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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING FIVE SMALL FUNCTIONS ON ANNULI

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(Communicated by Ali Abkar)

Abstract. The purpose of this article is to investigate the uniqueness of meromorphic functions sharing five small functions on annuli.
Keywords: Meromorphic function, Nevanlinna theory, small functions, uniqueness, annulus.
MSC(2010): Primary: 30D10; Secondary: 30D20, 30B10, 34M05.

1. Introduction and results

We assume that the reader is familiar with Nevanlinna's theory of meromorphic functions (for references, see [5,15]). The uniqueness of meromorphic functions in the complex plane $\mathbb{C}$ is an important subject in the value distribution theory (see [16]). We say that two meromorphic functions f and g share the value a ($a \in \widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$) in $\mathbb{X} \subset \mathbb{C}$ provided that in $\mathbb{X}$, we have $f(z) = a$ if and only if $g(z) = a$. We will state whether a shared value is by CM (counting multiplicities) or by IM (ignoring multiplicities).

In 1926, Nevanlinna [14] proved the following well-known five value theorem.
Theorem A ([14]). *Let f and g be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$, $a_j \in \overline{\mathbb{C}}$ (j=1,2,3,4,5) be five distinct values. If f and g share the values a_j (j=1,2,3,4,5) IM in $\mathbb{C}$, then $f \equiv g$.*

After his very work, the uniqueness theory of meromorphic functions in $\mathbb{C}$ attracted many investigations (for references, see [16]). For the uniqueness of meromorphic functions in the unit disc, refer to [4]. In [19], J.H. Zheng suggested first time to investigate the uniqueness of meromorphic functions in a precise subset of $\mathbb{C}$ and posed the following question.

Question 1.1. *Under what conditions, must two meromorphic functions on $\mathbb{X}(\neq \mathbb{C})$ be identical?*

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It is an interesting topic how to extend some important uniqueness results in the complex plane to an angular domain. In 2003, J. H. Zheng firstly took into account the uniqueness of meromorphic functions sharing values in an angular domain and extended five value theorem in the complex plane to an angular domain (see [19, 20]).

Theorem B ([21]). *Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$ $(0<\beta-\alpha<2\pi)$, and*

$$\limsup_{r\to\infty}\frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r}=\infty.$$

If f and g share five distinct values a_j (j=1,2,3,4,5) IM in $\Omega(\alpha,\beta)$, then $f\equiv g$.

The Nevanlinna five distinct value theorem has been extended in [11, 17] to the case of five IM shared small functions; see the following result.

Theorem C ([11, 17]). *Let f and g be two nonconstant meromorphic functions in complex plane $\mathbb{C}$, and α_j (j=1,2,3,4,5) be five distinct small functions with respect to f and g. If f and g share α_j (j=1,2,3,4,5) IM in $\mathbb{C}$, then $f(z)\equiv g(z)$.*

Also, we may raise the following natural question:

Question 1.2. *What is the analogous result for Theorem C in one angular domain?*

In [12], H. F. Liu and Z. Q. Mao firstly extended five small functions theorem in the complex plane to an angular domain.

Theorem D ([12]). *Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha,\beta)(0<\beta-\alpha<2\pi)$ such that*

$$\limsup_{r\to\infty}\frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r}=\infty$$

and let α_j $(j=1,2,3,4,5)$ be five distinct small functions with respect to f and g in $\Omega(\alpha,\beta)$. If f and g share α_j $(j=1,2,3,4,5)$ IM in $\Omega(\alpha,\beta)$, then $f\equiv g$.

However, all the above cases take place in simply connected domains. Thus it is very interesting to consider the uniqueness theory of meromorphic functions in doubly connected domains.

Here we shall mainly study the uniqueness of meromorphic functions in doubly connected domains of the complex plane $\mathbb{C}$. By the Doubly Connected Mapping Theorem [1] each doubly connected domain is conformally equivalent to an annulus $\{z: r<|z|<R\}$, $0\le r<R\le+\infty$. We consider only two cases: $r=0$, $R=+\infty$ simultaneously and $0<r<R<+\infty$. In the latter case, the homothety $z\mapsto\frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z:\frac{1}{R_0}<|z|<R_0\}$, where $R_0=\sqrt{\frac{R}{r}}$. Thus, in both cases every annulus is invariant with respect to the inversion $z\mapsto\frac{1}{z}$. Hence in this paper, we consider the uniqueness of meromorphic functions on the annulus $\Lambda=\{z:\frac{1}{R_0}<|z|<$

$R_0\}$, where $1 < R_0 \le +\infty$. We denote by S a subset of distinct elements in $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$. For a function f meromorphic in Λ, we define

$$E(S,f) = \bigcup_{a\in S}\{z \in \Lambda : f(z) - a = 0, \qquad \textit{counting multiplicity}\},$$

$$\overline{E}(S,f) = \bigcup_{a\in S}\{z \in \Lambda : f(z) - a = 0, \qquad \textit{ignoring multiplicity}\}.$$

The Nevanlinna characteristic $T_0(r,f)$ of a meromorphic function f on the annulus Λ shall be introduced in the next section.

