ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 3, pp. 771–783

Title:

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Author(s):

I. Y. Lee, H. S. Chung and S. J. Chang

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 3, pp. 771–783
Online ISSN: 1735-8515

INTEGRATION FORMULAS FOR THE CONDITIONAL TRANSFORM INVOLVING THE FIRST VARIATION

I. Y. LEE, H. S. CHUNG AND S. J. CHANG*

(Communicated by Hamid Pezeshk)

Abstract. In this paper, we show that the conditional transform with respect to the Gaussian process involving the first variation can be expressed in terms of the conditional transform without the first variation. We then use this result to obtain various integration formulas involving the conditional $\diamond$-product and the first variation.
Keywords: Brownian motion process, Gaussian process, simple formula, conditional transform with respect to Gaussian process, conditional $\diamond$-product, first variation.
MSC(2010): Primary: 60J25; Secondary: 28C20.

1. **Introduction**

Let $C_0[0,T]$ denote the one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m_w denote Wiener measure. $(C_0[0,T], \mathcal{M}, m_w)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm_w(x).$$

A subset $\mathcal{B}$ of $C_0[0,T]$ is said to be scale-invariant measurable provided $\rho\mathcal{B}$ is $\mathcal{M}$-measurable for all $\rho > 0$, and a scale-invariant measurable set $\mathcal{N}$ is said to be a scale-invariant null set provided $m_w(\rho\mathcal{N}) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e) [10].

Recently [7], the authors studied the transform with respect to the Gaussian process. They then examined various relationships of the transform with respect to the Gaussian process, $\diamond$-the product, and the first variation. In [9], the

Article electronically published on June 15, 2015.
Received: 13 January 2013, Accepted: 2 May 2014.
*Corresponding author.

authors introduced the concept of a conditional transform with respect to the Gaussian process. They also examined various relationships of the conditional transform with respect to the Gaussian process, the conditional $\diamond$-product, and the first variation for functionals F in a Banach algebra S_α [8].

In this paper, we show that the conditional transform with respect to the Gaussian process involving the first variation of a functional F can be expressed in terms of the ordinary Wiener integral of F multiplied by a linear factor. We then establish various relationships involving the conditional transform, the conditional $\diamond$-product, and the first variation. We also show that the conditional transform with respect to the Gaussian process involving two or three concepts can be expressed in terms of the Wiener integral of one concept.

2. Definitions and preliminaries

In this section, we state definition and notations which are needed to understand this paper [6–8, 11].

For $h \in L^2[0,T]$, we define the Gaussian process Z_h by $Z_h(x,t) = \int_0^t h(s)\tilde{d}x(s)$ where $\int_0^t h(s)\tilde{d}x(s)$ denotes the PWZ integral. For each $v \in L^2[0,T]$, let $\langle v, x\rangle = \int_0^T v(t)\tilde{d}x(t)$. From [5], we note that $\langle v, Z_h(x,\cdot)\rangle = \langle vh, x\rangle$ for $h \in L_\infty[0,T]$ and s-a.e. $x \in C_0[0,T]$. Thus, throughout this paper, we require h to be in $L_\infty[0,T]$ rather than simply in $L^2[0,T]$.

For all $v \in L^2[0,T]$, let

$$B_v = \frac{1}{T}\int_0^T v(t)dt. \tag{2.1}$$

Let $K_0[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t=0$ and whose real and imaginary parts are elements of $C_0[0,T]$.

Now, we state the definitions of the transform with respect to the Gaussian process, the $\diamond$-product and the first variation.

Definition 2.1. *Let F and G be functionals on $K_0[0,T]$ and let γ, β, ρ and τ be non-zero complex numbers. Then the transform with respect to the Gaussian process, the $\diamond$-product and the first variation are defined by formulas*

$$(T^{h_1,h_2}_{\gamma,\beta}(F))(y) = \int_{C_0[0,T]} F\Big(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot)\Big)dm_w(x),$$

$$((F \diamond G)^{s_1,s_2}_{\rho,\tau})(y) = \int_{C_0[0,T]} F\Big(\tau Z_{s_2}(y,\cdot) + \rho Z_{s_1}(x,\cdot)\Big) \cdot G\Big(\tau Z_{s_2}(y,\cdot) - \rho Z_{s_1}(x,\cdot)\Big)dm_w(x),$$

$$\delta F\Big(Z_h(x,\cdot)|Z_s(w_0,\cdot)\Big) = \frac{\partial}{\partial k}F\Big(Z_h(x,\cdot) + kZ_s(w_0,\cdot)\Big)\Big|_{k=0}$$

if they exist.

