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EXPECTED NUMBER OF LOCAL MAXIMA OF SOME GAUSSIAN RANDOM POLYNOMIALS

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Abstract. Let $Q_n(x) = \sum_{i=0}^{n} A_i x^i$ be a random algebraic polynomial where the coefficients $A_0, A_1, \cdots$ form a sequence of centered Gaussian random variables. Moreover, assume that the increments $\Delta_j = A_j - A_{j-1}$, $j = 0, 1, 2, \cdots$, with $A_{-1} = 0$, are independent. The coefficients can be considered as n consecutive observations of a Brownian motion. We study the asymptotic behaviour of the expected number of local maxima of $Q_n(x)$ below level $u = O(n^k)$, for some $k > 0$.

1. Introduction

The theory of the expected number of real zeros of random algebraic polynomials was addressed in the fundamental work of Kac [5]. The works of Wilkins [10], Farahmand [4] and Sambandham [9] are other fundamental contributions to the subject. For various aspects on random polynomials, see Bharucha-Reid and Sambandham [1], and Farahmand[3].

There has been recent interest in cases where the coefficients form certain random processes; see Rezakhah and Shemehsavar [6], and Rezakhah and Soltani [7].

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Let $A_0, A_1, \cdots$ be a mean zero Gaussian random sequence for which the increments $\Delta_i = A_i - A_{i-1},\ i = 1, 2, \cdots$, with $A_{-1} = 0$, are independent. The sequence $A_0, A_1 \cdots$ may be considered as successive Brownian points, that is, $A_j = W(t_j),\ j = 0, 1, \cdots$, where $t_0 < t_1 < \cdots$ and $\{W(t),\ t \geq 0\}$ is the standard Brownian motion. In this physical interpretation, $\mathrm{Var}(\Delta_j)$ is the distance between successive times t_{j-1} and t_j. Let

$$Q_n(x) = \sum_{i=0}^{n} A_i x^i, \ \ -\infty < x < \infty. \tag{1.1}$$

We note that $A_j = \Delta_0 + \Delta_1 + \cdots + \Delta_j, \quad j = 0, 1, \cdots$, where $\Delta_i \sim N(0, \sigma_i^2)$ and $\Delta_i,\ i = 0, 1, \cdots$ are independent. Thus, $Q_n(x) = \sum_{k=0}^{n}(\sum_{j=k}^{n} x^j)\Delta_k = \sum_{k=0}^{n} a_k(x)\Delta_k,\ \ Q_n'(x) = \sum_{k=0}^{n} b_k(x)\Delta_k$ and $Q_n''(x) = \sum_{k=0}^{n} d_k(x)\Delta_k$, where,

$$a_k(x) = \sum_{j=k}^{n} x^j, b_k(x) = \sum_{j=k}^{n} j x^{j-1}, d_k(x) = \sum_{j=k}^{n} j(j-1)x^{j-2}, k = 0, \cdots, n. \tag{1.2}$$

Here, we study the asymptotic behavior of the expected number of local maxima of $Q_n(x)$. We say $Q_n(x)$ has a local maxima at $t = t_i$ if $Q_n'(x)$ has a down-crossing of the level zero at t_i. A local maxima, which we consider here, is a maxima that occurs when $Q_n(x)$ is below level u. The total number of down-crossing of the level zero by $Q_n'(x)$ in (a, b) is defined as $M(a, b)$, and these occur at the points $a < t_1 < t_2 < \cdots < t_{M(a,b)} < b$. We define $M_u(a, b)$ as the number of zero-down crossing by $Q_n'(x)$ at those points $t_i \in (a,\ b)$, where $Q(t_i) \leq u$.

Rice [8, p 71] showed that for any function of the random variables $A_0, A_, \cdots A_n$ and x, say $U = Q_n(x)$, the expected number of maxima of U in the interval (a, b) is equal to

$$\int_a^b \int_{-\infty}^{\infty} \int_{-\infty}^{0} |t| p_x(r, 0, t)\, dt\, dr\, dx, \tag{1.3}$$

where $p_x(r, s, t)$ is the joint probability density function of $U = Q_n(x)$, $V = \partial Q_n(x)/\partial x$, and $W = \partial^2 Q_n(x)/\partial x^2$. We denote the covariance matrix of (U, V, W) by Σ and we find that

$$\det(\Sigma) = A^2B^2D^2 - A^2F^2 - B^2E^2 - C^2D^2 + 2CEF,$$

$$A^2 = \mathrm{Var}(Q_n(x)) = \sum_{k=1}^{n} a_k^2(x)\sigma_k^2, \qquad B^2 = \mathrm{Var}(Q_n'(x)) = \sum_{k=1}^{n} b_k^2(x)\sigma_k^2,$$