Let f be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. The function f is called a transcendental or admissible meromorphic function on the annulus Λ provided that

$$\limsup_{r\to\infty} \frac{T_0(r,f)}{\log r} = \infty, \quad 1 \le r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0} \frac{T_0(r,f)}{-\log(R_0 - r)} = \infty, \quad 1 \le r < R_0 < +\infty.$$

In 2009, T. B. Cao, H. X. Yi and H. Y. Xu [2] proved a general theorem on the multiple values and uniqueness of meromorphic functions in the annulus Λ. **Theorem E** ([2]). *Let f and g be two transcendental or admissible meromorphic functions on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $a_j (j = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\widehat{\mathbb{C}}$. If $\overline{E}(a_j, f) = \overline{E}(a_j, g)$ for $j = 1, 2, 3, 4, 5$, then $f(z) \equiv g(z)$.*

Let f and α be two meromorphic functions on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$. α is called a small function with respect to f on the annulus Λ if $T_0(r, \alpha) = o(T_0(r, f))$ as $r \to \infty$, possibly outside a set E of finite linear measure for $R_0 = +\infty$ or $T_0(r, \alpha) = o(T_0(r, f))$, as $r \to R_0$ possibly outside a set E with $\int_E dr/(R_0 - r) < \infty$ for $R_0 < +\infty$.

The following is the question we consider in this paper.

Question 1.3. *Do f and g coincide if $\overline{E}(a_j, f) = \overline{E}(a_j, g)$ for $j = 1, 2, 3, 4, 5$ on the annulus Λ, where a_j $(j = 1, 2, 3, 4, 5)$ are five distinct small functions with respect to f and g.*

Dealing with the above question, we obtain the following results which give an affirmative answer to Question 1.3.

Theorem 1.1. *Let f and g be two transcendental or admissible meromorphic functions on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $\alpha_j (j = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to f and g on the annulus Λ. If $\overline{E}(\alpha_j, f) = \overline{E}(\alpha_j, g)$ for $j = 1, 2, 3, 4, 5$, then $f(z) \equiv g(z)$.*

We complete the proof of Theorem 1.1 with the help of the method in Yi [17, 18] and Liu and Mao [12, 13]. These papers investigate the uniqueness of meromorphic functions sharing small functions in the complex plane or in the angular domains.

2. Preliminaries and some lemmas

Let f be a meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. We recall the classical notations of Nevanlinna theory as follows:

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad T(r,f) = N(r,f) + m(r,f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t,f)$ is the counting function of poles of function f in $\{z : |z| \leq t\}$. Here we give the notations of the Nevanlinna theory on annuli. Let

$$N_1(r,f) = \int_{\frac{1}{r}}^1 \frac{n_1(t,f)}{t} dt, \quad N_2(r,f) = \int_1^r \frac{n_2(t,f)}{t} dt,$$

$$m_0(r,f) = m(r,f) + m(\frac{1}{r}, f) - 2m(1,f),$$

$$N_0(r,f) = N_1(r,f) + N_2(r,f),$$

where $n_1(t,f)$ and $n_2(t,f)$ are the counting functions of poles of function f in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. Set

$$\begin{aligned}\overline{N_0}(r, \tfrac{1}{f-a}) &= \overline{N_1}(r, \tfrac{1}{f-a}) + \overline{N_2}(r, \tfrac{1}{f-a})\\ &= \int_{\frac{1}{r}}^1 \frac{\overline{n}_1(t, \frac{1}{f-a})}{t} dt + \int_1^r \frac{\overline{n}_2(t, \frac{1}{f-a})}{t} dt.\end{aligned}$$

in which each zero of the function $f - a$ is counted only once. The Nevanlinna characteristic of f on the annulus Λ is defined by

$$T_0(r,f) = m_0(r,f) + N_0(r,f).$$

Throughout, we denote by $S(r,*)$ quantities satisfying
(i) in the case $R_0 = \infty$,

$$S(r,*) = O(\log(rT_0(r,*)))$$

for $r \in (1, +\infty)$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1} < +\infty$;
(ii) if $R_0 < \infty$, then

$$S(r,*) = O(\log(\frac{T_0(r,*)}{R_0 - r}))$$

for $r \in (1, R_0)$ except for the set $\triangle'_r$ such that $\int_{\triangle'_r} \frac{dr}{(R_0-r)^{\lambda-1}} < +\infty$;

Thus for an admissible meromorphic function on the annulus Λ, $S(r,f) = o(T_0(r,f))$ holds for all $1 \le r < R_0$ except for the set $\triangle_r$ or the set $\triangle'_r$ mentioned above, respectively.