Remark 2.2. *(1) When $h_1 = h_2 = 1$, $T^{1,1}_{\gamma,\beta}$ is the integral transform used by Kim and Skoug [12]. In particular, $T^{1,1}_{1,i}(F)$ is the Fourier-Wiener transform introduced by Cameron in [1]. Also, $T^{1,1}_{\sqrt{2},i}(F)$ is the modified Fourier-Wiener transform used by Cameron and Martin [2].*

(2) If $s_1 = s_2 = 1$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{1}{\sqrt{2\lambda}}$ for $\lambda \in \tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0$ and $Re(\lambda) \geq 0\}$, then the $\diamond$-product $(F \diamond G)^{s_1,s_2}_{\rho,\tau}$ coincides with convolution product $(F * G)_\lambda$ [4, 8]; that is to say, $(F \diamond G)^{s_1,s_2}_{\rho,\tau} = (F * G)_\lambda$ for $\lambda \in \tilde{\mathbb{C}}_+$.

(3) If $h = s = 1$, then the first variation of F with respect to the Gaussian process coincides with the first variation of F [6].

Next, we give the definition of the conditional transform with respect to the Gaussian process.

Let X be a $\mathbb{R}$-valued function on $C_0[0, T]$ whose probability distribution μ_X is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Let F be a $\mathbb{C}$-valued μ-integrable functional on $C_0[0, T]$. Then the conditional integral of F given X, denoted by $E[F\|X](\eta)$, is a Lebesgue measurable function of η, unique up to null sets in $\mathbb{R}$, satisfying the equation

$$\int_{X^{-1}(B)} F(x)dm_w(x) = \int_B E[F\|X](\eta)d\mu_X(\eta)$$

for all Borel sets B in $\mathbb{R}$.

We will always condition by

$$X(x) = x(T). \tag{2.2}$$

In [13], Park and Skoug gave a simple formula for expression conditional Wiener integrals in terms of an ordinary Wiener integrals by the formula

$$E[F\|X](\eta) = \int_{C_0[0,T]} F\left(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\right)dm_w(x).$$

Definition 2.3. *Let F and G be functionals defined on $K_0[0, T]$ and let X be given by equation (2.2). For each non-zero complex numbers γ, β, ρ and τ, we define the conditional transform with respect to the Gussian process $T^{h_1,h_2}_{\gamma,\beta}(F\|X)$ of F given X by the formula*

$$\begin{aligned} &T^{h_1,h_2}_{\gamma,\beta}(F\|X)(y,\eta) \\ &= \int_{C_0[0,T]} F\left(\gamma Z_{h_1}\left(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\right) + \beta Z_{h_2}(y, \cdot)\right)dm_w(x) \end{aligned} \tag{2.3}$$

and we define the conditional $\diamond$-product $((F \diamond G)^{s_1,s_2}_{\rho,\tau}\|X)(y, \eta)$ of F and G given X by the formula

$$\begin{aligned}
&((F\diamond G)^{s_1,s_2}_{\rho,\tau}\|X)(y,\eta)\\
&=\int_{C_0[0,T]} F\bigg(\tau Z_{s_2}(y,\cdot)+\rho Z_{s_1}\bigg(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\bigg)\bigg)\\
&\qquad\cdot G\bigg(\tau Z_{s_2}(y,\cdot)-\rho Z_{s_1}\bigg(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\bigg)\bigg)dm_w(x)
\end{aligned}\tag{2.4}$$

for $y\in K_0[0,T]$ and $\eta\in\mathbb{R}$, if they exist.

Remark 2.4. *(1)When $h_1=h_2=1$, $T^{1,1}_{\gamma,\beta}(F\|X)(y,\eta)$ is the conditional integral transform $\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta)$ introduced in [6].*

(2)When $s_1=s_2=1$, $\tau=\frac{1}{\sqrt{2}}$ and $\rho=\frac{\gamma}{\sqrt{2}}$, $((F\diamond G)^{1,1}_{\rho,\tau}\|X)(y,\eta)$ is the conditional convolution product $((F*G)_\gamma\|X)(y,\eta)$ introduced in [6].

3. Conditional transform involving the first variation

In this section, we obtain the fundamental result that the conditional transform with respect to the Gaussian process for the first variation of F can be expressed in terms of the ordinary Wiener integral of F multiplied by a linear factor.

The following lemma was established in [3, Theorem T].

Lemma 3.1. (Translation theorem) *Let $x_0(t)=\int_0^t z(s)ds$ for some $z\in L^2[0,T]$ and let F be an integrable functional on $C_0[0,T]$. Then*

$$\begin{aligned}
&\int_{C_0[0,T]} F(x+x_0)dm_w(x)\\
&=\exp\Big\{-\frac{1}{2}\int_0^T z^2(t)dt\Big\}\int_{C_0[0,T]} F(x)\exp\{\langle z,x\rangle\}dm_w(x).
\end{aligned}\tag{3.1}$$

The next lemma plays a key role in obtaining Theorem 3.3 below.

