ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 5, pp. 1161–1172

Title:

Some approximate fixed point results for proximinal valued β-contractive multifunctions

Author(s):

M. A. Miandaragh, A. Pitea and Sh. Rezapour

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 5, pp. 1161–1172
Online ISSN: 1735-8515

SOME APPROXIMATE FIXED POINT RESULTS FOR PROXIMINAL VALUED β-CONTRACTIVE MULTIFUNCTIONS

M. A. MIANDARAGH, A. PITEA AND SH. REZAPOUR*

(Communicated by Antony To-Ming Lau)

Abstract. In this paper, we prove some approximate fixed point results for proximinal valued β-contractive multifunctions on metric spaces. We show that our results generalize some older fixed point results in the literature.
Keywords: β-contractive multifunction, approximate fixed point, proximinal, fixed point.
MSC(2010): Primary: 47H04; Secondary: 47H10.

1. Introduction

As it is well-known, there are mappings and multifunctions which have approximate fixed points but have no fixed points. A study has been done about approximate fixed points of several classes of mappings and many papers were published in this area (see [5, 9–12, 18, 22, 30–32] and [42]).

The technique of α-ψ-contractive mappings introduced by Samet, Vetro and Vetro in 2012 ([43]). Later, some authors used it for some subjects in fixed point theory (see [7, 25, 33] and [41]) or generalized it by using the method of β-ψ-contractive multifunctions (see [4, 24, 34]). By using and combining the idea of these references and main idea of [3, 29] and [44], we shall prove some approximate fixed point results for proximinal valued β-contractive multifunctions.

Let X be a set, $T : X \to 2^X$ a multifunction and $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping. We say that T is β-admissible whenever $\beta(A, B) \geq 1$ implies $\beta(Tx, Ty) \geq 1$ for all $x \in A$ and $y \in B$, where A and B are subsets of X. Denote by $\mathcal{R}$ the

Article electronically published on October 15, 2015.
Received: 8 March 2013, Accepted: 16 July 2014.
*Corresponding author.

set of all continuous mappings $g : [0, \infty)^5 \to [0, \infty)$ satisfying $g(1, 1, 1, 2, 0) = g(1, 1, 1, 0, 2) = h \in (0, 1)$, $g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5)$ for all nonnegative elements x_1, x_2, x_3, x_4, x_5 and $\alpha \geq 0$, $g(x_1, x_2, x_3, x_4, 0) < g(y_1, y_2, y_3, y_4, 0)$ and $g(x_1, x_2, x_3, 0, x_4) < g(y_1, y_2, y_3, 0, y_4)$ for all $x_i, y_i \in [0, \infty)$ with $x_i < y_i$ for $i = 1, ..., 4$ (see [3]). We need the next result.

Proposition 1.1. *([3]) If $g \in \mathcal{R}$ and $u, v \in [0, \infty)$ are such that*

$$u \leq \max\{g(v, v, u, v+u, o), g(v, v, u, o, v+u), g(v, u, v, v+u, o), g(v, u, v, o, v+u)\},$$

then $u \leq hv$.

Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and T a multifunction on X with closed and bounded values. We say that T is a generalized β-contractive multifunction whenever there exists $g \in \mathcal{R}$ such that

$$\beta(Tx, Ty)H(Tx, Ty) \leq g(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$

for all $x, y \in X$, where H is the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\{sup_{x\in A} d(x, B), sup_{y\in B} d(y, A)\}$$

for all closed and bounded subsets A and B of X. We say that T has approximate fixed points whenever $\inf_{x\in X} d(x, Tx) = 0$. Also, we say that T is lower semi-continuous at $x_0 \in X$ whenever for each sequence $\{x_n\}$ with $x_n \to x_0$ and every $y \in Tx_0$, there exists a sequence $\{y_n\}$ such that $y_n \to y$ and $y_n \in Tx_n$ for all n ([13]). Let C be a nonempty subset of a metric space (X, d) and $x \in X$. We say that T is lower semi-continuous whenever T is lower semi-continuous at each element of X. An element $y_0 \in C$ is said to be a best approximation of x whenever $d(x, y_0) = d(x, C) = \inf_{y\in C} d(x, y)$. The set C is called proximinal whenever every $x \in X$ has at least one best approximation in C ([1]). Every proximinal set is closed closed and bounded ([1]). Denote by $P(X)$ the set of all proximinal subsets of X.

2. **Main results**

We are ready to state and prove our main results.