Lemma 2.1 ([7,9]). *Let f be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Then*

$$(i) \quad T_0(r,f) = T_0(r, \tfrac{1}{f}),$$
$$(ii) \quad \max\{T_0(r, f_1 \cdot f_2), T_0(r, \tfrac{f_1}{f_2}), T_0(r, f_1 + f_2)\} \le T_0(r, f_1) + T_0(r, f_2) + O(1).$$

By Lemma 2.1, the first fundamental theorem on the annulus Λ is immediately obtained.

Lemma 2.2 ([7,9] The first fundamental theorem). *Let f be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Then*

$$T_0(r, \frac{1}{f-a}) = T_0(r,f) + O(1)$$

for every fixed $a \in \mathbb{C}$.

Lemma 2.3 ([8,9] The lemma of the logarithmic derivative). *Let f be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Then*

$$m_0(r, \frac{f^{(k)}}{f}) \le S(r,f)$$

for every $k \in \mathbb{N}$.

Khrystiyanyn and Kondratyuk also obtained the second fundamental theorem on the annulus Λ. We show here the reduced form due to Cao, Yi and Xu.

Lemma 2.4 ([2] The second fundamental theorem). *Let f be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in $\widehat{\mathbb{C}}$. Then*

$$(q-2)T_0(r,f) < \sum_{j=1}^{q} \overline{N_0}(r, \frac{1}{f-a_j}) + S(r,f).$$

Lemma 2.5 ([3]). *Let f be a nonconstant meromorphic function on the annulus Λ, $P_1(f)$ and $P_2(f)$ be two mutually prime polynomials in f with degrees m and n respectively. Then*

$$T_0(r, \frac{P_1(f)}{P_2(f)}) = \max\{m,n\}T_0(r,f) + S(r,f).$$

Lemma 2.6. *Let f be a nonconstant meromorphic function on the annulus $\Lambda = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \le r < R_0 \le +\infty$. Let β_j (j=1,2,3) be small functions with respect to f on the annulus Λ. Then*

$$T_0(r,f) \le \sum_{j=1}^{3} \overline{N_0}(r, \frac{1}{f-\beta_j}) + S(r,f) + o(T_0(r,f)).$$

Proof. Set

$$F(z) = \frac{f(z)-\beta_1(z)}{f(z)-\beta_2(z)} \cdot \frac{\beta_3(z)-\beta_2(z)}{\beta_3(z)-\beta_1(z)}.$$

Then combining Lemma 2.4 and Lemma 2.5, we obtain the result. □

Lemma 2.7 ([10]). *Let $g : (0,\infty) \to R$ and $h : (0,\infty) \to R$ be monotone nondecreasing functions such that $g(r) \le h(r)$ outside an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists r_0 such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Lemma 2.8 ([6]). *Let $h_1(r)$ and $h_2(r)$ be monotonically increasing and real valued functions on $[0, R_0)$ such that $h_1(r) \le h_2(r)$ possibly outside an exceptional set $E \subset [0, R_0)$, for which $\int_E dr/(R_0 - r) < \infty$. Then there exists a constant $b \in (0, R_0)$ such that if $s(r) = 1 - b(1-r)$, then $h_1(r) \le h_2(s(r))$ for all $r \in [0, R_0)$.*

3. Proof of Theorem 1.1.

The idea of the proof is from [17,18] and [12,13].

Suppose that $f \not\equiv g$. Set

$$L(w) = \frac{\omega-\alpha_1}{\omega-\alpha_2} \cdot \frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}. \tag{3.1}$$

Let $F(z) = L(f(z)), G(z) = L(g(z)), \beta_j = L(\alpha_j), (j = 1, \cdots, 5)$. By (3.1) and Lemma 2.5, we get $\beta_1 = 0, \beta_2 = \infty, \beta_3 = 1, \beta_4 \not\equiv 0, \infty, 1, \quad \beta_5 \not\equiv 0, \infty, 1, \quad \beta_4 \not\equiv \beta_5$, and $\beta_1, \cdots, \beta_5$ are small functions with respect to F and G. By the assumption of Theorem 1.1 and (3.1), we know that F and G share $0, 1, \infty$ IM . Then by the second fundamental theorem on Λ, we get

$$\begin{aligned} T_0(r,F) &\le \overline{N_0}(r,\frac{1}{F}) + \overline{N_0}(r,\frac{1}{F-1}) + \overline{N_0}(r,F) + S(r,F) \\ &\le \overline{N_0}(r,\frac{1}{G}) + \overline{N_0}(r,\frac{1}{G-1}) + \overline{N_0}(r,G) + S(r,F) \\ &\le 3T_0(r,G) + S(r,F). \end{aligned} \tag{3.2}$$