$$D^2=\text{Var}(Q''_n(x))=\sum_{k=1}^{n} d_k^2(x)\sigma_k^2, C=\text{Cov}(Q_n(x), Q'_n(x))=\sum_{k=1}^{n} a_k(x)b_k(x)\sigma_k^2,$$

$$E = \text{Cov}(Q_n(x), Q''_n(x)) = \sum_{k=1}^{n} a_k(x)d_k(x)\sigma_k^2,$$

$$F = \text{Cov}(Q'_n(x), Q''_n(x)) = \sum_{k=1}^{n} b_k(x)d_k(x)\sigma_k^2,$$

where $a_k(x)$, $b_k(x)$ and $d_k(x)$ are defined as in (1.2).

Using (1.3) and the above notations we find that the expected number of local maxima of $Q_n(x)$ below level u, and inside any interval $(a,\, b), EM_u(a, b)$ is equal to

$$EM_u(a, b) = \int_a^b f_n(x)dx, \quad \text{where} \quad f_n(x) = \int_{-\infty}^{u}\int_{-\infty}^{0} |t|p_x(r, 0, t)dtdr, \tag{1.4}$$

where,

$$p_x(r, 0, t) = \frac{\exp(-Lr^2 - 2Mrt - Kt^2)}{(2\pi)^{3/2}\det(\Sigma)^{1/2}},$$

$$K = \frac{A^2B^2 - C^2}{2\det(\Sigma)}, \qquad L = \frac{B^2D^2 - F^2}{2\det(\Sigma)} \tag{1.5}$$

$$M = \frac{CF - B^2E}{2\det(\Sigma)}, \qquad S = K - \frac{M^2}{4L}.$$

By simultaneous change of variables as $s = -t$ and $q = -r$, the integrand in (1.4) is unchanged. So, the expected number of maxima below a level u is exactly equal to the expected number of minima above level $-u$ in any interval (a, b). Thus, our result is valid for the expected number of minima above a level $-u$ as well.

Using (1.4), (1.5) and the function $\text{erf}(t) = 2\Phi(t\sqrt{2}) - 1$, we find that

$$EM_u(a, b) = \int_a^b f_n(x)dx, \tag{1.6}$$

where,

$$f_n(x) := (4\pi)^{-1}\left[G_1(\text{erf}(G_2) + 1) - G_1G_3\Big(\text{erf}(G_4) + 1\Big)\exp(G_5)\right],$$

in which,

$$G_1 = \left(2S\sqrt{2L\det(\Sigma)}\right)^{-1}, \quad G_3 = |M|\left(\sqrt{LK}\right)^{-1}, \quad G_4 = u\sqrt{K^{-1}M^2},$$

$$G_2 = u\sqrt{L}, \quad G_5 = -K^{-1}LSu^2.$$

Farahmand and Hannigan [4] obtained a similar formula for the case where the coefficients are independent with standard normal distribution.

2. Asymptotic behaviour of EM_u

Here, we obtain the asymptotic behaviour of the expected number of local maxima of $Q_n(x) = 0$ given by (1.1). We prove the following theorem for the case that the increments $\Delta_1, \cdots, \Delta_n$ are independent and have the same distribution. Also we assume that $\sigma_k^2 = 1$, for $k = 1, \cdots, n$.

Theorem 2.1. *Let $Q_n(x)$ be the random algebraic polynomial given by (1.1) for which $A_j = \Delta_1 + ... + \Delta_j$, where $\Delta_i, i = 1, ..., n$ are independent and $\Delta_j \sim N(0, 1)$. Then the expected number of local maxima of $Q_n(x)$ below level u satisfies:*

- *For* $u = O(n^{5/4})$:

i) $EM_u(1, \infty) = \frac{0.0013074}{4\pi} + \frac{(0.0350655)u}{2(n\pi)^{3/2}} + O(n^{-1/2}),$

ii) $EM_u(0, 1) = \frac{2(\sqrt{35}-5)}{345\pi} ln\left(\frac{n^{3/2}}{u}\right) - 0.001648 - \frac{(2.033388)u}{2(n\pi)^{3/2}} + O(n^{-1/2}).$

- *For* $u = O(n^{1/4})$:

iii) $EM_u(-\infty, -1) = \frac{0.0162552}{4\pi} + \frac{(0.0997677)u}{2\pi\sqrt{n\pi}} + O(n^{-1/2}),$

iv) $EM_u(-1, 0) = \frac{2(\sqrt{3}-1)}{11\pi} ln\left(\frac{n^{1/2}}{u}\right) + .0735255 - \frac{(0.6777048)u}{2\pi\sqrt{n\pi}} + O(n^{-1/2}).$

Proof. The asymptotic behaviour is treated separately on the intervals $1 < x < \infty$, $-\infty < x < -1$, $0 < x < 1$, and $-1 < x < 0$.