Theorem 2.1. *Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to P(X)$ a β-admissible generalized β-contractive multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has approximate fixed points.*

Proof. Choose $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Since T is proximinal valued, we can choose a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) = d(x_n, Tx_n)$ for all $n \geq 0$. Since T is β-admissible and $\beta(A, Tx_0) \geq 1$, it is easy to see that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all $n \geq 0$. Choose $g \in \mathcal{R}$ such that

$$\beta(Tx, Ty)H(Tx, Ty) \leq g(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$

for all $x, y \in X$. Fix $1 > r > h$, where $h = g(1,1,1,2,0)$. Then, we have

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) \\ &\leq \beta(Tx_0, Tx_1)H(Tx_0, Tx_1) \\ &\leq g(d(x_0, x_1), d(x_1, Tx_1), d(x_0, Tx_0), d(x_0, Tx_1), d(x_1, Tx_0)) \\ &\leq g(d(x_0, x_1), d(x_1, x_2), d(x_0, x_1), d(x_0, x_1) + d(x_1, x_2), 0). \end{aligned}$$

By using Proposition 1.1, we obtain $d(x_1, Tx_1) \leq hd(x_0, x_1) < rd(x_0, x_1)$. On the other hand, we have

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) \\ &\leq \beta(Tx_1, Tx_2)H(Tx_1, Tx_2) \\ &\leq g(d(x_1, x_2), d(x_2, Tx_2), d(x_1, Tx_1), d(x_1, Tx_2), d(x_2, Tx_1)) \\ &\leq g(d(x_1, x_2), d(x_2, x_3), d(x_1, x_2), d(x_1, x_2) + d(x_2, x_3), 0). \end{aligned}$$

Again by using Proposition 1.1, we get

$$d(x_2, x_3) \leq hd(x_1, x_2) < rd(x_1, x_2) < r^2 d(x_0, x_1).$$

Continuing the same process, we conclude that

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) < rd(x_{n-1}, x_n) < r^n d(x_0, x_1)$$

for all $n \geq 0$. But, $d(x_n, Tx_n) \leq d(x_n, x_{n+1})$ for all $n \geq 0$. This implies that $\inf_{x \in X} d(x, Tx) = 0$ and so T has an approximate fixed point. □

Corollary 2.2. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and $T : X \to P(X)$ a β-admissible lower semi-continuous generalized β-contractive multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.*

Proof. By using a similar argument as in the in proof of Theorem 2.1, we obtain

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) < rd(x_{n-1}, x_n) < r^n d(x_0, x_1).$$

Then for each natural numbers m and n with $m < n$, we have

$$d(x_m, x_n) \leq (r^m + r^{m+1} + ... + r^{n-1})d(x_0, x_1) < \frac{r^m}{1 - r} d(x_0, x_1).$$

Hence, $\{x_n\}$ is a Cauchy sequence. Choose $x^* \in X$ such that $x_n \to x^*$. Since T is lower semi-continuous at x^*, for each $y \in Tx^*$ there exists a sequence $\{y_n\}$ such that $y_n \to y$ and $y_n \in Tx_n$ for all n. Let $n \geq 1$ be given. Then, for each $u \in Tx_n$ we have

$$d(x^*, Tx^*) \leq d(x^*, y) \leq d(x^*, x_{n+1}) + d(x_{n+1}, u) + d(u, y_n) + d(y_n, y).$$

This implies that

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, y_n) + d(y_n, y).$$

Since $x_{n+1} \in Tx_n$ and $y_n \in Tx_n$, $d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(y_n, y)$ for all n. Hence, $d(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. □

As one may know, Banach proved his contraction principle in 1922 ([8]). Also, Nadler extended the Banach contraction principle to set-valued mappings in 1969 ([36]). In fact, he proved that if (X,d) is a complete metric space and there exists $k \in (0,1)$ such that $H(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$, then T has a fixed point. Let $\beta : 2^X \times 2^X \to [0,\infty)$ be a mapping. We say that T is a β-contraction whenever there exists $k \in (0,1)$ such that $\beta(Tx,Ty)H(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. It is clear that each Nadler type contractive multifunction is a β-contraction. The next example shows that there exist β-contraction multifunctions which are not Nadler type contractions.

Example 2.1. *Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Define T on X defined by $Tx = [x,4]$ whenever $x \leq 4$ and $Tx = [4,x]$ whenever $x > 4$. Let $\lambda \in (0,1)$ be given. Put $x = 4$ and $y = 4+2\lambda$. Then, we have $H(Tx,Ty) = 2\lambda > \lambda d(x,y)$. Now, define $\beta : 2^X \times 2^X \to [0,\infty)$ by $\beta(A,B) = \frac{1}{4}$ whenever $A \subseteq (-\infty,4]$ and $B \subseteq [4,\infty)$ and $\beta(A,B) = 0$ otherwise. Then, it is easy to see that $\beta(Tx,Ty)H(Tx,Ty) \leq \frac{1}{2}d(x,y)$ for all $x,y \in X$. Hence, T is a β-contraction while is not a Nadler type contraction.*

Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and T a multifunction on X. We say that T is β-convergent whenever for each convergent sequence $\{x_n\}$ with $x_n \to x$, there exists a natural number N such that $\beta(Tx_n,Tx) \geq 1$ for all $n \geq N$.