Lemma 3.2. *Let F be an integrable functional on $K_0[0,T]$ and let $\frac{1}{h},\frac{1}{s}\in L_\infty[0,T]$. Then*

$$\begin{aligned}
&\int_{C_0[0,T]} F\Big(Z_h\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+Z_s(w_0,\cdot)\Big)dm_w(x)\\
&=\exp\Big\{-\frac{1}{2}\int_0^T (z(t)-B_z)^2dt\Big\}\\
&\qquad\cdot\int_{C_0[0,T]} F\Big(Z_h\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)\Big)\exp\{\langle z-B_z,x\rangle\}dm_w(x)
\end{aligned}$$

for $\eta\in\mathbb{R}$ where $w_0(t)=\int_0^t\frac{h(u)}{s(u)}(z(u)-B_z)du$ for some $z\in L^2[0,T]$ and B_z is given by equation (2.1).

Proof. The proof is straightforward by applying the translation theorem (Lemma 3.1) with $G(x) = F\Big(Z_h\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big)\Big)$ and $x_0(t) = \int_0^t \frac{s(u)}{h(u)} dw_0(u)$. □

The following theorem tells us that the conditional transform involving the first variation of a functional F can be expressed in terms of the ordinary Wiener integral of F multiplied by a linear factor.

Theorem 3.3. *Let F, h and s be as in Lemma 3.2 and let $\frac{1}{h_1} \in L_\infty[0,T]$. Let X be given by equation (2.2). Assume that $\delta F(Z_h(x,\cdot)| Z_s(w_0,\cdot))$ is m_w-integrable on $K_0[0,T]$. Then for all non-zero complex numbers γ and β*

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\|X\Big)(y,\eta) \\ &= \frac{1}{\gamma}\int_{C_0[0,T]} \langle z - B_z, x\rangle \\ &\quad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big) + \beta Z_{hh_2}(y,\cdot)\Big) dm_w(x) \end{aligned} \tag{3.2}$$

for $y \in K_0[0,T]$ and $\eta \in \mathbb{R}$ where $w_0(t) = \int_0^t \frac{h(u)h_1(u)}{s(u)}(z(u) - B_z)du$ for some $z \in L^2[0,T]$ and B_z is given by the equation (2.1). Furthermore, the last expression in equation (3.2) above can be expressed by the formula

$$\frac{1}{\gamma^2} T_{\gamma,\beta}^{hh_1,hh_2}(H\|X)(y,\eta) - \frac{\beta}{\gamma^2}\Big\langle \frac{h_2}{h_1}(z - B_z), y\Big\rangle T_{\gamma,\beta}^{hh_1,hh_2}(F\|X)(y,\eta) \tag{3.3}$$

for $y \in K_0[0,T]$ and $\eta \in \mathbb{R}$ where $H(\cdot) = \langle \frac{z - B_z}{hh_1}, \cdot\rangle F(\cdot)$.

Proof. By using the equation (2.3) and Lemma 3.2, we have

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\|X\Big)(y,\eta) \\ &= \frac{\partial}{\partial k}\Big[\exp\Big\{-\frac{k^2}{2\gamma^2}\int_0^T (z(u) - B_z)^2 du\Big\} \\ &\qquad \cdot \int_{C_0[0,T]} F\Big(\gamma Z_{hh_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big) + \beta Z_{hh_2}(y,\cdot)\Big) \\ &\qquad\qquad \cdot \exp\Big\{\frac{k}{\gamma}\langle z - B_z, x\rangle\Big\} dm_w(x)\Big]\Big|_{k=0} \\ &= \frac{1}{\gamma}\int_{C_0[0,T]} \langle z - B_z, x\rangle \\ &\qquad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta, \cdot\Big) + \beta Z_{hh_2}(y,\cdot)\Big) dm_w(x). \end{aligned}$$

Hence we obtain equation (3.2). To establish equation (3.3), note that

$$
\begin{aligned}
&\frac{1}{\gamma}\int_{C_0[0,T]}\langle z-B_z, x\rangle \\
&\qquad\qquad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)dm_w(x) \\
&=\frac{1}{\gamma^2}\int_{C_0[0,T]}\Big\langle \frac{z-B_z}{hh_1}, \gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big\rangle \\
&\qquad\qquad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)dm_w(x) \\
&\quad+\frac{1}{\gamma^2}\int_{C_0[0,T]}\Big\langle \frac{z-B_z}{hh_1}, \gamma Z_{hh_1}\Big(\frac{\cdot}{T}x(T),\cdot\Big)\Big\rangle \\
&\qquad\qquad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)dm_w(x) \\
&\quad-\frac{1}{\gamma^2}\int_{C_0[0,T]}\Big\langle \frac{z-B_z}{hh_1}, \gamma Z_{hh_1}\Big(\frac{\cdot}{T}\eta,\cdot\Big)\Big\rangle \\
&\qquad\qquad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)dm_w(x) \\
&\quad-\frac{1}{\gamma^2}\int_{C_0[0,T]}\Big\langle \frac{z-B_z}{hh_1}, \beta Z_{hh_2}(y,\cdot)\Big\rangle \\
&\qquad\qquad \cdot F\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)dm_w(x) \\
&=\frac{1}{\gamma^2}T_{\gamma,\beta}^{hh_1,hh_2}(H\|X)(y,\eta)-\frac{\beta}{\gamma^2}\Big\langle\frac{h_2}{h_1}(z-B_z),y\Big\rangle T_{\gamma,\beta}^{hh_1,hh_2}(F\|X)(y,\eta)
\end{aligned}
$$