For $u = O(n^{5/4})$ and $1 < x < \infty$, make the change of variable $x = 1 + \frac{t}{n}$ and the equality $(1 + \frac{t}{n})^n = e^t \left(1 - \frac{t^2}{2n}\right) + O\left(n^{-2}\right)$. Using (1.6), we find that

$$EM_u(1, \infty) = \frac{1}{n}\int_0^\infty f_n(1 + \frac{t}{n})dt,$$

where by (1.5) and (1.6), and a tedious manipulation we have,

$$n^{-1}G_1\left(1 + \frac{t}{n}\right) = H_{11}(t) + O(n^{-1}),\ G_3\left(1 + \frac{t}{n}\right) = H_{13}(t) + O(n^{-1}) \quad (2.1)$$

$$G_2\left(1+\frac{t}{n}\right)=\frac{2u}{n^{3/2}\sqrt{\pi}}H_{12}(t)+O(n^{-5/4}),$$

$$G_4\left(1+\frac{t}{n}\right)=\frac{2u}{n^{3/2}\sqrt{\pi}}H_{14}(t)+O(n^{-5/4}),\quad G_5\left(1+\frac{t}{n}\right)=1+O(n^{-1/2}),$$

where $H_{11}(t)=\dfrac{q_{11}(t)}{p_{11}(t)}$, with

$$\begin{aligned} q_{11}(t)=&\frac{1}{192}\Bigg(-4+\left(32t+16+32t^2\right)e^t \\ &+\left(32t^5+208t^4+472t+124+1040t^2+736t^3\right)e^{2t} \\ &-32\left(9-6t^4+24t-8t^3+36t^2\right)e^{3t}+4\left(5-20t^2+4t+44t^4-16t^5-176t^3\right)e^{4t} \\ &+\left(272+224t+160t^2\right)e^{5t}+(-140+24t)e^{6t}\Bigg) \\ &\times\Bigg[35-32\left(1-t+5t^2\right)e^t-(294-588t-324t^2+600t^3+216t^4+80t^5)e^{2t} \\ &\qquad+32(17-51t+33t^2+16t^3-6t^4)e^{3t} \\ &\qquad-4(253/4-253t+350t^2-184t^3+43t^4-4t^5)e^{4t}\Bigg]^{1/2}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} p_{11}(t)=&\Bigg[\frac{115}{192}-\left(\frac{7}{8}+\frac{35}{12}t^2-\frac{37}{8}t\right)e^t \\ &-\left(\frac{733}{48}+\frac{485}{24}t+\frac{523}{48}t^2+\frac{1073}{24}t^3+\frac{5}{3}t^5+\frac{73}{12}t^4\right)e^{2t} \\ &+\left(\frac{1043}{24}-\frac{232}{3}t^4-\frac{125}{8}t+\frac{364}{3}t^3-\frac{34}{3}t^6+162t^2-14t^5\right)e^{3t} \\ &+\left(43t^7-\frac{32777}{48}t^2+\frac{2161}{12}t^5+\frac{31}{3}t^8+\frac{507}{8}t+\frac{5177}{16}t^4+\frac{1239}{32}+\frac{1949}{12}t^6+\frac{1043}{12}t^3\right)e^{4t} \\ &-\left(\frac{6887}{24}-\frac{7153}{24}t-\frac{12731}{12}t^2+\frac{5627}{6}t^3-26t^7+\frac{2335}{6}t^4+\frac{173}{3}t^6+\frac{1697}{6}t^5\right)e^{5t} \\ &+\left(\frac{20383}{48}+\frac{68}{3}t^7-\frac{4023}{16}t^2-\frac{8}{3}t^8-\frac{7297}{8}t+\frac{29851}{24}t^3-\frac{3907}{24}t^4-\frac{547}{6}t^6+\frac{397}{4}t^5\right)e^{6t} \\ &+\left(-\frac{2018}{3}t^2-\frac{2141}{8}+\frac{527}{2}t^4+6t^6+\frac{6749}{8}t-\frac{1243}{6}t^3-\frac{401}{6}t^5\right)e^{7t} \\ &+\left(\frac{12155}{192}-\frac{6281}{24}t+\frac{1385}{16}t^4+\frac{19097}{48}t^2+t^6-\frac{787}{3}t^3-\frac{29}{2}t^5\right)e^{8t}\Bigg]t, \end{aligned}$$