Corollary 2.3. *Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and $T : X \to P(X)$ a β-admissible and β-contraction multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. Then T has approximate fixed points. If (X,d) is a complete metric space and T is β-convergent, then T has a fixed point.*

Proof. Choose $k \in (0,1)$ such that $\beta(Tx,Ty)H(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Define $g : [0,\infty)^5 \to [0,\infty)$ by $g(x_1,x_2,x_3,x_4,x_5) = kx_1$. Then, $g \in \mathcal{R}$ and

$$\beta(Tx,Ty)H(Tx,Ty) \leq g(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))$$

for all $x,y \in X$, that is, T is a generalized β-contractive multifunction. Now by using Theorem 2.1, T has approximate fixed points. Now, we show that T is lower semi-continuous. Let $x \in X$, $\{x_n\}$ be a sequence with $x_n \to x$ and $y \in Tx$. Choose $y_n \in Tx_n$ for all $n \geq 1$. We have to show that $y_n \to y$. Since T is β-convergent, there exists a natural number N such that $\beta(Tx_n,Tx) \geq 1$ for all $n \geq N$. Thus,

$$d(y_n,y) \leq H(Tx_n,Tx) \leq \beta(Tx_n,Tx)H(Tx_n,Tx) \leq kd(x_n,x)$$

for all $n \geq N$. Hence, $y_n \to y$ and so T is lower semi-continuous. If (X,d) is a complete metric space, then by using Corollary 2.2, T has a fixed point. □

In 1968, the notion of Kannan type contraction mappings introduced ([28]). Later, some authors extended the notion for multifunctions (see for example, [14] and [21]). Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and T a multifunction on X. We say that T is a Kannan type contraction whenever there exists $\alpha \in (0,\frac{1}{2})$ such that

$$H(Tx,Ty) \leq \alpha(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$. Also, we say that T is a β-Kannan multifunction whenever there exists $\alpha \in (0,\frac{1}{2})$ such that

$$\beta(Tx,Ty)H(Tx,Ty) \leq \alpha(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$. The next example shows that there exist β-Kannan multifunctions which are not Kannan type contraction.

Example 2.2. *Let $X = [0,3]$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Define T on X defined by $Tx = [\frac{x}{3},\frac{x}{2}]$ for all $x \in X$. Let $\alpha \in (0,\frac{1}{2})$ be given. Put $x = 0$ and $y = 6\alpha$. Then, we have $H(Tx,Ty) = 3\alpha > \alpha(d(x,Tx) + d(y,Ty)) = 4\alpha^2$. Now, define $\beta : 2^X \times 2^X \to [0,\infty)$ by $\beta(A,B) = \frac{1}{4}$ whenever $A,B \subseteq [0,\frac{1}{2}]$ and $\beta(A,B) = 0$ otherwise. Then, it is easy to see that*

$$\beta(Tx,Ty)H(Tx,Ty) \leq \frac{1}{4}(d(x,Tx) + d(y,Ty))$$

for all $x,y \in X$. Hence, T is a β-Kannan multifunction while is not a Kannan type contraction.

If we consider the map $g : [0,\infty)^5 \to [0,\infty)$ by $g(x_1,x_2,x_3,x_4,x_5) = \alpha x_2 + \alpha x_3$, then by using Theorem 2.1 and Corollary 2.2 it is easy to obtain next result.

Corollary 2.4. *Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and $T : X \to P(X)$ a β-admissible and β-Kannan multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. Then T has approximate fixed points. If (X,d) is a complete metric space and T is lower semi-continuous, then T has a fixed point.*

In 1971, the notion of Reich type contraction mappings introduced ([39]). Later, the notion was extended for multifunctions ([38]). Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and T a multifunction on X. We say that T is a Reich type contraction whenever there exist nonnegative real numbers α,β,γ with $\alpha + \lambda + \gamma < 1$ such that

$$H(Tx,Ty) \leq \alpha d(x,y) + \lambda d(x,Tx) + \gamma d(y,Ty)$$

for all $x,y \in X$. Also, we say that T is a β-Reich multifunction whenever there exists there exist nonnegative real numbers α,β,γ with $\alpha + \lambda + \gamma < 1$ such that $\beta(Tx,Ty)H(Tx,Ty) \leq \alpha d(x,y) + \lambda d(x,Tx) + \gamma d(y,Ty)$ for all $x,y \in X$. The next example shows that there exist β-Reich multifunctions which are not Reich type contraction.