since

$$\Big\langle \frac{z-B_z}{hh_1}, \gamma Z_{hh_1}\Big(\frac{\cdot}{T}x(T),\cdot\Big)\Big\rangle = \frac{\gamma x(T)}{T}\int_0^T (z(t)-B_z)dt = 0$$

and

$$\Big\langle \frac{z-B_z}{hh_1}, \gamma Z_{hh_1}\Big(\frac{\cdot}{T}\eta,\cdot\Big)\Big\rangle = \frac{\gamma\eta}{T}\int_0^T (z(t)-B_z)dt = 0.$$

Thus we have the desired results. □

In our next theorem we obtain an integration by parts formula for the conditional transform with respect to the Gaussian process.

Theorem 3.4. *Let F, X, h and w_0 be as in Theorem 3.3. Let G be a complex-valued Borel measurable functional on $K_0[0,T]$. Assume that* $T_{\gamma,\beta}^{h_1,h_2}(\delta(FG)\|X)$

exists. Then

(3.4)

$$
\begin{aligned}
T_{\gamma,\beta}^{h_1,h_2}\Big(F\Big(Z_h(\cdot,\cdot)\Big)\delta G\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\Big)\\
+\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)G\Big(Z_h(\cdot,\cdot)\Big)\|X\Big)(y,\eta)\\
=\frac{1}{\gamma}\int_{C_0[0,T]}\langle z-B_z,x\rangle F\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)\\
\cdot G\Big(\gamma Z_{hh_1}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta,\cdot\Big)+\beta Z_{hh_2}(y,\cdot)\Big)dm_w(x)
\end{aligned}
$$

for $y\in K_0[0,T]$ and $\eta\in\mathbb{R}$. Furthermore, the last expression in the equation (3.4) *above can be expressed by the formula*

$$
\frac{1}{\gamma^2}T_{\gamma,\beta}^{hh_1,hh_2}(H_1\|X)(y,\eta)-\frac{\beta}{\gamma^2}\Big\langle\frac{h_2}{h_1}(z-B_z),y\Big\rangle T_{\gamma,\beta}^{hh_1,hh_2}(FG\|X)(y,\eta)
$$

where $H_1(\cdot)=\langle\frac{z-B_z}{hh_1},\cdot\rangle F(\cdot)G(\cdot)$.

Proof. The proof is straightforward by replacing F with $K=FG$ in Theorem 3.3. □

4. Relationships involving the conditional transform, the conditional ⋄-product and the first variation

In this section, we obtain all possible relationships involving the conditional transform, the conditional ⋄-product, and the first variation.

The following two lemmas play a key role in obtaining the results of this section.

Lemma 4.1. *[4, Theorem 3.5] Let F be an integrable functional on $K_0[0,T]$. Then for all non-zero complex numbers γ and β*

$$
\begin{aligned}
&\int_{C_0^2[0,T]}F\Big(Z_h(\gamma w+\beta z,\cdot)\Big)d(m_w\times m_w)(w,z)\\
&=\int_{C_0[0,T]}F\Big(\sqrt{\gamma^2+\beta^2}Z_h(x,\cdot)\Big)dm_w(x).
\end{aligned}
$$

Lemma 4.2. *Let F be as in Theorem 3.3. Let X be given by the equation* (2.2)*. Let h, s, l, m and $h_j(j=1,2,3,4)$ satisfy the following conditions:*

(1) $h_3(t)=h(t)h_1(t)$
(2) $l(t)h_4(t)=h(t)h_2(t)$
(3) $m(t)h_4(t)=s(t)$ *on* $[0,T]$.

Then for all non-zero complex numbers γ *and* β*,*

$$T^{h_1,h_2}_{\gamma,\beta}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\|X\Big)(y,\eta) = \delta T^{h_3,h_4}_{\gamma,\beta}(F\|X)\Big(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(w_0,\cdot),\eta\Big)$$

for $y\in K_0[0,T]$ *and* $\eta\in\mathbb{R}$*.*

For a more detailed study of this, see [9].

Now we show that the conditional transform with respect to the Gaussian process of the conditional $\diamond$-product is a product of their conditional transforms.