and $H_{12}(t)=\sqrt{\dfrac{q_{12}(t)}{p_{12}(t)}}$ with

$$\begin{aligned}q_{12}(t)\ =&-80\Bigg((\frac{253}{80}-\frac{46}{5}t^3+\frac{35}{2}t^2-1/5t^5+\frac{43}{20}t^4-\frac{253}{20}t)e^{-2t}\\&+(\frac{147}{40}+\frac{27}{10}t^4+t^5-\frac{81}{20}t^2+15/2t^3-\frac{147}{20}t)e^{-4t}\\&+(\frac{102}{5}t-\frac{34}{5}-\frac{66}{5}t^2-\frac{32}{5}t^3+\frac{12}{5}t^4)e^{-3t}-\frac{7}{16}e^{-6t}\\&+(2t^2-2/5t+2/5)e^{-5t}\Bigg)t^3,\end{aligned}\tag{2.3}$$

$$\begin{aligned}p_{12}(t)\ =&\Bigg(\left(-176\,t^3-20t^2-16t^5+44t^4+4t+5\right)e^{-2t}\\&+\left(184t^3+260t^2+52t^4+31+118t+8t^5\right)e^{-4t}\\&+\left(64t^3-192\,t-288t^2-72+48t^4\right)e^{-3t}\\&+\left(56t+68+40t^2\right)e^{-t}+6t-35-e^{-6t}+\left(8\,t+8t^2+4\right)e^{-5t}\Bigg).\end{aligned}$$

Also, $H_{13}(t)=\dfrac{q_{13}(t)}{\sqrt{p_{13}(t)}}$ with

$$\begin{aligned}q_{13}(t)\ =&\Bigg(5+\left(4+36t-8t^2\right)e^{t}-\left(12t-24t^4-52t^3+78-150t^2\right)e^{2t}\\&-\left(120t^2-124-24t^3+156t\right)e^{3t}+\left(132t-58t^2+8t^3-55\right)e^{4t}\Bigg),\end{aligned}\tag{2.4}$$

$$\begin{aligned}p_{13}(t)\ =&\Bigg(\left(253-736t^3+172t^4+1400t^2-1012t-16t^5\right)e^{4t}\\&+\left(294-324t^2-588t+600t^3+80t^5+216t^4\right)e^{2\,t}\\&-\left(512t^3+1056t^2+544-1632t-192t^4\right)e^{3t}-35+\left(160t^2+32-32t\right)e^{t}\Bigg)\\&\times\Bigg((15-4t)e^{4t}-(32+24t)e^{3t}+\left(36t+18+12t^2+8t^3\right)e^{2t}-8e^{t}t-1\Bigg),\end{aligned}$$

and

$H_{14}(t)=\dfrac{q_{14}(t)}{\sqrt{p_{14}(t)}}$ with

$$q_{14}(t) = \Big(\big(-78+150t^2+52t^3-12t+24\,t^4\big)\,e^{-3t} + \big(-120t^2-156t+124+24t^3\big)\,e^{-2t} + \big(132t+8\,t^3-58t^2-55\big)\,e^{-t}+5\,e^{-5t}+\big(36\,t+4-8t^2\big)\,e^{-4t}\Big)\,t^{3/2}, \tag{2.5}$$

$$p_{14}(t) = \Big((4t-15)+(32+24t)e^{-t}-(18+36t+12t^2+8t^3)e^{-2t}+8te^{-3t}+e^{-4t}\Big) \Big(6t-35+(68+56t+40t^2)e^{-t}+(5+14t-20t^2-176t^3+44t^4-16t^5)e^{-2t} +(64t^3-72-192t+48t^4-288t^2)e^{-3t} +(31+118t+260t^2+184t^3+8t^5+52t^4)e^{-4t}+(4+8t+8t^2)e^{-5t}-e^{-6t}\Big).$$

As $t\to\infty$ we have that asymptotically $q_{11}(t)\sim t^{3.5}e^{8t}/2,\ p_{11}(t)\sim t^7e^{8t}$, $q_{12}(t)\sim 16t^8e^{-2t}$ and $p_{12}(t)\sim 6t$. Also, $q_{13}(t)\sim 8t^3e^{4t},\ p_{13}(t)\sim 64t^6e^{8t}$, $q_{14}(t)\sim 8t^{4.5}e^{-t}$ and $p_{14}(t)\sim 24t^2$. These asymptotics are the terms of the biggest degree of each function. Thus as $t\to\infty$,

$$H_{11}(t)\sim\frac{1}{2t^{7/2}},\ H_{13}(t)\sim 1,\ H_{12}(t)=O(t^{7/2}e^{-t}),\ H_{14}(t)=O(t^{7/2}e^{-t}).$$

These calculations show that the following integrals are finite. One can also use mathematical softwares like Maple to obtain these and next asymptotic results. For more on asymptotic calculations see Bingham et al. [2].

