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Author(s):

M. Darus, I. Aldawish and R. W. Ibrahim

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SOME CONCAVITY PROPERTIES FOR GENERAL INTEGRAL OPERATORS

M. DARUS*, I. ALDAWISH AND R. W. IBRAHIM

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Abstract. Let $C_0(\alpha)$ denote the class of concave univalent functions defined in the open unit disk $\mathbb{D}$. Each function $f \in C_0(\alpha)$ maps the unit disk $\mathbb{D}$ onto the complement of an unbounded convex set. In this paper, we study the mapping properties of this class under integral operators.
Keywords: Unit disk, univalent function, concave function, integral operator.
MSC(2010): Primary: 30C45.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$. Also, let $\wp$ be the class of all functions which are analytic in $\mathbb{D}$ and satisfy $p(0) = 1$, $Re\, p(z) > 0$.

In 1915, Alexander [1] introduced the first integral operator defined by

$$A(z) = \int_0^z \frac{f(t)}{t} dt,$$

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*Corresponding author.

while the Libera operator [24] is defined by

$$L(z) = \frac{2}{z} \int_0^z f(t)dt,$$

and the Bernardi operator [5], which generalizes the Libera operator, is defined for $\gamma = 1, 2, 3, ...,$

$$L_\gamma(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt.$$

Many authors have studied special integral operators by employing different methods (see [12, 20, 25, 27]). All of these special integral operators are the particular cases of the other general operator of the form

$$I[f](z) = \left[\frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t)t^{\delta-1}\varphi(t)dt\right]^{\frac{1}{\beta}}, \tag{1.2}$$

where $\varphi, \phi \in \mathbb{D}$, $\gamma, \beta, \delta, \alpha$ are complex numbers. Note that, the operator I given in (1.2) was defined by Miller, Mocanu and Reade [26] in 1978. In 2002, Breaz and Breaz [8] defined the following operator:

$$F_n[f](z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} .... \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt, \tag{1.3}$$

for $f_i(z) \in \mathcal{A}$ and $\gamma_i > 0$, for all $i \in 1, 2, 3, ..., n$.

By using the concept of convolution, many authors had generalized the operator by Breaz and Breaz (1.3) in several directions (see [2,9,11,28]). Frasin [21] (see also, [7]) introduced and studied a generalized operator $I(f_i, g_i)$ as follows

$$I(f_i, g_i)(z) = \int_0^z \left(\frac{(f_i * g_i)(t)}{t}\right)^{\alpha_i} dt, \quad i = 1, ..., m. \tag{1.4}$$

The operator $I(f_i, g_i)$ reduces to many well-known integral operators by varying the parameters α_i and by choosing suitable functions instead of $g_i(z)$. For example if we take $m = 1, g_1(z) = \frac{z}{1-z}, \alpha_1 = \alpha \in [0, 1]$ in (1.4), we obtain the integral operator studied in [26] given as

$$\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt. \tag{1.5}$$

Other types of integral operators are also studied by researchers in this area. For example, integral operators of the types $J(z), H(z), F(z), G(z)$, $F_{\alpha\beta,n}$ and $G_{\alpha,n}(z)$ which were also proved by Breaz et al., [10]. In addition, Ibrahim and Darus [19] defined and studied integral operator as follows:

$$J^{k,\alpha,\beta}_{\lambda,\delta}$$

$$= \left\{ \underbrace{\phi(z) * .... * \phi(z)}_{k-times} * \underbrace{\frac{z\beta}{(1-z)^2} * .... * \frac{z\beta}{(1-z)^2}}_{\alpha k-times} * \frac{z}{(1-z)^{(\delta+1)}} \right\}^{-1} * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{a_n}{\left[(\beta_n)^\alpha + (n-1)(\beta_n)^\alpha \lambda\right]^k C(\delta, n)} z^n,$$

where $k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta \geq 1, \lambda \geq 0, \delta \geq 0 \;\; z \in \mathbb{D}$.

Many results were obtained by the authors of [14]- [31] related to this integral operator.