Example 2.3. *Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define T on X defined by $Tx = [\frac{x}{3}, x]$ for all $x \in X$. Let $\alpha, \beta, \gamma \in [0, \infty)$ with $\alpha + \lambda + \gamma < 1$ be given. Put $x = 0$ and $y = 2$. Then, we have*

$$H(Tx, Ty) = 2 > \alpha d(x, y) + \lambda d(x, Tx) + \gamma d(y, Ty).$$

Now, define $\beta : 2^X \times 2^X \to [0, \infty)$ by $\beta(A, B) = \frac{\alpha}{2}$ for all subsets A and B. Then, it is easy to see that

$$\beta(Tx, Ty)H(Tx, Ty) \leq \alpha d(x, y) + \lambda d(x, Tx) + \gamma d(y, Ty)$$

for all $x, y \in X$. Hence, T is a β-Reich multifunction while is not a Reich type contraction.

If we consider the map $g : [0, \infty)^5 \to [0, \infty)$ by $g(x_1, x_2, x_3, x_4, x_5) = \alpha x_1 + \lambda x_2 + \gamma x_3$, then by following the proof of Corollary 2.4, one can obtain next result.

Corollary 2.5. *Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and $T : X \to P(X)$ a β-admissible and β-Reich multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has approximate fixed points. If (X, d) is a complete metric space and T is lower semi-continuous, then T has a fixed point.*

In 1972, the notion of Chatterjea type contraction mappings introduced ([17]). Later, the notion extended for multifunctions ([21]). Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and T a multifunction on X. We say that T is a Chatterjea type contraction whenever there exists $\alpha \in (0, \frac{1}{2})$ such that $H(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$ for all $x, y \in X$. Also, we say that T is a β-Chatterjea multifunction whenever there exists $\alpha \in (0, \frac{1}{2})$ such that $\beta(Tx, Ty)H(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$ for all $x, y \in X$. The next example shows that there exist β-Chatterjea multifunctions which are not Chatterjea type contraction.

Example 2.4. *Let $X = [0, 4]$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define T on X defined by $Tx = [\frac{x}{2}, x]$ for all $x \in X$. Let $\alpha \in (0, \frac{1}{2})$ be given. Put $x = 0$ and $y = 2\alpha$. Then, we have $H(Tx, Ty) = 2\alpha > \alpha(d(x, Ty) + d(y, Tx)) = 3\alpha^2$. Now, define $\beta : 2^X \times 2^X \to [0, \infty)$ by $\beta(A, B) = \frac{1}{16}$ for all subsets A and B. Then, it is easy to see that $\beta(Tx, Ty)H(Tx, Ty) \leq \frac{1}{4}(d(x, Ty) + d(y, Tx))$ for all $x, y \in X$. Hence, T is a β-Chatterjea multifunction while is not a Chatterjea type contraction.*

One can easily conclude the next result.

Corollary 2.6. *Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and $T : X \to P(X)$ a β-admissible and β-Chatterjea multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has approximate fixed points. If (X, d) is a complete metric space and T is lower semi-continuous, then T has a fixed point.*

In 1972, the notion of Zamfirescu type contraction mappings introduced ([45]). Later, the notion extended for multifunctions ([37]). Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and T a multifunction on X. We say that T is a Zamfirescu type contraction whenever there exists $k \in [0, 1)$ such that $H(Tx, Ty) \le kM_T(x, y)$ for all $x, y \in X$, where

$$M_T(x, y) = \max\{d(x, y), \frac{1}{2}[d(x, Ty) + d(y, Tx)], \frac{1}{2}[d(x, Tx) + d(y, Ty)]\}.$$

Also, we say that T is a β-Zamfirescu multifunction whenever there exists $k \in [0, 1)$ such that $\beta(Tx, Ty)H(Tx, Ty) \le kM_T(x, y)$ for all $x, y \in X$. The next example shows that there exist β-Zamfirescu multifunctions which are not Zamfirescu type contraction.

Example 2.5. *Let $X = [0, 2]$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define T on X defined by $Tx = [\frac{x}{3}, x]$ for all $x \in X$. Let $k \in [0, 1)$ be given. Put $x = 0$ and $y = 2k$. Then, we have $H(Tx, Ty) = 2k > kM_T(x, y) = 2k^2$. Now, define $\beta : 2^X \times 2^X \to [0, \infty)$ by $\beta(A, B) = \frac{1}{6}$ for all subsets A and B. Then, it is easy to see that $\beta(Tx, Ty)H(Tx, Ty) \le \frac{1}{2}M_T(x, y)$ for all $x, y \in X$. Hence, T is a β-Zamfirescu multifunction while is not a Zamfirescu type contraction.*

Again, one can obtain next result.