Now using (2.1), (1.6), the fact that $exp(a/n)\sim 1+a/n+O(n^{-2})$ and $\mathrm{erf}(a/n)\sim 2a(n\sqrt{\pi})^{-1}+O(n^{-3})$, we have,

$$\begin{aligned} EM_u(1,\infty) &= \frac{1}{n}\int_0^\infty f_n\Big(1+\frac{t}{n}\Big)dt \\ &= \frac{1}{4\pi}\int_0^\infty H_{11}(t)[1-H_{13}(t)]dt \\ &+\frac{u}{2n^{3/2}\pi\sqrt{\pi}}\int_0^\infty H_{11}(t)[H_{12}(t)-H_{13}(t)H_{14}(t)]dt+O(n^{-1/2}), \end{aligned}$$

where by (2.2)-(2.5) and by numerical calculations we find,

$$\int_0^\infty H_{11}(t)[1-H_{13}(t)]dt = 0.0013074,$$

$$\int_0^\infty H_{11}(t)[H_{12}(t)-H_{13}(t)H_{14}(t)]dt = 0.0350655.$$

This gives the first assertion of the theorem.

For $u = O(n^{1/4})$ and $-\infty < x < -1$, let $x = -1 - \frac{t}{n}$. Then, by (1.6), $EM_u(-\infty, -1) = \dfrac{1}{n}\displaystyle\int_0^\infty f_n\left(-1 - \frac{t}{n}\right) dt$. Using (1.5) and (1.6), we have,

$$n^{-1}G_1\left(-1-\frac{t}{n}\right) = H_{21}(t)+O(n^{-1}), G_2\left(-1-\frac{t}{n}\right) = \frac{2u}{\sqrt{n\pi}}H_{22}(t)+O(n^{-5/4}), \tag{2.6}$$

$$G_3\left(-1-\frac{t}{n}\right) = H_{23}(t)+O(n^{-1}), \qquad G_5\left(-1-\frac{t}{n}\right) = 1+O(n^{-1/2}),$$

and

$$G_4\left(-1-\frac{t}{n}\right) = \frac{2u}{\sqrt{n\pi}}H_{24}(t) + O(n^{-5/4}),$$

where,

$$\begin{aligned} H_{21}(t) = \frac{1}{6t} & \left(\left(-3/8 - 2\,t^5 - 9/2\,t^2 - 2\,t^3 + 3/2\,t^4 - 3/2\,t\right) e^{4\,t} \right. \\ & \left. + \left(t^5 + \frac{9}{2}t^4 + \frac{3}{4}t + 7t^3 + \frac{9}{2}t^2 + \frac{3}{8}\right) e^{2t} + \frac{3}{4}e^{6t}t + \frac{1}{8}e^{6t} - \frac{1}{8}\right) \\ & \left[3 + (12t - 6 - 12t^2 - 56t^3 - 56t^4 - 16t^5)e^{2t} \right. \\ & \qquad \left. + (3 - 12t + 24t^2 + 32t^3 - 44t^4 + 16t^5)e^{4t}\right]^{1/2} \\ & \left[\left(\frac{7}{3}t^8 + \frac{221}{12}t^5 + \frac{1}{8}t + \frac{11}{32} + \frac{259}{48}t^4 + \frac{7}{4}t^3 + \frac{35}{3}t^7 + \frac{257}{12}t^6 + \frac{29}{16}t^2\right) e^{4\,t} \right. \\ & + \left(-\frac{15}{8}t^3 - \frac{55}{48}t^2 - 1/24\,t - 5/4t^4 - 1/3t^5 - \frac{11}{48}\right) e^{2t} \\ & + \left(\frac{13}{6}t^6 - \frac{1}{8}t - \frac{49}{24}t^3 - \frac{51}{4}t^5 - \frac{8}{3}t^8 + \frac{8}{3}t^7 - \frac{211}{24}t^4 - \frac{11}{48} - \frac{3}{16}t^2\right) e^{6t} \\ & \left. + \frac{11}{192} - \left(\frac{5}{2}t^5 - \frac{1}{24}t + \frac{23}{48}t^2 - t^6 - \frac{13}{6}t^3 - \frac{11}{192} - \frac{25}{16}t^4\right) e^{8t}\right]^{-1}, \end{aligned} \tag{2.7}$$