In 2013, Frasin [23] obtained the general integral operator $B_\beta(z)$ defined by

$$B_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} \rho_i^{\zeta_i}(t) dt\right]^{\frac{1}{\beta}}, \tag{1.6}$$

where $f_i \in \mathcal{A}, \rho_i \in \wp, \beta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and $\gamma_i, \zeta_i \in \mathbb{C}$ for all $i = 1, 2, ..., n$.

Note that the integral operator B_β generalized special operators introduced and studied by several authors including [8] and [22].

The aim of this work is to investigate some mapping properties of the integral operators (1.5) and

$$F(z) = \frac{2}{z^{\lambda+1}} \int_0^z t^\lambda f(t) dt, \lambda > 0, \tag{1.7}$$

for the class of concave univalent functions.

Remark 1.1. *Note that when $\lambda = 0$ in (1.7) we get Libera operator.*

We need the following definition in the sequel.

Definition 1.2. *A function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:*

(i) f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.

(ii) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.

(iii) The opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha, \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as concave univalent functions and for a detailed discussion about concave functions, we may refer to [3,4,13] and the references therein.

We recall the analytic characterization for the functions in $C_0(\alpha), \alpha \in (1,2]$: $f \in C_0(\alpha)$ if and only if $\Re\left(P_{f(z)}(z)\right) > 0, z \in \mathbb{D}$, where

$$P_f(z) = \frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

In [6], the authors used this characterization for their work.

2. **Main results**

One of our main results is contained in the following theorem:

Theorem 2.1. *If* $\alpha \in (1,2]$ *and* $f \in \mathcal{A}$ *satisfy the inequality*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, z \in \mathbb{D}, \tag{2.1}$$

then for all $\lambda \in \mathbb{R}$

$$F(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\lambda dt \in C_0(\alpha). \tag{2.2}$$

Proof. Our aim is to prove that for any complex number w we have (see [10])

$$|w-1| < 1 \Leftrightarrow \Re\frac{1}{w} > \frac{1}{2} \quad . \tag{2.3}$$

If we let

$$\frac{1}{w} = \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{F''(z)}{F'(z)}\right],$$

then

$$|w-1| = \left|\frac{2z\left((1-z)F''(z) - (1+\alpha)F'(z)\right)}{(\alpha+\alpha z+3z-1)F'(z) - 2z(1-z)F''(z)}\right|. \tag{2.4}$$

From (2.2), we obtain

$$F'(z) = \left(\frac{f(z)}{z}\right)^\lambda,$$

and

$$F''(z) = \lambda\left(\frac{f(z)}{z}\right)^{\lambda-1}\left(\frac{zf'(z)-f(z)}{z^2}\right).$$

We substitute $F'(z)$ and $F''(z)$ in (2.4), and make a simple computation to get

$$\begin{aligned}
&\left|\frac{2\lambda(1-z)\left(\frac{zf'(z)}{f(z)}-1\right)-(1+\alpha)}{(\alpha+\alpha z+3z-1)-2\lambda(1-z)\left(\frac{zf'(z)}{f(z)}-1\right)}\right| \\
&\leq \frac{2|\lambda||1-z|\left|\frac{zf'(z)}{f(z)}-1\right|+(1+\alpha)}{|\alpha+\alpha z+3z-1|-2|\lambda||1-z|\left|\frac{zf'(z)}{f(z)}-1\right|} \\
&\leq \frac{2|\lambda||1-z|+(1+\alpha)}{|\alpha+\alpha z+3z-1|-2|\lambda||1-z|},
\end{aligned}$$

where

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1.$$

Taking $z \longrightarrow 1^+$ (by the maximum principle theorem), yields

$$\left|\frac{2\lambda(1-z)\left(\frac{zf'(z)}{f(z)}-1\right)-(1+\alpha)}{(\alpha+\alpha z+3z-1)-2\lambda(1-z)\left(\frac{zf'(z)}{f(z)}-1\right)}\right| \leq \frac{1}{2}<1.$$

In view of conclusion (2.3), we have $F(z) \in C_0(\alpha)$. This completes the proof. $\square$

When $\lambda = 1$, we obtain the next result.