Theorem 4.3. *Let* F *and* G *be complex-valued Borel measurable functionals on* $K_0[0,T]$*. Let* X *be given by the equation* (2.2)*. Assume that* $\tau\gamma=\rho$*,* $h_1(t)s_2(t)=s_1(t)$ *on* $[0,T]$ *and* $T^{h_1,h_2}_{\gamma,\beta}(((F\diamond G)^{s_1,s_2}_{\rho,\tau}\|X)(\cdot,\eta_1)\|X)(\cdot,\eta_2)$ *exists. Then for all non-zero complex numbers* γ*,* β*,* ρ *and* τ

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}(((F\diamond G)^{s_1,s_2}_{\rho,\tau}\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\ &= T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(F\|X)\Big(y,\frac{\eta_2+\eta_1}{\sqrt{2}}\Big)T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(G\|X)\Big(y,\frac{\eta_2-\eta_1}{\sqrt{2}}\Big)\end{aligned} \tag{4.1}$$

for $y\in K_0[0,T]$ *and* $\eta_1,\eta_2\in\mathbb{R}$*.*

Proof. By using the equations (2.3) and (2.4), we have

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}(((F\diamond G)^{s_1,s_2}_{\rho,\tau}\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\
&=\int_{C_0^2[0,T]}F\Big(\tau\gamma Z_{h_1s_2}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_2,\cdot\Big)\\
&\qquad\qquad+\rho Z_{s_1}\Big(w(\cdot)-\frac{\cdot}{T}w(T)+\frac{\cdot}{T}\eta_1,\cdot\Big)+\tau\beta Z_{h_2s_2}(y,\cdot)\Big)\\
&\quad\cdot G\Big(\tau\gamma Z_{h_1s_2}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_2,\cdot\Big)\\
&\quad-\rho Z_{s_1}\Big(w(\cdot)-\frac{\cdot}{T}w(T)+\frac{\cdot}{T}\eta_1,\cdot\Big)+\tau\beta Z_{h_2s_2}(y,\cdot)\Big)d(m_w\times m_w)(w,x)\\
&=\int_{C_0^2[0,T]}F\Big(\rho Z_{s_1}\Big(x(\cdot)+w(\cdot)-\frac{\cdot}{T}(x(T)+w(T))\\
&\qquad\qquad+\frac{\cdot}{T}(\eta_2+\eta_1),\cdot\Big)+\tau\beta Z_{h_2s_2}(y,\cdot)\Big)d(m_w\times m_w)(w,x)\\
&\quad\cdot\int_{C_0^2[0,T]}G\Big(\rho Z_{s_1}\Big(x(\cdot)-w(\cdot)-\frac{\cdot}{T}(x(T)-w(T))\\
&\qquad\qquad+\frac{\cdot}{T}(\eta_2-\eta_1),\cdot\Big)+\tau\beta Z_{h_2s_2}(y,\cdot)\Big)d(m_w\times m_w)(w,x).\end{aligned}$$

Since $\tau\gamma=\rho$ and $h_1(t)s_2(t)=s_1(t)$ on $[0,T]$, the processes

$$\tau\gamma Z_{h_1s_2}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_2,\cdot\Big)+\rho Z_{s_1}\Big(w(\cdot)-\frac{\cdot}{T}w(T)+\frac{\cdot}{T}\eta_1,\cdot\Big)$$

and

$$\tau\gamma Z_{h_1 s_2}\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_2,\cdot\Big)-\rho Z_{s_1}\Big(w(\cdot)-\frac{\cdot}{T}w(T)+\frac{\cdot}{T}\eta_1,\cdot\Big)$$

are independent processes. So, the second equality follows. Hence applying Lemma 4.1 to the last expression in the above equation, we can easily obtain the equation (4.1). □

We now consider relationships involving the conditional transform with respect to the Gaussian process, the conditional $\diamond$-product, and the first variation. Here we simply state the formulas, and no proofs are included.

In the following formula, the double conditional transform for the first variation of F can be calculated from the conditional transform of F without the first variation.

Formula 1. Let F, X, l, m and $h_j (j=1,2,3,4)$ be as in Lemma 4.2. Assume that $T^{h_1,h_2}_{\gamma,\beta}(T^{h_1,h_2}_{\gamma,\beta}(\delta F\|X)(\cdot,\eta_1)\|X)(\cdot,\eta_2)$ exists. Then for all non-zero complex numbers γ and β

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}\Big(T^{h_1,h_2}_{\gamma,\beta}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\|X\Big)(\cdot,\eta_1)\|X\Big)(y,\eta_2)\\&=\frac{1}{\gamma^2}T^{lh_1,lh_2}_{\gamma,\beta}(H_3\|X)(y,\eta_2)\\&\qquad-\frac{\beta}{\gamma^2}\Big\langle\frac{h_2}{h_1}(z-B_z),y\Big\rangle T^{lh_1,lh_2}_{\gamma,\beta}(T^{h_3,h_4}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\end{aligned}$$

for $y\in K_0[0,T]$ and $\eta_1,\eta_2\in\mathbb{R}$ where $w_0(u)=\beta\int_0^u\frac{l(t)h_1(t)}{m(t)}(z(t)-B_z)dt$ and $H_3(\cdot)=\langle\frac{z-B_z}{lh_1},\cdot\rangle T^{h_3,h_4}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)$.