$$H_{22}(t) = 2\sqrt{t}\Big(3 - (16t^5 + 56t^4 + 56t^3 + 12t^2 - 12t + 6)e^{2t} \\ +(16t^5 - 44t^4 + 32t^3 + 24t^2 - 12t + 3)e^{4t}\Big)^{1/2} \\ \times\Big[(1 + 6t)e^{6t} + (12t^4 - 16t^5 - 16t^3 - 36t^2 - 12t - 3)e^{4t} \\ +(3 + 6t + 36t^2 + 56t^3 + 36t^4 + 8t^5)e^{2t} - 1\Big]^{-1/2}, \tag{2.8}$$

and

$$H_{23}(t) = \frac{1}{8}\Big(1+(8t^4+20t^3+14t^2+4t-2)e^{2t}+(1-4t-10t^2+8t^3)e^{4t}\Big) \\ \times\Big[\Big((-2t^3 - 3t^2 - t - 1/2)e^{2t} + 1/4 + (1/4 + t)e^{4t}\Big) \\ \times\Big((t^5 - 11/4t^4 + 3/16 + 2t^3 + 3/2t^2 - 3/4t)e^{4t} + 3/16 \\ +(3/4t - 3/4t^2 - 7/2t^3 - 7/2t^4 - t^5 - 3/8)e^{2t}\Big)\Big]^{-1/2}, \tag{2.9}$$

$$H_{24}(t) = 2\sqrt{t}\Big(1+(8t^4+20t^3+14t^2+4t-2)e^{2t}+(1-4t-10t^2+8t^3)e^{4t}\Big) \\ \times\Big[\Big((1 + 6t)e^{6t}-(16t^5-12t^4+16t^3+36t^2+12t+3)e^{4t} \\ +(3+6t+36t^2+56t^3+36t^4+8t^5)e^{2t}-1\Big) \\ \times\Big(1-(2+4t+12t^2+8t^3)e^{2t}+(1+4t)e^{4t}\Big)\Big]^{-1/2}. \tag{2.10}$$

As $t \to \infty$, we have,

$$H_{21}(t) \sim \frac{1}{2t^{7/2}},\ H_{23}(t) \sim 1,\ H_{22}(t) = O(t^{5/2}e^{-t}),\ H_{24}(t) = O(t^{5/2}e^{-t}).$$

Thus, by (1.6) and (2.6) we have,

$$EM_u(-\infty, -1) = \frac{1}{n}\int_0^\infty f_n(-1 - \frac{t}{n}) = \frac{1}{4\pi}\int_0^\infty H_{21}(t)\,[1 - H_{23}(1)]\,dt \\ +\frac{u}{2\pi\sqrt{n\pi}}\int_0^\infty H_{21}(t)\,[H_{22}(t)-H_{23}(t)H_{24}(t)]\,dt + O(n^{-1/2}),$$

where by (2.7)-(2.10) and by numerical calculations we find,

$$\int_0^\infty H_{21}(t)\,[1-H_{23}(t)]\,dt=0.01625525,$$

$$\int_0^\infty H_{21}(t)\,[H_{22}(t)-H_{23}(t)H_{24}(t)]\,dt = 0.09976775.$$

This gives the third assertion of the theorem.

For $u = O(n^{5/4})$ and $0 < x < 1$, let $x = 1 - \frac{t}{n+t}$. Thus, by (1.6), $EM_u(0,1) = \left(\frac{n}{(n+t)^2}\right)\int_0^\infty f_n\left(1-\frac{t}{n+t}\right)dt$, where by (1.5) and (1.6) we have,

$$\frac{n}{(n+t)^2}\,G_1\Big(1-\frac{t}{n+t}\Big) = H_{31}(t) + O(n^{-1}),$$

$$G_3\left(1-\frac{t}{n+t}\right) = H_{33}(t) + O(n^{-1}),$$

$$G_2\Big(\frac{n}{n+t}\Big)=\frac{2u}{n\sqrt{n\pi}}\,H_{32}(t)+O(n^{-5/4}),\ G_5\Big(\frac{n}{n+t}\Big)=1+O(n^{-1/2}), \quad (2.11)$$

$$G_4\left(1-\frac{t}{n+t}\right) = \frac{2u}{n^{3/2}\sqrt{\pi}}H_{34}(t) + O(n^{-5/4}),$$

where using (2.2)-(2.5) one can verify that

$$H_{31}(t)=H_{11}(\text{-}t),\quad H_{32}(t)=H_{12}(\text{-}t),\quad H_{33}(t)=H_{13}(\text{-}t),\quad H_{34}(t)=H_{14}(\text{-}t).$$