Corollary 2.7. *Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and $T : X \to P(X)$ a β-admissible and β-Zamfirescu multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \ge 1$. Then T has approximate fixed points. If (X, d) is a complete metric space and T is lower semi-continuous, then T has a fixed point.*

In 1972, the notion of Ciric type contraction mappings introduced ([15]). Later, the notion extended for multifunctions ([19]). Let (X, d) be a metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and T a multifunction on X. We say that T is a Ciric type contraction whenever there exists $\lambda \in (0, 1)$ such that $H(Tx, Ty) \le \lambda N_T(x, y)$ for all $x, y \in X$, where

$$N_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

We say that T is a β-Ciric multifunction whenever there exists $\lambda \in (0, 1)$ such that $\beta(Tx, Ty)H(Tx, Ty) \le \lambda N_T(x, y)$ for all $x, y \in X$. The next example shows that there exist β-Ciric multifunctions which are not Ciric type contraction.

Example 2.6. *Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define T on X defined by $Tx = [\frac{x}{4}, x]$ for all $x \in X$. Let $\lambda \in (0, 1)$ be given. Put $x = 0$ and $y = \frac{\lambda}{2}$. Then, we have $H(Tx, Ty) = \frac{\lambda}{2} > \lambda N_T(x, y) = \lambda d(x, y) = \frac{\lambda^2}{2}$. Now, define $\beta : 2^X \times 2^X \to [0, \infty)$ by $\beta(A, B) = \frac{1}{6}$ for all subsets A and B. Then, it is easy to see that $\beta(Tx, Ty)H(Tx, Ty) \le \frac{1}{2}N_T(x, y)$ for all $x, y \in X$. Hence, T is a β-Ciric multifunction while is not a Ciric type contraction.*

The reader can get a similar result to Corollary 2.7 for β-Ciric multifunctions. In 1974, the notion of quasi-contractive mappings introduced by Ciric ([16]). Later, the notion extended for multifunctions (see for example [6, 23, 26, 27] and [40]). Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and T a multifunction on X. We say that T is a quasi-contraction whenever there exists $\lambda \in (0,1)$ such that

$$H(Tx,Ty) \leq \lambda \max\{d(x,y), d(y,Ty), d(x,Tx), d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$. We say that T is a β-quasi-contraction whenever there exists $\lambda \in (0,1)$ such that

$$\beta(Tx,Ty)H(Tx,Ty) \leq \lambda \max\{d(x,y), d(y,Ty), d(x,Tx), d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$. The next example shows that there exist β-quasi-contractions which are not quasi-contraction.

Example 2.7. *Let $X = [0,5]$ and $d(x,y) = |x-y|$ for all $x, y \in X$. Define T on X defined by $Tx = [\frac{x}{2}, x]$ for all $x \in X$. Let $\lambda \in (0,1)$ be given. Put $x = 0$ and $y = \lambda$. Then, we have*

$$H(Tx,Ty) = \lambda > \lambda \max\{d(x,y), d(y,Ty), d(x,Tx), d(x,Ty), d(y,Tx)\} = \lambda^2.$$

Now, define $\beta : 2^X \times 2^X \to [0,\infty)$ by $\beta(A,B) = \frac{1}{5}$ for all subsets A and B. Then, it is easy to see that

$$\beta(Tx,Ty)H(Tx,Ty) \leq \frac{1}{2} \max\{d(x,y), d(y,Ty), d(x,Tx), d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$. Hence, T is a β-quasi-contraction while is not a quasi-contraction.

If we consider the map $g : [0,\infty)^5 \to [0,\infty)$ by $g(x_1, x_2, x_3, x_4, x_5) = \lambda \max\{x_1, x_2, x_3, x_4, x_5\}$, then by following the proof of Corollary 2.4, one can obtain the next result.

Corollary 2.8. *Let (X,d) be a metric space, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and $T : X \to P(X)$ a β-quasi-contraction and β-admissible multifunction. Suppose that there exist $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has approximate fixed points. If (X,d) is a complete metric space and T is lower semi-continuous, then T has a fixed point.*

In 2008, Suzuki introduced a new type of mappings and a generalization of the Banach contraction principle in which the completeness can be also characterized by the existence of fixed points of these mappings ([44]). Consider the non-increasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ by $\theta(r) = 1$ whenever $r \leq \frac{\sqrt{5}-1}{2}$, $\theta(r) = \frac{1-r}{r^2}$ whenever $\frac{\sqrt{5}-1}{2} < r \leq \frac{1}{\sqrt{2}}$ and $\theta(r) = \frac{1}{1+r}$ whenever $\frac{1}{\sqrt{2}} < r < 1$. Let (X,d) be a metric space, $r \in [0,1)$ and T be a mapping on X such that $\theta(r)d(x,Tx) \leq d(x,y)$ implies$d(Tx,Ty) \leq rd(x,y)$ for all $x, y \in X$. Suzuki proved that T has a unique fixed point ([44]). Later, some authors tried to generalize the results of Suzuki for mappings and multifunctions (see for example, [29, 35] and [20]). In 2011, Aleomraninejad, et. al. collected these type results in a result ([2]). By using the main idea of [2], we give our last result about fixed point of β-generalized Suzuki type proximinal valued multifunctions.

