ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 4, pp. 945–954

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 4, pp. 945–954
Online ISSN: 1735-8515

ANALYTIC EXTENSION OF A NTH ROOTS OF M-HYPONORMAL OPERATOR

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(Communicated by Behzad Djafari-Rouhani)

Abstract. In this paper, we study some properties of analytic extension of a nth roots of M-hyponormal operator. We show that every analytic extension of a nth roots of M-hyponormal operator is subscalar of order $2k+2n$. As a consequence, we get that if the spectrum of such operator T has a nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace. Finally, we show that the sum of a nth roots of M-hyponormal operator and an algebraic operator of order k which are commuting is subscalar of order $2kn+2$.
Keywords: nth roots of M-hyponormal operator, Bishop's property (β), subscalar operator, invariant subspace.
MSC(2010): Primary: 47B20; Secondary: 47A15.

1. Introduction and Preliminaries

Let H be a complex separable Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. If $T \in B(H)$, we shall write $R(T)$ for the range space of T.

One of recent trends in operator theory is studying natural extensions of hyponormal operators. So we introduce some of these non-hyponormal operators. The operator T is said to be M-hyponormal if there exists a real positive number M such that

$$M^2(T-\lambda)^*(T-\lambda) \geq (T-\lambda)(T-\lambda)^* \text{ for all } \lambda \in \mathbb{C}.$$

It is known that the class of M-hyponormal operators contains the class of hyponormal operators. There is a vast literature concerning M-hyponormal operators (see [3, 10, 12], etc.). We say that an operator $T \in B(H)$ is a nth roots of M-hyponormal operator, if T^n is an M-hyponormal operator for some positive integer n. In Example 2.1, we give an example of a nth roots of

Article electronically published on August 16, 2015.
Received: 22 September 2013, Accepted: 17 June 2014.
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M-hyponormal operator which is not an M-hyponormal operator. Therefore, this class gives good reasons for the future studied. In order to generalize these classes we introduce analytic extension of a nth roots of M-hyponormal operator defined as follows:

Definition 1.1 An operator $T \in B(H_1 \oplus H_2)$ is said to be an analytic extension of a nth roots of M-hyponormal operator, if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $H_1 \oplus H_2$ where T_1 is a nth roots of M-hyponormal operator and $F(T_3) = 0$ for a nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$.

Let z be the coordinate in the complex plane $\mathbb{C}$ and let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of $\mathbb{C}$. We shall denote by $L^2(U, H)$ the Hilbert space of measurable functions $f : U \to H$, such that

$$||f||_{2,U} = (\int_U ||f(z)||^2 d\mu(z))^{\frac{1}{2}} < \infty.$$

The Bergman space for U is defined by $A^2(U, H) = L^2(U, H) \cap O(U, H)$ where $O(U, H)$ denotes the Fréchet space of H-valued analytic functions on U with respect to uniform topology. Note that $A^2(U, H)$ is a Hilbert space. Now we define a special Sobolev type space. Let U be a bounded open subset of $\mathbb{C}$ and m be a fixed nonnegative integer. The vector valued Sobolev space $W^m(U, H)$ with respect to $\overline{\partial}$ and of order m will be the space of those functions $f \in L^2(U, H)$ whose derivatives $\overline{\partial} f, \ldots, \overline{\partial}^m f$ in the sense of distributions still belong to $L^2(U, H)$. Endowed with the norm

$$||f||^2_{W^m} = \sum_{i=0}^{m} ||\overline{\partial}^i f||^2_{2,U}$$

$W^m(U, H)$ becomes a Hilbert space contained continuously in $L^2(U, H)$. A bounded linear operator S on H is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to B(H)$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. Let U be a (connected) bounded open subset of $\mathbb{C}$ and let m be a non-negative integer. The linear operator M_f of multiplication by f on $W^m(U, H)$ is continuous and it has a spectral distribution of order m, defined by the functional calculus

$$\Phi_M : C_0^m(\mathbb{C}) \to B(W^m(U, H)), \, \Phi_M(f) = M_f.$$

Therefore, M_z is a scalar operator of order m.