In the following formula, the conditional transform for the conditional $\diamond$-product with respect to the first variation of F and G can be calculated from the conditional transform of F and G without the first variation and conditional $\diamond$-product.

Formula 2. Let F, X, h, h_1 and s be as in Theorem 3.3. Assume that $T^{h_1,h_2}_{\gamma,\beta}(((\delta F\diamond\delta G)^{s_1,s_2}_{\rho,\tau}\|X)(\cdot,\eta_1)\|X)(\cdot,\eta_2)$ exists and $\tau\gamma=\rho$ and $h_1(t)s_2(t)=s_1(t)$ on $[0,T]$. Then for all non-zero complex numbers γ, β, ρ and τ

$$\begin{aligned}&T^{h_1,h_2}_{\gamma,\beta}\Big(\Big(\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\diamond\delta G\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\Big)^{s_1,s_2}_{\rho,\tau}\|X\Big)(\cdot,\eta_1)\|X\Big)(y,\eta_2)\\&=\Big[\tfrac{1}{2\rho^2}T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(H_4\|X)\Big(y,\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)-\tfrac{\tau\beta}{\gamma^2}\Big\langle\tfrac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(F\|X)\Big(y,\tfrac{\eta_2+\eta_1}{\sqrt{2}}\Big)\Big]\\&\cdot\Big[\tfrac{1}{2\rho^2}T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(H_5\|X)\Big(y,\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big)-\tfrac{\tau\beta}{\gamma^2}\Big\langle\tfrac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(G\|X)\Big(y,\tfrac{\eta_2-\eta_1}{\sqrt{2}}\Big)\Big]\end{aligned}$$

for $y\in K_0[0,T]$ and $\eta_1,\eta_2\in\mathbb{R}$ where $w_0(u)=\int_0^u\frac{h(t)s_1(t)}{s(t)}(z(t)-B_z)dt$, $H_4(\cdot)=\langle\frac{z-B_z}{hs_1},\cdot\rangle F(\cdot)$ and $H_5(\cdot)=\langle\frac{z-B_z}{hs_1},\cdot\rangle G(\cdot)$.

In the following formula, the conditional transform for the conditional $\diamond$-product with respect to the first variation of the conditional transforms can be calculated from the conditional transform of F and G without the first variation and conditional $\diamond$-product.

Formula 3. Let F, X, h, h_1 and s be as in Theorem 3.3. Assume that $T^{h_1,h_2}_{\gamma,\beta}(((\delta T^{h_1,h_2}_{\gamma,\beta}(F\|X)\diamond\delta T^{h_1,h_2}_{\gamma,\beta}(G\|X))^{s_1,s_2}_{\rho,\tau}\|X)(\cdot,\eta_3)\|X)(\cdot,\eta_4)$ exists and $\tau\gamma=\rho$ and $h_1(t)s_2(t)=s_1(t)$ on $[0,T]$. Then for all non-zero complex numbers γ, β, ρ and τ

$$
\begin{aligned}
&T^{h_1,h_2}_{\gamma,\beta}\Big(\Big(\Big(\delta T^{h_1,h_2}_{\gamma,\beta}(F\|X)\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot),\eta_1\Big)\\
&\qquad\diamond\,\delta T^{h_1,h_2}_{\gamma,\beta}(G\|X)\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot),\eta_2\Big)\Big)^{s_1,s_2}_{\rho,\tau}\|X\Big)(\cdot,\eta_3)\|X\Big)(y,\eta_4)\\
&=\Big[\frac{1}{2\rho^2}T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(H_6\|X)\Big(y,\frac{\eta_4+\eta_3}{\sqrt{2}}\Big)\\
&-\frac{\tau\beta}{2\rho^2}\Big\langle\frac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(T^{h_1,h_2}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)\|X)\Big(y,\frac{\eta_4+\eta_3}{\sqrt{2}}\Big)\Big]\\
&\cdot\Big[\frac{1}{2\rho^2}T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(H_7\|X)\Big(y,\frac{\eta_4+\eta_3}{\sqrt{2}}\Big)\\
&-\frac{\tau\beta}{2\rho^2}\Big\langle\frac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(T^{h_1,h_2}_{\gamma,\beta}(G\|X)(\cdot,\eta_2)\|X)\Big(y,\frac{\eta_4-\eta_3}{\sqrt{2}}\Big)\Big]
\end{aligned}
$$

for $y\in K_0[0,T]$ and $\eta_1,\eta_2\in\mathbb{R}$ where $H_6(\cdot)=\langle\frac{z-B_z}{hs_1},\cdot\rangle T^{h_1,h_2}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)$, $H_7(\cdot)=\langle\frac{z-B_z}{hs_1},\cdot\rangle T^{h_1,h_2}_{\gamma,\beta}(G\|X)(\cdot,\eta_2)$ and $w_0(u)=\int_0^u\frac{h(t)s_1(t)}{s(t)}(z(t)-B_z)dt$.