Theorem 2.9. *Let (X, d) be a complete metric space, α a constant in $(0, 1)$, $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping and $g \in \mathcal{R}$ with $\alpha(h + 1) \leq 1$, where $h = g(1, 1, 1, 2, 0)$. Suppose that T is a β-admissible and β-convergent proximinal valued multifunction on X such that $\alpha d(x, Tx) \leq d(x, y)$ implies*

$$\beta(Tx, Ty)H(Tx, Ty) \leq g(d(x, y), d(y, Ty), d(x, Tx), d(x, Ty), d(y, Tx))$$

for all $x, y \in X$. Assume that there exits a subset A of X and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Choose the subset A of X and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Since T is proximinal valued, for each $n \geq 0$ there exists $x_{n+1} \in Tx_n$ such that $d(x_n, x_{n+1}) = d(x_n, Tx_n)$. Since T is β-admissible and $\beta(A, Tx_0) \geq 1$, it is easy to see that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all $n \geq 1$. Fix $1 > r > h$. Since $\alpha d(x_0, Tx_0) < d(x_0, x_1)$, by using the assumption we have

$$\begin{aligned} d(x_1, Tx_1) &\leq H(Tx_0, Tx_1) \\ &\leq \beta(Tx_0, Tx_1)H(Tx_0, Tx_1) \\ &\leq g(d(x_0, x_1), d(x_1, Tx_1), d(x_0, Tx_0), d(x_0, Tx_1), d(x_1, Tx_0)) \\ &\leq g(d(x_0, x_1), d(x_1, Tx_1), d(x_0, x_1), d(x_0, x_1) + d(x_1, Tx_1), 0). \end{aligned}$$

By using Proposition 1.1, we get $d(x_1, Tx_1) \leq hd(x_0, x_1) < rd(x_0, x_1)$. By continuing this process, it is easy to see that $d(x_{n+1}, x_n) < r^n d(x_0, x_1)$ and $d(x_{n+1}, Tx_{n+1}) \leq hd(x_{n+1}, x_n)$ for all $n \geq 0$. If $x_m = x_{m+1}$ for some $m \geq 1$, then x_m is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$. Choose $x \in X$ such that $x_n \to x$. We claim that either $\alpha d(x_n, Tx_n) \leq d(x_n, x)$ or $\alpha d(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, x)$ hold for all n. If $\alpha d(x_n, Tx_n) > d(x_n, x)$ and $\alpha d(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, x)$ for some $n \geq 1$, then

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, x) + d(x, x_n) \\ &< \alpha d(x_{n+1}, Tx_{n+1}) + \alpha d(x_n, Tx_n) \\ &\leq \alpha hd(x_n, x_{n+1}) + \alpha d(x_n, x_{n+1}) \end{aligned}$$

and so $\alpha(h + 1) > 1$ which is a contradiction. Thus, either

$$\beta(Tx_n, Tx)H(Tx_n, Tx) \leq g(d(x_n, x), d(x, Tx), d(x_n, Tx_n), d(x_n, Tx), d(x, Tx_n))$$

or

$$\beta(Tx_{n+1}, Tx)H(Tx_{n+1}, Tx)$$
$$\leq g(d(x_{n+1}, x), d(x, Tx), d(x_{n+1}, Tx_{n+1}), d(x_{n+1}, Tx), d(x, Tx_{n+1}))$$

hold for all n. Hence, either there exists an infinite subset $I \subseteq \mathbb{N}$ such that

$$\beta(Tx_n, Tx)H(Tx_n, Tx) \le g(d(x_n, x), d(x, Tx), d(x_n, Tx_n), d(x_n, Tx), d(x, Tx_n))$$

for all $n \in I$, or there exists an infinite subset $J \subseteq \mathbb{N}$ such that

$$\beta(Tx_{n+1}, Tx)H(Tx_{n+1}, Tx)$$
$$\le g(d(x_{n+1}, x), d(x, Tx), d(x_{n+1}, Tx_{n+1}), d(x_{n+1}, Tx), d(x, Tx_{n+1}))$$