Similarly, we have

$$T_0(r,G) \le 3T_0(r,F) + S(r,G). \tag{3.3}$$

Hence by (3.2) and (3.3), we get

$$S(r,F) = S(r,G). \tag{3.4}$$

We claim that at least three among $\overline{N_0}(r, \frac{1}{F-\beta_j})(j = 1, 2, 3, 4, 5)$ are not equal to $S(r, F) + o(T_0(r, F))$. Otherwise, by Lemma 2.6, we get

$$T_0(r, F) \le S(r, F) + o(T_0(r, F)). \tag{3.5}$$

By (3.5) and Lemma 2.7 and Lemma 2.8, we get

$$\limsup_{r\to\infty} \frac{T_0(r, f)}{\log r} < \infty, \quad 1 \le r < R_0 = +\infty,$$

or

$$\limsup_{r\to R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} < \infty, \quad 1 \le r < R_0 < +\infty.$$

which contradicts f is transcendental or admissible.

Without loss of generality, we assume that

$$\overline{N_0}(r, \frac{1}{F - \beta_5}) \ne S(r, F) + o(T_0(r, F)). \tag{3.6}$$

Set

$$H = \frac{F'(\beta_4' G - \beta_4 G')(F - G)}{F(F - 1)G(G - \beta_4)} - \frac{G'(\beta_4' F - \beta_4 F')(F - G)}{G(G - 1)F(F - \beta_4)}. \tag{3.7}$$

Then by (3.7), we get

$$H = \frac{(F - G)H_1}{F(F - 1)(F - \beta_4)G(G - 1)(G - \beta_4)}, \tag{3.8}$$

where

(3.9)

$$\begin{aligned} H_1 &= F'(\beta_4' G - \beta_4 G')(G - 1)(F - \beta_4) - G'(\beta_4' F - \beta_4 F')(F - 1)(G - \beta_4) \\ &= \beta_4' F F' G^2 - \beta_4' F F' G - \beta_4(\beta_4 - 1)F F' G' - \beta_4 \beta_4' F' G^2 + \beta_4 \beta_4' F' G \\ &\quad - \beta_4' F^2 G G' + \beta_4' F G G' + \beta_4(\beta_4 - 1)F' G G' + \beta_4 \beta_4' F^2 G' - \beta_4 \beta_4' F G'. \end{aligned}$$

Noting that $f \not\equiv g$, by (3.1), we have

$$F \not\equiv G. \tag{3.10}$$

We discuss the following two cases.

Case 1. $H \equiv 0$. By (3.7) and (3.10), we get

$$\frac{F'(\beta_4' G - \beta_4 G')}{(F - 1)(G - \beta_4)} \equiv \frac{G'(\beta_4' F - \beta_4 F')}{(G - 1)(F - \beta_4)}. \tag{3.11}$$

If β_4 is a constant, then by $\beta_4 \ne 1$ and (3.11), we get $F \equiv G$, which contradicts (3.10). So β_4 is not a constant. By (3.11), we get

$$\frac{F'(\beta_4' G - \beta_4 G')}{G'(\beta_4' F - \beta_4 F')} - 1 \equiv \frac{(F - 1)(G - \beta_4)}{(G - 1)(F - \beta_4)} - 1.$$

Hence we get

$$\frac{F'-G'}{F-G} \equiv \frac{(1-\beta_4)G'(\beta_4' F-\beta_4 F')}{\beta_4' G(G-1)(F-\beta_4)} + \frac{G'}{G}. \tag{3.12}$$

By (3.6), we know that there is a point z_0 such that z_0 is a common zero of $F-\beta_5$ and $G-\beta_5$, but is not a zero or a pole of $\beta_4, \beta_4', \beta_5, \beta_5-1, \beta_5-\beta_4$. It is obvious that z_0 is a pole of the left side of (3.12), and not a pole of the right side of (3.12), which is a contradiction.