An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G.

An operator $T \in B(H)$ is said to be analytic if there exists a nonconstant analytic function F on a neighborhood of $\sigma(T)$ such that $F(T) = 0$. We say that an operator $T \in B(H)$ is algebraic if there is a nonconstant polynomial p such that $p(T) = 0$. If an operator $T \in B(H)$ is analytic, then $F(T) = 0$ for some nonconstant analytic function F on a neighborhood D of $\sigma(T)$. Since F cannot have infinitely many zeros in D, we write $F(z) = G(z)p(z)$ where G is a function that is analytic and does not vanish on D and p is a nonconstant polynomial with zeros in D. By Riesz-Dunford calculus, $G(T)$ is invertible and then $p(T) = 0$, which means that T is algebraic (see [2]). When p has degree k, we say that T is analytic with order k throughout this paper.

In 1984, Putinar showed in [11] that every hyponormal operator is subscalar, and then in 1987, Brown used this result to prove that a hyponormal operator with rich spectrum has a nontrivial invariant subspace (see [1]). There have been a lot of generalizations of such beautiful consequences (see [5–8]). In this paper, we study various properties of analytic extension of a nth roots of M-hyponormal operator. We show that every analytic extension of a nth roots of M-hyponormal operator is subscalar of order $2k + 2n$. As a consequence, we get that if the spectrum of such operator has a nonempty interior in $\mathbb{C}$, then it has a nontrivial invariant subspace. Finally, we show that the sum of a nth roots of M-hyponormal operator and an algebraic operator of order k which are commuting is subscalar of order $2kn + 2$.

2. Analytic Extension of a nth Roots of M-hyponormal Operator

In this section we show that every analytic extension of a nth roots of M-hyponormal operator has a scalar extension. For this we start with an example of a nth roots of M-hyponormal operator which is not an M-hyponormal operator.

Example 2.1. If $T \neq 0$, consider the following operator matrix

$$B = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}.$$

Then B is a nth roots of M-hyponormal operator, but it is easy to show B is not an M-hyponormal operator.

Lemma 2.1. (See [8, Theorem 3.1].) *For a bounded disk D in the complex plane $\mathbb{C}$, there is a constant C_D such that for an arbitrary operator $T \in B(H)$*

and $f \in W^{2k}(D, H)$, we have

$$||(I-P)f||_{2,D} \leq C_D \sum_{i=k}^{2k} ||(T-z^k)^* \overline{\partial}^i f||_{2,D}$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

Lemma 2.2. *[11] Let $T \in B(H)$ be a hyponormal operator and let D be a bounded disk in $\mathbb{C}$. If $\{f_n\}$ is a sequence in $W^m(D, H)(m > 2)$ such that*

$$\lim_{n\to\infty} ||(z-T)\overline{\partial}^i f_n||_{2,D} = 0$$

for $i = 1, 2, \dots, m$, then $\lim_{n\to\infty} ||\overline{\partial}^i f_n||_{2,D_0} = 0$ for $i = 1, 2, \dots, m-2$ where D_0 is a disk strictly contained in D.

Lemma 2.3. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of a nth roots of M-hyponormal operator, i.e., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $H_1 \oplus H_2$ where T_1 is a nth roots of M-hyponormal operator and T_3 is analytic with order k and let D be a bounded disk in $\mathbb{C}$ containing $\sigma(T)$. Define the map $V : H_1 \oplus H_2 \to H(D)$ by*

$$Vh = 1 \otimes h + \overline{(T-z)W^{2k+2n}(D, H_1) \oplus W^{2k+2n}(D, H_2)} (= \widetilde{1 \otimes h})$$

where

$$H(D) := W^m(D, H_1) \oplus W^m(D, H_2) / \overline{(T-z)W^m(D, H_1) \oplus W^m(D, H_2)}$$

and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h, where m=$2k$+$2n$. Then V is one-to-one and has closed range.