for all $n \in J$. Since T is β-convergent, in the first case we obtain

$$\begin{aligned} &d(x, Tx) \\ &\quad\le\ d(x, Tx_n) + H(Tx_n, Tx) \\ &\quad\le\ d(x, Tx_n) + \beta(Tx_n, Tx)H(Tx_n, Tx) \\ &\quad\le\ d(x, x_{n+1}) + g(d(x_n, x), d(x, Tx), d(x_n, Tx_n), d(x_n, Tx), d(x, Tx_n)) \\ &\quad\le\ d(x, x_{n+1}) + g(d(x_n, x), d(x, Tx), d(x_n, x_{n+1}), d(x_n, x) + d(x, Tx), d(x, x_{n+1})) \end{aligned}$$

for sufficiently large $n \in I$. Since g is continuous, we get

$$d(x, Tx) \le g(0, d(x, Tx), 0, 0 + d(x, Tx), 0)$$

and so by using Proposition 1.1, we conclude that $d(x, Tx) = 0$. Since T is β-convergent, in the second case we obtain

$$\begin{aligned} &d(x, Tx) \\ &\quad\le\ d(x, Tx_{n+1}) + H(Tx_{n+1}, Tx) \\ &\quad\le\ d(x, Tx_{n+1}) + \beta(Tx_{n+1}, Tx)H(Tx_{n+1}, Tx) \\ &\quad\le\ d(x, x_{n+2}) + g(d(x_{n+1}, x), d(x, Tx), d(x_{n+1}, Tx_{n+1}), d(x_{n+1}, Tx), d(x, Tx_{n+1})) \\ &\quad\ \ d(x, x_{n+2}) + g(d(x_{n+1}, x), d(x, Tx), d(x_{n+1}, x_{n+2}), d(x_{n+1}x) + d(x, Tx), d(x, x_{n+2})) \end{aligned}$$

for sufficiently large $n \in J$. Since g is continuous, we get

$$d(x, Tx) \le g(0, d(x, Tx), 0, 0 + d(x, Tx), 0)$$

and so by using Proposition 1.1, we obtain $d(x, Tx) = 0$. Thus, $x \in Tx$ and so T has a fixed point. □

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, Fixed point theory for Lipschitzian-type mappings with applications, Springer, New York, 2009.

[2] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, Some fixed point results on a metric space with a graph, *Topol. Appl.* **159** (2012), no. 3, 659–663.

[3] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, On fixed point generalizations of suzuki's method, *Appl. Math. Lett.* **24** (2011), no. 7, 1037–1040.

[4] H. Alikhani, Sh. Rezapour and N. Shahzad, Fixed points of a new type contractive mappings and multifunctions, *Filomat* **27** (2013), no. 7, 1315–1319.

[5] A. Almudevar, Approximate fixed point iteration with an application to infinite horizon Markov decision processes, *SIAM J. Control Optim.* **47** (2008), no. 5, 2303–2347.

[6] A. Amini-Harandi, Fixed point theory for set-valued quasi-contraction maps in metric spaces, *Appl. Math. Lett.* **24** (2011), no. 11, 1791–1794.