Case 2. $H \not\equiv 0$. By (3.7), we get

$$\begin{aligned} H = {} & \frac{F'}{F-1}\cdot\frac{\beta_4' G-\beta_4 G'}{G(G-\beta_4)} - \left(\frac{F'}{F-1}-\frac{F'}{F}\right)\cdot\frac{\beta_4' G-\beta_4 G'}{G-\beta_4} \\ & -\left(\frac{G'}{G-1}-\frac{G'}{G}\right)\cdot\frac{\beta_4' F-\beta_4 F'}{F-\beta_4} + \frac{G'}{G-1}\cdot\frac{\beta_4' F-\beta_4 F'}{F(F-\beta_4)}. \end{aligned} \tag{3.13}$$

Since

$$\frac{\beta_4' G-\beta_4 G'}{G(G-\beta_4)} = \frac{G'}{G} - \frac{G'-\beta_4'}{G-\beta_4}, \quad \frac{\beta_4' G-\beta_4 G'}{G-\beta_4} = \beta_4' - \frac{\beta_4(G'-\beta_4')}{G-\beta_4}, \tag{3.14}$$

then by Lemma 2.3 and (3.4), we get

$$\begin{aligned} m_0\left(r, \frac{\beta_4' G-\beta_4 G'}{G(G-\beta_4)}\right) & \leq m_0\left(r, \frac{G'}{G}\right) + m_0\left(r, \frac{G'-\beta_4'}{G-\beta_4}\right) \\ & = S(r,F) + o(T_0(r,F)), \end{aligned} \tag{3.15}$$

$$\begin{aligned} m_0\left(r, \frac{\beta_4' G-\beta_4 G'}{G-\beta_4}\right) & \leq m_0(r,\beta_4') + m_0\left(r, \frac{\beta_4(G'-\beta_4')}{G-\beta_4}\right) \\ & = S(r,F) + o(T_0(r,F)). \end{aligned} \tag{3.16}$$

Combining (3.13),(3.15) and (3.16), we get

$$m_0(r,H) = S(r,F) + o(T_0(r,F)). \tag{3.17}$$

Next we estimate $N_0(r,H)$. By (3.7), we know that the poles of H only possibly occur from the zeros of $F, G, F-1, G-1, F-\beta_4$ and $G-\beta_4$, the poles of F, G and β_4. Let E_0 be the set of all zeros, 1-points and poles of β_4. We discuss the following four subcases.

Subcase 1. Suppose that z_1 is a zero of F with multiplicity m_1 and G with multiplicity n_1, but $z_1 \notin E_0$. Then by (3.9), we know that z_1 is a zero of H_1 with multiplicity at least m_1+n_1-1. Noting that z_1 is a zero of $F-G$ with multiplicity $\min\{m_1, n_1\}$, by (3.8), we deduce that z_1 is not a pole of H.

Subcase 2. Suppose that z_2 is a pole of F with multiplicity m_2 and G with multiplicity n_2, but $z_2 \notin E_0$. Then by (3.9), we know that z_2 is a pole of H_1 with multiplicity at most $2m_2+2n_2-1$. Noting that z_2 is a pole of $F-G$ with multiplicity at most $\max\{m_2, n_2\}$, by (3.8), we deduce that z_2 is not a pole of H.

Subcase 3. Suppose that z_3 is a zero of $F-1$ with multiplicity m_3 and $G-1$ with multiplicity n_3, but $z_3 \notin E_0$. Noting that z_3 is a zero of $F-G$ with multiplicity $\min\{m_3, n_3\}$, a simple pole of $\frac{F'}{F-1}$ and $\frac{G'}{G-1}$, by (3.7), we deduce that z_3 is not a pole of H.

Subcase 4. Suppose that z_4 is a zero of $F-\beta_4$ with multiplicity m_4 and $G-\beta_4$ with multiplicity n_4, but $z_4 \notin E_0$. By (3.14), we know that z_4 is a simple pole of $\frac{\beta_4' G-\beta_4 G'}{G(G-\beta_4)}$ and $\frac{\beta_4' F-\beta_4 F'}{F(F-\beta_4)}$. Noting that z_4 is a zero of $F-G$, by (3.7), we deduce that z_1 is not a pole of H.

From the above, we get

$$N_0(r, H) = o(T_0(r, F)). \tag{3.18}$$

Thus by (3.17) and (3.18), we get

$$T_0(r, H) = S(r, F) + o(T_0(r, F)). \tag{3.19}$$

Since F and G share β_5 IM, by (3.7) and (3.19), we get

$$\bar{N}_0(r, \frac{1}{F-\beta_5}) \le \overline{N_0}(r, \frac{1}{H}) \le S(r, F) + o(T_0(r, F)),$$

which contradicts (3.6). Theorem 1.1 is now completely proved.

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