Proof. Let $h_n = h_n^1 \oplus h_n^2 \in H_1 \oplus H_2$ and $f_n = f_n^1 \oplus f_n^2 \in W^{2k+2n}(D, H_1) \oplus W^{2k+2n}(D, H_2)$ be sequences such that

$$\lim_{n\to\infty} ||(z-T)f_n + 1 \otimes h_n||_{W^{2k+2n}} = 0. \tag{2.1}$$

Then (2.1) implies

$$\begin{cases} \lim_{n\to\infty} ||(z-T_1)f_n^1 + T_2 f_n^2 + 1 \otimes h_n^1||_{W^{2k+2n}} = 0 \\ \lim_{n\to\infty} ||(z-T_3)f_n^2 + 1 \otimes h_n^2||_{W^{2k+2n}} = 0. \end{cases} \tag{2.2}$$

By the definition of the norm of Sobolev space and (2.2), we obtain

$$\begin{cases} \lim_{n\to\infty} ||(z-T_1)\overline{\partial}^i f_n^1 + T_2 \overline{\partial}^i f_n^2||_{2,D} = 0 \\ \lim_{n\to\infty} ||(z-T_3)\overline{\partial}^i f_n^2||_{2,D} = 0 \end{cases} \tag{2.3}$$

for $i = 1, 2, \dots, 2k + 2n$. Since T_3 is analytic with order k, there exists a nonconstant analytic function F on a neighborhood of $\sigma(T_3)$ such that $F(T_3) =$

0. As remarked in section one, let $F(z) = G(z)p(z)$ where G is an analytic function and does not vanish on a neighborhood of $\sigma(T_3)$ and $p(z) = (z-z_1)(z-z_2)\cdots(z-z_k)$ is a polynomial of order k. Set $q_j(z) = (z-z_{j+1})\cdots(z-z_k)$ for $j = 0, 1, 2, \ldots, k-1$ and $q_k(z) = 1$.

Claim. For every $j = 0, 1, 2, \ldots, k$

$$\lim_{n\to\infty} ||q_j(T_3)\overline{\partial}^i f_n^2||_{2,D_j} = 0$$

for $i = 1, 2, \ldots, 2k-2j+2n$, where $\sigma(T) \subsetneqq D_k \subsetneqq \cdots \subsetneqq D_2 \subsetneqq D_1 \subsetneqq D$.

To prove the claim, we will use the induction on j. Since $0 = F(T_3) = G(T_3)p(T_3)$ and $G(T_3)$ is invertible, then $q_0(T_3) = p(T_3) = 0$, we have the claim holds when $j = 0$. Assume that the claim is true for some $j = r$ where $0 \leq r < k$, i.e.,

$$\lim_{n\to\infty} ||q_r(T_3)\overline{\partial}^i f_n^2||_{2,D_r} = 0 \tag{2.4}$$

for $i = 1, 2, \ldots, 2k-2r+2n$, where $\sigma(T) \subsetneqq D_r \subsetneqq \cdots \subsetneqq D_2 \subsetneqq D_1 \subsetneqq D$. By the second equation of (2.3) and (2.4), we get that

$$\begin{aligned} 0 &= \lim_{n\to\infty} ||q_{r+1}(T_3)(T_3 - z)\overline{\partial}^i f_n^2||_{2,D_r} \\ &= \lim_{n\to\infty} ||q_{r+1}(T_3)(T_3 - z_{r+1} + z_{r+1} - z)\overline{\partial}^i f_n^2||_{2,D_r} \\ &= \lim_{n\to\infty} ||(z_{r+1} - z)q_{r+1}(T_3)\overline{\partial}^i f_n^2||_{2,D_r} \end{aligned}$$

for $i = 1, 2, \ldots, 2k-2r+2n$. Since $z_{r+1}I$ is hyponormal, by applying Lemma 2.2 we obtain that

$$\lim_{n\to\infty} ||q_{r+1}(T_3)\overline{\partial}^i f_n^2||_{2,D_{r+1}} = 0$$

for $i = 1, 2, \ldots, 2k-2r+2n-2$, where $\sigma(T) \subsetneqq D_{r+1} \subsetneqq D_r$. Hence we complete the proof of our claim.

