n(x,t)\right)$$

$$= \psi\left(\left|f_x\left(x - \frac{x}{\mu(n)}\right)\right|\right)\lambda_n\left(x, x - \frac{x}{\mu(n)}\right) - \int_0^{x-x/\mu(n)} \lambda_n(x,t)d_t\left(\psi\left(|f_x(t)|\right)\right).$$

Let $y = x - x/\mu(n)$. By (4.2), it is clear that

(5.4) $$\lambda_n(x,y) \le B_n(x)x^{-\beta}\mu^{\beta-1}(n).$$

Here we note that

$$\psi\left(\left|f_x\left(x - \frac{x}{\mu(n)}\right)\right|\right) = \left|\psi\left(\left|f_x\left(x - \frac{x}{\mu(n)}\right)\right|\right) - \psi\left(|f_x(x)|\right)\right| \le \bigvee_{x-x/\mu(n)}^{x} \psi(|f_x|).$$

Using partial integration and applying (5.4), we obtain

$$|I_1(n,x)| \le \bigvee_{x-x/\mu(n)}^{x} \psi(|f_x|)\left|\lambda_n\left(x, x - \frac{x}{\mu(n)}\right)\right| + \int_0^{x-x/\mu(n)} \lambda_n(x,t)d_t\left(-\bigvee_{t}^{x}\psi(|f_x|)\right)$$

$$\le \bigvee_{x-x/\mu(n)}^{x} \psi(|f_x|)B_n(x)x^{-\beta}\mu^{\beta-1}(n) + \frac{B_n(x)}{\mu(n)}\int_0^{x-x/\mu(n)}(x-t)^{-\beta}d_t\left(-\bigvee_{t}^{x}\psi(f_x)\right)$$

$$= \bigvee_{x-x/\mu(n)}^{x} \psi(f_x)B_n(x)x^{-\beta}\mu^{\beta-1}(n) + \frac{B_n(x)}{\mu(n)}\left[-x^{-\beta}\mu^{\beta}(n)\bigvee_{x-x/\mu(n)}^{x}\psi(|f_x|)\right.$$

$$\left. +x^{-\beta}\bigvee_{0}^{x}\psi(|f_x|) + \int_0^{x-x/\mu(n)}\bigvee_{t}^{x}\psi(|f_x|)\frac{\beta}{(x-t)^{\beta+1}}dt\right]$$

$$= \frac{B_n(x)}{\mu^{\beta}(n)}\left[x^{-\beta}\bigvee_{0}^{x}\psi(|f_x|) + \int_0^{x-x/\mu(n)}\bigvee_{t}^{x}\psi(|f_x|)\frac{\beta}{(x-t)^{\beta+1}}dt\right].$$

Changing the variable t by $x - x/u^{1/\beta}$ in the last integral, we have

$$\int_0^{x-x/\mu(n)}\bigvee_{t}^{x}\psi(|f_x|)\frac{\beta}{(x-t)^{\beta+1}}dt = \frac{1}{x^{\beta}}\int_1^{\mu^{\beta}(n)}\bigvee_{x-x/u^{1/\beta}}^{x}\psi(|f_x|)\,du$$

$$\le \frac{1}{x^{\beta}}\sum_{k=1}^{[\mu^{\beta}(n)]}\bigvee_{x-x/k^{1/\beta}}^{x}\psi(|f_x|).$$

Consequently, we obtain

$$|I_1(n,x)| \leq \frac{B_n(x)}{\mu^\beta(n)} x^{-\beta} \left[\bigvee_0^x \psi(|f_x|) + \sum_{k=1}^{[\mu^\beta(n)]} \bigvee_{x-x/k^{1/\beta}}^x \psi(|f_x|) \right].$$

Using a similar method, we can find

$$|I_3(n,x)| \leq \frac{B_n(x)}{\mu^\beta(n)} (1-x)^{-\beta} \left[\bigvee_x^1 \psi(|f_x|) + \sum_{k=1}^{[\mu^\beta(n)]} \bigvee_x^{x+(1-x)/k^{1/\beta}} \psi(|f_x|) \right].$$

Collecting the above estimates we get the required result. □

Acknowledgments

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