The following simple examples illustrate the Formulas $1-3$ in this section.

Let $F,G:K_0[0,T]\to\mathbb{R}$ be defined by the formulas

$$
F(x)=\langle u,x\rangle,\quad G(x)=\langle v,x\rangle\qquad\text{for}\quad u,v\in L^2[0,T]. \tag{4.2}
$$

Then for all non-zero complex numbers γ, β, ρ and τ, direct calculations show that

$$
T^{h_1,h_2}_{\gamma,\beta}(F\|X)(y,\eta)=\beta\langle uh_2,y\rangle+\gamma\eta B_{uh_1}, \tag{4.3}
$$

$$
\begin{aligned}
&T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\
&=\beta_1\beta_2\langle uh_2h_4,y\rangle+\gamma_1\beta_2\eta_2B_{uh_1h_4}+\gamma_2\eta_1B_{uh_3},
\end{aligned} \tag{4.4}
$$

(4.5)

$$
\begin{aligned}
&T^{h_1,h_2}_{\gamma,\beta}(\langle z,\cdot\rangle F(\cdot)\|X)(y,\eta)\\
&=\beta^2\langle zh_2,y\rangle\langle uh_2,y\rangle+\gamma\beta\eta B_{zh_1}\langle uh_2,y\rangle+\gamma\beta\eta B_{uh_1}\langle zh_2,y\rangle\\
&\quad+\gamma^2\eta^2B_{zh_1}B_{uh_1}+\gamma^2\int_0^T\Big(z(t)h_1(t)-B_{zh_1}\Big)\Big(u(t)h_1(t)-B_{uh_1}\Big)dt
\end{aligned}
$$

and
(4.6)

$$
\begin{aligned}
&T^{h_1,h_2}_{\gamma_1,\beta_1}(\langle z,\cdot\rangle T^{h_3,h_4}_{\gamma_2,\beta_2}(F\|X)(\cdot,\eta_1))(y,\eta_2)\\
&= \beta_1^2\beta_2\langle zh_2,y\rangle\langle uh_2h_4,y\rangle + \gamma_1\beta_1\beta_2\eta_2(B_{zh_1}\langle uh_2h_4,y\rangle + B_{uh_1h_4}\langle zh_2,y\rangle)\\
&\quad + \gamma_2\beta_1\eta_1B_{uh_3}\langle zh_2,y\rangle + \gamma_1^2\beta_2\eta_2^2B_{zh_1}B_{uh_1h_4} + \gamma_1\gamma_2\eta_1\eta_2B_{zh_1}B_{uh_3}\\
&\quad + \gamma_1^2\beta_2\int_0^T\Big(z(t)h_1(t)-B_{zh_1}\Big)\Big(u(t)h_1(t)h_4(t)-B_{uh_1h_4}\Big)dt.
\end{aligned}
$$

Example for the Formula 1. Let F be given by the equation (4.2). Then by using the equations (4.3), (4.4) and (4.6), it follows that

$$
\begin{aligned}
&\frac{1}{\gamma^2}T^{lh_1,lh_2}_{\gamma,\beta}(H_3\|X)(y,\eta_2)\\
&\qquad\qquad -\frac{\beta}{\gamma^2}\Big\langle\frac{h_2}{h_1}(z-B_z),y\Big\rangle T^{lh_1,lh_2}_{\gamma,\beta}(T^{h_3,h_4}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\
&= \beta\int_0^T u(t)h_2(t)h_3(t)\Big(z(t)-B_z\Big)dt
\end{aligned}
$$

and hence from Formula 1

$$
\begin{aligned}
&T^{h_1,h_2}_{\gamma,\beta}\Big(T^{h_1,h_2}_{\gamma,\beta}\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\|X\Big)(\cdot,\eta_1)\|X\Big)(y,\eta_2)\\
&= \beta\int_0^T u(t)h_2(t)h_3(t)\Big(z(t)-B_z\Big)dt.
\end{aligned}
$$