[7] D. Baleanu, H. Mohammadi and Sh. Rezapour, Some existence results on nonlinear fractional differential equations, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **371** (2013), no. 1990, 20120144, 7 pages.
[8] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, *Fund. Math.* **3** (1922) 133–181.
[9] C. S. Barroso, O. F. K. Kalenda and M. P. Reboucas, Optimal approximate fixed point results in locally convex spaces, *J. Math. Anal. Appl.* **401** (2013), no. 1, 1–8.
[10] C. S. Barroso, O. F. K. Kalenda and P. K. Lin, On the approximate fixed point property in abstract spaces, *Math. Z.* **271** (2012), no. 3-4, 1271–1285.
[11] I. Beg and M. Abbas, Fixed point, approximate fixed point and Kantorovich-Rubinstein maximum principle in convex metric spaces, *J. Appl. Math. Comput.* **27** (2008), no. 1-2, 211–226.
[12] M. Berinde, Approximate fixed point theorems, *Stud. Univ. Babeş-Bolyai Math.* **51** (2006), no. 1, 11-25.
[13] J. M. Borwein and A. S. Lewis, Convex Analysis and Nonlinear Optimization, Theory and Examples, Springer-Verlag, New York, 2000.
[14] D. Bosko and D. Doric, Multivalued generalizations of the Kannan fixed point theorem, *Filomat* **25** (2011), 125–131.
[15] Lj. B. Ćirić, Fixed points for generalized multi-valued contractions, *Mat. Vesnik* **9(24)** (1972) 265–272.
[16] Lj. B. Ciric, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* **45** (1974) 267–273.
[17] S. K. Chatterjea, Fixed-point theorems, *C. R. Acad. Bulgare Sci.* **25** (1972) 727–730.
[18] H. H. Cuenya, Approximate fixed points and best proximity pairs, *Fixed Point Theory* **11** (2010), no. 2, 251–258.
[19] P. Z. Daffer and H. Kaneko, Fixed points of generalized contractive multi-valued mappings, *J. Math. Anal. Appl.* **192** (1995), no. 2, 655–666.
[20] S. Dhompongsa and H. Yingtaweesittikul,Fixed point for multivalued mappings and the metric completeness, *Fixed Point Theory Appl.* (2009), Article ID 972395, 15 pages.
[21] W. S. Du, Fixed point theorems for generalized Hausdorff metrics, *Int. Math. Forum* **3** (2008) no. 21-24, 1011–1022.
[22] W. S. Du, On generalized weakly directional contractions and approximate fixed point property with applications, *Fixed Point Theory Appl.* (2012) 2012:6, 22 pages.
[23] R. H. Haghi, Sh. Rezapour and N. Shahzad, On fixed points of quasi-contraction type multifunctions, *Appl. Math. Lett.* **24** (2011)), no. 5, 843–846.
[24] J. Hasanzade Asl, Sh. Rezapour and N. Shahzad, On fixed points of α-ψ-contractive multifunctions, *Fixed Point Theory Appl.* (2012) 2012:212, 6 pages.
[25] M. Jleli and B. Samet, Best proximity points for α-ψ-proximal contractive type mappings and applications, *Bull. Sci. Math.* **137** (2013), no. 8, 977–995.
[26] D. Ilic and V. Rakocevic, Quasi-contraction on a cone metric space, *Appl. Math. Lett.* **22** (2009), no. 5, 728–731.
[27] Z. Kadelburg, S. Radenovic and V. Rakocevic, Remarks on "quasi-contraction on a cone metric space", *Appl. Math. Lett.* **22** (2009), no. 11, 1674–1679.
[28] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.* **60** (1968) 71–76.
[29] M. Kikkawa and T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, *Nonlinear Anal.* **69** (2008), no. 9, 2942–2949.
[30] W. A. Kirk, Approximate fixed points for mappings in Banach spaces. Nonlinear functional analysis and its applications (Maratea, 1985), 299–303, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 173, Reidel, Dordrecht, 1986.
[31] W. A. Kirk, Approximate fixed points of nonexpansive maps, *Fixed Point Theory* **10** (2009), no. 2, 275–288.

[32] W. A. Kirk, Remarks on approximate fixed points, *Nonlinear Anal.* **75** (2012), no. 12, 4632–4636.

[33] M. A. Miandaragh, M. Postolache and Sh. Rezapour, Some approximate fixed point results for generalized α-contractive mappings, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **75** (2013), no. 2, 3–10.

[34] B. Mohammadi, Sh. Rezapour and N. Shahzad, Some results on fixed points of α-ψ-Ciric generalized multifunctions, *Fixed Point Theory Appl.* **2013**, 2013:24, 10 pages.

[35] G. Mot and A. Petrusel, Fixed point theory for a new type of contractive multivalued operators, *Nonlinear Anal.* **70** (2009), no. 9, 3371–3377.

[36] S. B. Nadler, Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969) 475–488.

[37] K. Neammanee and A. Kaewkhao, Fixed point theorems of multivalued Zamfirescu mapping, *J. Math. Research* **2** (2010) 150–156.

[38] I. A. Rus, A. Petrusel and A. Sintamarian, Data dependence of the fixed points set of multivalued weakly picard operators, *Studia Univ. Babeş-Bolyai Math.* (2001), no. 2, 111–121.

[39] S. Reich, Kannan's fixed point theorem, *Boll. Un. Mat. Ital. (4)* **4** (1971) 1–11.

[40] Sh. Rezapour, R. H. Haghi, N. Shahzad, Some Notes on fixed points of quasi-contraction maps, *Appl. Math. Lett.* **23** (2010), no. 4, 498–502.

[41] Sh. Rezapour and J. Hasanzade Asl, A simple method for obtaining coupled fixed points of α-ψ-contractive type mappings, *Int. J. Anal.* (2013), Article ID 438029, 7 pages.

[42] G. S. Saluja, Approximation of common random fixed point for a finite family of non-self asymptotically nonexpansive random mappings, *Demonstratio Math.* **42** (2009), no. 3, 581–598.

[43] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-cotractive type mappings, *Nonlinear Anal.* **75** (2012), no. 4, 2154–2165.

[44] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.* **136** (2008), no. 5, 1861–1869.

[45] T. Zamfirescu, Fix point theorems in metric spaces, *Arch. Math. (Basel)* **23** (1972) 292–298.

(M. A. Miandaragh) Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

E-mail address: mehdi59ir@yahoo.com

(A. Pitea) Department of Mathematics, University Politehnica of Bucharest, Bucharest, Romania

E-mail address: arianapitea@yahoo.com

(Sh. Rezapour) Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

E-mail address: sh.rezapour@azaruniv.edu