As $t \to \infty$, we have,

$$H_{31}(t) \sim \frac{4\sqrt{35}}{115t},\quad H_{33}(t) \sim \frac{5}{\sqrt{35}},\quad H_{32}(t) \sim \sqrt{35}\,t^{3/2},\quad H_{34}(t) \sim 5t^{3/2}.$$

For any real numbers A and B we have,

$$\frac{A}{t} - \frac{B\sqrt{t}}{n^{3/2}} = \frac{A}{t} - \frac{B\sqrt{t}}{n^{3/2} + (B/A)t^{3/2}} + O(n^{-3}). \quad (2.12)$$

Let $a = \frac{\sqrt{35}}{115\pi}$, $b = \frac{10u}{23\pi^{3/2}}$, $c = \frac{1}{23\pi}$ and $d = \frac{14u}{23\pi^{3/2}}$. Now by (1.6), (2.11), (2.12), and above calculations we have,

$$
\begin{aligned}
EM_u(0,1) &= \frac{n}{(n+t)^2}\int_0^\infty f_n\Big(\frac{n}{n+t}\Big)dt \\
&= \frac{1}{4\pi}\int_0^\infty\left(H_{11}(-t)[1-H_{13}(-t)] - \frac{4(\sqrt{35}-5)\,I_{[t\geq 1]}}{115t}\right)dt \\
&+ \frac{u}{2(n\pi)^{3/2}}\int_0^\infty\left(H_{11}(-t)H_{12}(-t) - \frac{28\sqrt{t}I_{[t\geq 1]}}{23}\right)dt \\
&- \frac{u}{2(n\pi)^{3/2}}\int_0^\infty\left(H_{11}(-t)H_{13}(-t)H_{14}(-t) - \frac{20\sqrt{t}I_{[t\geq 1]}}{23}\right)dt \\
&+ \int_1^\infty \frac{a}{t} - \frac{b\sqrt{t}}{n^{3/2}+(b/a)t^{3/2}}\,dt - \int_1^\infty \frac{c}{t} - \frac{d\sqrt{t}}{n^{3/2}+(d/c)t^{3/2}}\,dt + O(n^{-1/2})
\end{aligned}
$$

where by (2.2)-(2.5) and by some numerical calculations we find,

$$
\int_0^\infty\left(H_{11}(-t)[1-H_{13}(-t)] - \frac{4(\sqrt{35}-5)I_{[t\geq 1]}}{115t}\right)dt = -0.0460436,
$$

$$
\int_0^\infty\left(H_{11}(-t)[H_{12}(-t) - H_{13}(-t)H_{14}(-t)] - \frac{8\sqrt{t}I_{[t\geq 1]}}{23}\right)dt
$$

$$
= -2.033388159.
$$

The assumed values for a, b, c, and d lead to:

$$
\int_1^\infty \frac{a}{t} - \frac{b\sqrt{t}}{n^{3/2}+\frac{b}{a}t^{3/2}}dt = \frac{2a}{3}\ln\Big(\frac{a}{b}n^{3/2}+1\Big) = \frac{2\sqrt{35}}{345\pi}\ln\Big(\frac{\sqrt{35\pi}}{50u}n^{3/2}+1\Big),
$$

$$
\int_1^\infty\left(\frac{c}{t} - \frac{d\sqrt{t}}{n^{3/2}+(d/c)t^{3/2}}\right)dt = \frac{2c}{3}\ln\Big(\frac{c}{d}n^{3/2}+1\Big) =
$$

$$
\frac{2}{69\pi}\ln\Big(\frac{\sqrt{\pi}}{14u}n^{3/2}+1\Big).
$$

This gives the second assertion of the theorem.