Example for the Formula 2. Let F and G be given by the equation (4.2). Then by using the equations (4.3) and (4.5), it follows that

$$
\begin{aligned}
&\Big[\frac{1}{2\rho^2}T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(H_4\|X)\Big(y,\frac{\eta_2+\eta_1}{\sqrt{2}}\Big)\\
&\qquad\qquad -\frac{\tau\beta}{\gamma^2}\Big\langle\frac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(F\|X)\Big(y,\frac{\eta_2+\eta_1}{\sqrt{2}}\Big)\Big]\\
&\quad\cdot\Big[\frac{1}{2\rho^2}T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(H_5\|X)\Big(y,\frac{\eta_2-\eta_1}{\sqrt{2}}\Big)\\
&\qquad\qquad -\frac{\tau\beta}{\gamma^2}\Big\langle\frac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T^{hs_1,hh_2s_2}_{\sqrt{2}\rho,\tau\beta}(G\|X)\Big(y,\frac{\eta_2-\eta_1}{\sqrt{2}}\Big)\Big]\\
&= \int_0^T u(t)h(t)s_1(t)\Big(z(t)-B_z\Big)dt\int_0^T v(t)h(t)s_1(t)\Big(z(t)-B_z\Big)dt
\end{aligned}
$$

and hence from Formula 2

$$T_{\gamma,\beta}^{h_1,h_2}\Big(\Big(\Big(\delta F\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big) \\ \diamond\, \delta G\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot)\Big)\Big)_{\rho,\tau}^{s_1,s_2}\|X\Big)(\cdot,\eta_1)\|X\Big)(y,\eta_2) \\ = \int_0^T u(t)h(t)s_1(t)\Big(z(t)-B_z\Big)dt\int_0^T v(t)h(t)s_1(t)\Big(z(t)-B_z\Big)dt.$$

Example for the Formula 3. Let F and G be given by the equation (4.2). Then by using the equations (4.3), (4.4) and (4.6), it follows that

$$\Big[\frac{1}{2\rho^2}T_{\sqrt{2}\rho,\tau\beta}^{hs_1,hh_2s_2}(H_6\|X)\Big(y,\frac{\eta_4+\eta_3}{\sqrt{2}}\Big) \\ -\frac{\tau\beta}{2\rho^2}\Big\langle\frac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T_{\sqrt{2}\rho,\tau\beta}^{hs_1,hh_2s_2}(T_{\gamma,\beta}^{h_1,h_2}(F\|X)(\cdot,\eta_1)\|X)\Big(y,\frac{\eta_4+\eta_3}{\sqrt{2}}\Big)\Big] \\ \cdot\Big[\frac{1}{2\rho^2}T_{\sqrt{2}\rho,\tau\beta}^{hs_1,hh_2s_2}(H_7\|X)\Big(y,\frac{\eta_4+\eta_3}{\sqrt{2}}\Big) \\ -\frac{\tau\beta}{2\rho^2}\Big\langle\frac{h_2s_2}{s_1}(z-B_z),y\Big\rangle T_{\sqrt{2}\rho,\tau\beta}^{hs_1,hh_2s_2}(T_{\gamma,\beta}^{h_1,h_2}(G\|X)(\cdot,\eta_2)\|X)\Big(y,\frac{\eta_4-\eta_3}{\sqrt{2}}\Big)\Big] \\ =\beta^2\int_0^T u(t)h(t)h_2(t)s_1(t)\Big(z(t)-B_z\Big)dt \\ \cdot\int_0^T v(t)h(t)h_2(t)s_1(t)\Big(z(t)-B_z\Big)dt$$

and hence from Formula 3

$$T_{\gamma,\beta}^{h_1,h_2}\Big(\Big(\Big(\delta T_{\gamma,\beta}^{h_1,h_2}(F\|X)\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot),\eta_1\Big) \\ \diamond\, \delta T_{\gamma,\beta}^{h_1,h_2}(G\|X)\Big(Z_h(\cdot,\cdot)|Z_s(w_0,\cdot),\eta_2\Big)\Big)_{\rho,\tau}^{s_1,s_2}\|X\Big)(\cdot,\eta_3)\|X\Big)(y,\eta_4) \\ =\beta^2\int_0^T u(t)h(t)h_2(t)s_1(t)\Big(z(t)-B_z\Big)dt \\ \cdot\int_0^T v(t)h(t)h_2(t)s_1(t)\Big(z(t)-B_z\Big)dt.$$

Acknowledgments

The authors thank the referees for their helpful suggestions which led to the present version of this paper.

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(Il Yong Lee) Department of Mathematics, Dankook University, Cheonan 330-714, Korea
E-mail address: iylee@dankook.ac.kr

(Hyun Soo Chung) Department of Mathematics, Dankook University, Cheonan 330-714, Korea
E-mail address: hschung@dankook.ac.kr

(Seung Jun Chang) Department of Mathematics, Dankook University, Cheonan 330-714, Korea
E-mail address: sejchang@dankook.ac.kr