From the claim with $j = k$, we derive

$$\lim_{n\to\infty} ||\overline{\partial}^i f_n^2||_{2,D_k} = 0 \tag{2.5}$$

for $i = 1, 2 \ldots, 2n$, which implies by Lemma 2.1 that

$$\lim_{n\to\infty} ||(I - P_2)f_n^2||_{2,D_k} = 0 \tag{2.6}$$

where P_2 denotes the orthogonal projection of $L^2(D_k, H_2)$ onto $A^2(D_k, H_2)$. By combining (2.5) with the first equation of (2.3), we have that

$$\lim_{n\to\infty} ||(z - T_1)\overline{\partial}^i f_n^1||_{2,D_k} = 0 \tag{2.7}$$

for $i = 1, 2 \ldots, 2n$. From (2.7), we get

$$\lim_{n\to\infty} ||(z^n - T_1^n)\overline{\partial}^i f_n^1||_{2,D_k} = 0 \tag{2.8}$$

for $i = 1, 2, \dots, 2n$. Since T_1^n is an M-hyponormal operator, we obtain from (2.8) that

$$\lim_{n\to\infty} ||(z^n - T_1^n)^* \overline{\partial}^i f_n^1||_{2,D_k} = 0 \tag{2.9}$$

for $i = 1, 2 \dots, 2n$. By using Lemma 2.1 and (2.9)

$$\lim_{n\to\infty} ||(I - P_1) f_n^1||_{2,D_k} = 0 \tag{2.10}$$

where P_1 denotes the orthogonal projection of $L^2(D_k, H_1)$ onto $A^2(D_k, H_1)$.

Set $Pf_n := \begin{pmatrix} P_1 f_n^1 \\ P_2 f_n^2 \end{pmatrix}$. Combining (2.6) and (2.10) with (2.2), we have

$$\lim_{n\to\infty} ||(z - T)Pf_n + 1 \otimes h_n||_{2,D_k} = 0.$$

Let Γ be a curve in D_k surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty} ||Pf_n(z) + (z - T)^{-1}(1 \otimes h_n)(z)|| = 0$$

uniformly. Hence, by Riesz functional calculus,

$$\lim_{n\to\infty} ||\frac{1}{2\pi i} \int_\Gamma Pf_n(z)dz + h_n|| = 0.$$

But $\frac{1}{2\pi i} \int_\Gamma Pf_n(z)dz = 0$ by Cauchy's theorem. Hence, $\lim_{n\to\infty} h_n = 0$, and so V is one-to-one and has closed range. □

Now we are ready to prove that every analytic extension of a nth roots of M-hyponormal operator has a scalar extension.

Theorem 2.4. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of a nth roots of M-hyponormal operator. Then T is subscalar of order $2k + 2n$.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ be an operator matrix on $H_1 \oplus H_2$ where T_1 is a nth roots of M-hyponormal operator and T_3 is analytic with order k and let D be a bounded disk in $\mathbb{C}$ containing $\sigma(T)$. As in Lemma 2.3, if we define the map $V : H_1 \oplus H_2 \to H(D)$ by

$$Vh = 1 \otimes h + \overline{(T - z)W^{2k+2n}(D, H_1) \oplus W^{2k+2n}(D, H_2)} (= \widetilde{1 \otimes h}),$$