For $u = O(n^{1/4})$ and $-1 < x < 0$, let $x = -1 + \frac{t}{n+t}$. Thus, by (1.6), $EM_u(-1,0) = \left(\frac{n}{(n+t)^2}\right)\int_0^\infty f_n\left(-1+\frac{t}{n+t}\right)dt$, where by (1.5) and (1.6) we have,

$$
\frac{n}{(n+t)^2}G_1\left(\frac{-n}{n+t}\right) = H_{41}(t) + O(n^{-1}),
$$

$$G_2\left(\frac{-n}{n+t}\right)=\frac{2u}{\sqrt{n\pi}}H_{42}(t)+O(n^{-5/4}),$$

$$G_3\left(-1+\frac{t}{n+t}\right)=H_{43}(t)+O(n^{-1}),\quad G_5\left(-1+\frac{t}{n+t}\right)=1+O(n^{-1/2}),\tag{2.13}$$

$$G_4\left(-1+\frac{t}{n+t}\right)=\frac{2u}{\sqrt{n\pi}}H_{44}(t)+O(n^{-5/4}),$$

where using (2.7)-(2.10) we find,

$$H_{41}(t)=H_{21}(-t),\ H_{42}(t)=H_{22}(-t),\ H_{43}(t)=H_{23}(-t),\ H_{44}(t)=H_{24}(-t).$$

As $t\to\infty$, we have,

$$H_{41}(t)\sim\frac{4\sqrt{3}}{11t},\qquad H_{43}(t)\sim\frac{1}{\sqrt{3}},\qquad H_{42}(t)\sim 2\sqrt{3t},\qquad H_{44}(t)\sim 2\sqrt{t}.$$

For any real numbers a and b we have,

$$\frac{a}{t}-\frac{b/\sqrt{t}}{n^{1/2}}=\frac{a}{t}-\frac{b/\sqrt{t}}{n^{1/2}+(b/a)t^{1/2}}+O(n^{-1}).\tag{2.14}$$

Now, let $a=\frac{\sqrt{3}}{11\pi}$, $b=\frac{4u}{11\pi^{3/2}}$, $c=\frac{1}{11\pi}$ and $d=\frac{12u}{11\pi^{3/2}}$. We know that the expected number of local maxima and thus the whole integral is finite. Thus by subtracting some terms we make the integrands integrable. Finally, we add these terms to the whole integrand by using the equality (2.14). This way, we can break the whole integrand to different integrable parts. Thus, by (1.6), (2.13), (2.14) and above calculations we have,

$$\begin{aligned}
EM_u(-1,0)&=\frac{n}{(n+t)^2}\int_0^\infty f_n(-1+\frac{t}{n+t})dt\\
&=\frac{1}{4\pi}\int_0^\infty\left(H_{21}(-t)[1-H_{23}(-t)]-\frac{4(\sqrt{3}-1)I_{[t\geq 1]}}{11t}\right)dt\\
&+\frac{u}{2\pi\sqrt{n\pi}}\int_0^\infty\left(H_{21}(-t)[H_{22}(-t)-H_{23}(-t)H_{24}(-t)]-\frac{16I_{[t\geq 1]}}{11\sqrt{t}}\right)dt\\
&+\int_1^\infty\frac{a}{t}-\frac{b/\sqrt{t}}{n^{1/2}+(b/a)t^{1/2}}dt-\int_1^\infty\frac{c}{t}-\frac{d/\sqrt{t}}{n^{1/2}+(d/c)t^{1/2}}dt+O(n^{-1/2}),
\end{aligned}$$

where by (2.7)-(2.10) and some numerical calculations we find,

$$\int_0^\infty\left(H_{21}(-t)[1-H_{23}(-t)]-\frac{4(\sqrt{3}-1)I_{[t\geq 1]}}{11t}\right)dt=-0.1336511689,$$

$$\int_0^\infty \left(H_{21}(-t)[H_{22}(-t) - H_{23}(-t)H_{24}(-t)] - \frac{16 I_{[t\geq 1]}}{11\sqrt{t}} \right) dt$$
$$= -0.6777048198.$$

The assumed values for a, b, c, and d lead to:

$$\int_1^\infty \frac{a}{t} - \frac{b/\sqrt{t}}{n^{1/2} + \frac{b}{a} t^{1/2}} dt = 2a\ln\left(\frac{a}{b} n^{1/2} + 1\right) = \frac{2\sqrt{3}}{11\pi} \ln\left(\frac{\sqrt{3\pi}}{4u} n^{1/2} + 1\right),$$

$$\int_1^\infty \left(\frac{c}{t} - \frac{d/\sqrt{t}}{n^{1/2} + (d/c) t^{1/2}}\right) dt = 2c\ln\left(\frac{c}{d} n^{1/2} + 1\right) = \frac{2}{11\pi} \ln\left(\frac{\sqrt{\pi}}{12u} n^{1/2} + 1\right).$$

Thus, we get to the last assertion of the theorem.

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