Corollary 2.2. *If $\lambda = 1$ in Theorem 2.1, then the Alexander operator is a concave univalent function.*

Theorem 2.3. *Let $f \in \mathcal{A}$ and $\alpha \in (1, 2]$. If $f \in C_0(\alpha)$, then the integral operator*

$$F(z)=\frac{2}{z^{\lambda+1}}\int_0^z t^\lambda f(t)dt, \quad \lambda>0, \tag{2.5}$$

satisfies the inequality

$$\Re\left\{\frac{(z^2F''(z))'}{F'(z)}\right\} \leq \frac{(\alpha-1)(\lambda+2)}{2},$$

whenever

$$\Re\{\frac{zF''(z)}{F'(z)}\}>0. \tag{2.6}$$

Proof. Since

$$F(z)=\frac{2}{z^{\lambda+1}}\int_0^z t^{\lambda}f(t)dt,$$

we have

$$\frac{z^{\lambda+1}}{2}F(z)=\int_0^z t^{\lambda}f(t)dt.$$

By taking the derivatives, we obtain

$$\frac{\lambda+1}{2}F(z)+\frac{z}{2}F'(z)=f(z),$$

which implies that

$$\frac{\lambda+2}{2}F'(z)+\frac{z}{2}F''(z)=f'(z). \tag{2.7}$$

We deduce

$$\frac{\lambda+3}{2}F''(z)+\frac{z}{2}F'''(z)=f''(z). \tag{2.8}$$

Since

$$\Re\left(\frac{2}{\alpha-1}\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{f''(z)}{f'(z)}\right)\right)>0,$$

this is equivalent to

$$\Re\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{f''(z)}{f'(z)}\right)>0.$$

Since

$$\Re\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-1\right)\leq\left|\frac{\alpha+1}{2}\frac{1+z}{1-z}-1\right|,$$

by letting $z\to 0$, we conclude that

$$\Re\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-1\right)\leq\frac{\alpha-1}{2}.$$

Hence we obtain

$$0<\Re\left(\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{f''(z)}{f'(z)}\right)\leq\Re\left(\frac{\alpha-1}{2}-z\frac{f''(z)}{f'(z)}\right),$$

which yields the inequality

$$\Re(\frac{zf''(z)}{f'(z)})\leq\frac{\alpha-1}{2}. \tag{2.9}$$

Substituting (2.7) and (2.8) in (2.9), we have

$$\Re\left\{(7-\alpha+2\lambda)F''(z)+2zF'''(z)\right\}\leq(\alpha-1)(\lambda+2)\Re\{\frac{F'(z)}{z}\}.$$

Consequently, we have

$$\Re\left\{\frac{(7-\alpha+2\lambda)}{2}\frac{zF''(z)}{F'(z)}+\frac{z^2F'''(z)}{F'(z)}\right\}\leq\frac{(\alpha-1)(\lambda+2)}{2}.$$

Since for $\alpha\in(1,2]$, $\lambda>0$ and $3-\alpha+2\lambda>0$, we have

$$\frac{7-\alpha+2\lambda}{2}>2,$$

then, by the assumption (2.6), we conclude that

$$\begin{aligned}\Re\left\{2\frac{zF''(z)}{F'(z)}+\frac{z^2F'''(z)}{F'(z)}\right\}&\leq\Re\left\{\frac{(7-\alpha+2\lambda)}{2}\frac{zF''(z)}{F'(z)}+\frac{z^2F'''(z)}{F'(z)}\right\}\\&\leq\frac{(\alpha-1)(\lambda+2)}{2}.\end{aligned}$$

This completes the proof. □

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(Maslina Darus) School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi Selangor DE, Malaysia
E-mail address: `maslina@ukm.edu.my`

(Ibtisam Aldawish) School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi Selangor DE, Malaysia
E-mail address: `ibtisamaldawish@gmail.com`

(Rabha W. Ibrahim) Institute of Mathematical Sciences, Universiti Malaya, 50603, Malaysia
E-mail address: `rabhaibrahim@yahoo.com`