then V is one-to-one and has closed range. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, $\widetilde{S}$ respectively. Let M be the operator of multiplication by z on $W^{2k+2n}(D, H_1) \oplus W^{2k+2n}(D, H_2)$. Then M is a scalar operator of order $2k + 2n$ and has a spectral distribution Φ. Since $R(T - z)$ is invariant under M, $\widetilde{M}$ can be well-defined. In addition, consider the spectral distribution $\Phi : C_0^{2k+2n}(\mathbb{C}) \to B(W^{2k+2n}(D, H_1) \oplus W^{2k+2n}(D, H_2))$ defined by the following relation: for $\varphi \in C_0^{2k+2n}(\mathbb{C})$ and $f \in W^{2k+2n}(D, H_1) \oplus W^{2k+2n}(D, H_2)$, $\Phi(\phi)f = \phi f$. Then the spectral distribution Φ of M commutes with $T - z$, and so $\widetilde{M}$ is still a scalar operator of order $2k + 2n$ with $\widetilde{\Phi}$ as a spectral distribution. Since

$$VTh = \widetilde{1 \otimes Th} = \widetilde{z \otimes h} = \widetilde{M}(\widetilde{1 \otimes h}) = \widetilde{M}Vh$$

for all $h \in H_1 \oplus H_2$, $VT = \widetilde{M}V$. In particular, $R(V)$ is invariant under $\widetilde{M}$. Since $R(V)$ is closed, it is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{R(V)}$ and $\widetilde{M}$ is scalar of order $2k + 2n$, T is a subscalar operator of order $2k + 2n$. □

In the next corollary, we obtain a partial solution to the invariant subspace problem for analytic extension of a nth roots of M-hyponormal operator, which is a generalization of Brown's result mentioned in section one.

Corollary 2.5. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of a nth roots of M-hyponormal operator. If $\sigma(T)$ has nonempty interior in the complex plane $\mathbb{C}$, then T has a nontrivial invariant subspace.*

Proof. It suffices to apply Theorem 2.4 and [4]. □

Corollary 2.6. *Let $T \in B(H_1 \oplus H_2)$ be an analytic extension of a nth roots of M-hyponormal operator. Then T has property (β).*

Proof. Since property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 2.4 to the case of a scalar operator. Since every scalar operator has property (β)(see [11]), T has property (β). □

Corollary 2.7. *Let $T \in B(H)$ be a nth roots of M-hyponormal operator. Then T is subscalar of order $2n$.*

3. The Sum of a nth Roots of M-hyponormal Operator and an Algebraic Operator

In this section, we show that the sum of a nth roots of M-hyponormal operator and an algebraic operator of order k is subscalar of order $2kn + 2$.

Lemma 3.1. *Let $T = C + A$ where C is a nth roots of M-hyponormal operator, $CA = AC$, A is an algebraic operator of order k, and let D be a bounded disk in $\mathbb{C}$ containing $\sigma(T)$. Define the map $V : H \to H(D)$ by*

$$Vh = \widetilde{1 \otimes h}(\equiv 1 \otimes h + \overline{(T - z)W^{2kn+2}(D, H)})$$

where

$$H(D) := W^{2kn+2}(D, H)/\overline{(T - z)W^{2kn+2}(D, H)}$$

and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h. Then V is one-to-one and has closed range.

Proof. Let $f_n \in W^{2kn+2}(D,H)$ and h_n be sequences in H which satisfy

$$\lim_{n\to\infty} ||(T-z)f_n + 1\otimes h_n||_{W^{2kn+2}(D,H)} = 0. \tag{3.1}$$

It follows from the definition of the norm for the Sobolev space and (3.1) that

$$\lim_{n\to\infty} ||(T-z)\overline{\partial}^i f_n||_{2,D} = 0$$

for $i = 1, 2, \dots, 2kn+2$. Since $T = C + A$,

$$\lim_{n\to\infty} ||(C-z)\overline{\partial}^i f_n + A\overline{\partial}^i f_n||_{2,D} = 0 \tag{3.2}$$

for $i = 1, 2, \dots, 2kn+2$. Since A is algebraic of order k, $q(A) = 0$ for some nonconstant polynomial $q(z) = (z-z_1)(z-z_2)\cdots(z-z_k)$. Set $q_1(z) = (z-z_2)(z-z_3)\cdots(z-z_k)$. Then it follows from (3.2) that

$$\lim_{n\to\infty} ||(C+z_1-z)q_1(A)\overline{\partial}^i f_n||_{2,D} = 0 \tag{3.3}$$

for $i = 1, 2, \dots, 2kn+2$. From (3.3), we have

$$\lim_{n\to\infty} ||(C^n-(z-z_1)^n)q_1(A)\overline{\partial}^i f_n||_{2,D} = 0 \tag{3.4}$$

for $i = 1, 2, \dots, 2kn+2$. Since C^n is an M-hyponormal operator, we obtain from (3.4) and the definition of M-hyponormal operator that

$$\lim_{n\to\infty} ||(C^n-(z-z_1)^n)^* q_1(A)\overline{\partial}^i f_n||_{2,D} = 0 \tag{3.5}$$

for $i = 1, 2, \dots, 2kn+2$. By using Lemma 2.1 and (3.5)

$$\lim_{n\to\infty} ||(I-P)q_1(A)\overline{\partial}^i f_n||_{2,D} = 0$$

for $i = 1, 2, \dots, 2n(k-1)+2$, where P denotes the orthogonal projection of $L^2(D,H)$ onto $A^2(D,H)$. Thus from (3.4), we get

$$\lim_{n\to\infty} ||(C^n-(z-z_1)^n)Pq_1(A)\overline{\partial}^i f_n)||_{2,D} = 0$$

for $i = 1, 2, \dots, 2n(k-1)+2$. Since C^n has Bishop's property β [9],

$$\lim_{n\to\infty} ||Pq_1(A)\overline{\partial}^i f_n)||_{2,D_1} = 0$$

for $i = 1, 2, \dots, 2n(k-1)+2$ where $\sigma(T) \subset \overline{D_1} \subset D$. Hence

$$\lim_{n\to\infty} ||q_1(A)\overline{\partial}^i f_n||_{2,D_1} = 0 \tag{3.6}$$

for $i = 1, 2, \dots, 2n(k-1)+2$. By using the induction (that is the same procedure from (3.2) to (3.6), we obtain

$$\lim_{n\to\infty} ||\overline{\partial}^i f_n||_{2,D'} = 0$$

for $i = 1, 2$, where $\sigma(T) \subset \overline{D'} \subset D$. It holds by Lemma 2.1 that

$$\lim_{n\to\infty} ||(I-P)f_n||_{2,D'} = 0. \tag{3.7}$$

By (3.1) and (3.7), we have

$$\lim_{n\to\infty} ||(T-z)Pf_n + 1\otimes h_n||_{2,D'} = 0.$$

As the proof of Lemma 2.3, V is one-to-one and has closed range. □

Theorem 3.2. *Let $T = C + A$ where C is a nth roots of M-hyponormal operator, $CA = AC$, and A is an algebraic operator of order k. Then T is subscalar of order $2kn + 2$.*

Proof. The proof is similar to Theorem 2.4. □

Corollary 3.3. *Let $T = C + A$ where C is a nth roots of M-hyponormal operator, $AC = CA$, and A is an algebraic operator of order k. If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.*

Proof. It suffices to apply Theorem 3.2 and [4]. □

With the same proof as Corollary 2.6, we can get the following Corollary 3.4.

Corollary 3.4. *Let $T = C + A$ where C is a nth roots of M-hyponormal operator, $AC = CA$, and A is an algebraic operator of order k. Then T has Bishop's property (β).*

Acknowledgments

Research supported by the National Natural Science Foundation of China (11371185) and Major subject of Natural Science Foudation of Inner Mongolia of China (2013ZD01). The authors wish to thank the anonymous referee for his or her valuable suggestions which help to improve this manuscript.

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