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AUTOMATIC CONTINUITY OF SURJECTIVE n-HOMOMORPHISMS ON BANACH ALGEBRAS

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Abstract. In this paper, we show that every surjective n-homomorphism (n-anti-homomorphism) from a Banach algebra A into a semisimple Banach algebra B is continuous.
Keywords: Banach algebra, n-homomorphism, semisimple algebra.
MSC(2010): Primary: 46H05.

1. **Introduction**

Let A and B be complex Banach algebras. The linear mapping $\theta : A \longrightarrow B$ is called an n-homomorphism, if $\theta(a_1a_2\cdots a_n) = \theta(a_1)\theta(a_2)\cdots\ \theta(a_n)$, for all $a_1a_2\cdots a_n \in A$. A linear mapping $\theta : A \longrightarrow B$ is called an n-anti-homomorphism if $\theta(a_1a_2\cdots a_n) = \theta(a_n)\cdots\theta(a_2)\theta(a_1)$, for all $a_1a_2\cdots a_n \in A$. The algebra B is called factorizable if for every $a \in B$ there are $b, c \in B$ such that $a = bc$. The concept of n-homomorphisms was studied for complex algebras by Hejazian, Mirzavaziri, and Moslehian [6]. Bračič and Moslehian [1] investigated. 3-homomorphisms on Banach algebras with bounded approximate identities and established that every involution preserving 3-homomorphism between C^*-algebras is continuous and norm decreasing. It is due to Park and Trout that every $*$-preserving n-homomorphism between C^*-algebras is continuous [7]. Automatic continuity of n-homomorphisms considered for factorizable Banach algebras in [4]. A similar problem was studied for topological algebras in [5]. A linear mapping $\theta : A \longrightarrow B$ is called an n-Jordan homomorphism if $\theta(a^n) = [\theta(a)]^n$ for all $a \in A$. Some results about automatic continuity of n-Jordan homomorphisms on Banach algebras and C^*-algebras are investigated in [3].

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Let A be a Banach algebra. If A has a unit e_A, the spectrum of $a \in A$ is defined by

$$sp_A(a) = \{\lambda \in \mathbb{C} : \lambda e_A - a \notin Inv\ A\},$$

and the spectral radius of a is defined as follows

$$\rho_A(a) = \sup\{|\lambda| : \lambda \in sp_A(a)\},$$

where $InvA$ is the set of all invertible elements of A.

If A is non-unital, we consider the quasi product "$\diamond$" on A as follows

$$a \diamond b = a + b - ab \qquad (a, b \in A).$$

An element $a \in A$ is called left (right) quasi-invertible if there is $b \in A$ such that $b \diamond a = 0$ ($a \diamond b = 0$). Then an element $a \in A$ is quasi-invertible if it is both left and right quasi-invertible. The set of all quasi-invertible elements of A denoted by $q - InvA$.

Let A be a non-unital (complex) Banach algebra. Then $A^{\#} = A \oplus \mathbb{C}$ is a unital Banach algebra with the product and norm given by

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta),$$

$$\|(a, \alpha)\| = \|a\| + |\alpha|$$

for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$. We denote the identity of $A^{\#}$ by $e_{A^{\#}}(= (0, 1))$.

Let A be a non-unital (complex) Banach algebra. Then, obviously for every $x, y \in A$, $a \diamond b = 0$ if and only if

$$(e_{A^{\#}} - a)(e_{A^{\#}} - b) = e_{A^{\#}}.$$

The definition of spectrum in the non-unital Banach algebras is different from the unital case, and we define it as follows

$$sp_A(a) = \{0\} \cup \{\lambda \in \mathbb{C}\backslash\{0\} : \frac{1}{\lambda}a \notin\ q - InvA\},$$

and it is easy to see that $sp_A(a) = sp_{A^{\#}}((a, 0))$ and $\rho_A(a) = \rho_{A^{\#}}((a, 0))$. By $\partial sp(a)$, we mean the boundary set of $sp(a)$. The radical of A, denoted by $Rad(A)$, is the intersection of all maximal left ideals of A. The algebra A is called semisimple if $Rad(A) = \{0\}$ (for more details see Section 1.5 of [2]). If A is a semisimple Banach algebra, given $a \in A$, if $axy = 0$ (or $xay = 0$ or $xya = 0$) for all $x, y \in A$, then it is easy to show that $a = 0$.

2. **Automatic continuity**

In this section we extend Johnson's techniques [8] for n-homomorphism on non-unital Banach algebras. Our results differ from those obtained in [4, 5, 8] and [7].

We state [8, Lemma 1], which is valid for non-unital Banach algebras.

Lemma 2.1. *([8, Lemma 1]) Let A be a Banach algebra, $a \in A$, and suppose that $\rho_A(a_1 a) = 0$ for all $a_1 \in A$. Then $a \in Rad(A)$.*

We can generalize this result for non-unital Banach algebras as follows:

Lemma 2.2. *Let A be a Banach algebra. Then*

(1) *given $a \in A$ satisfies $\rho_A(a_1 a_2 \cdots a_{n-1} a) = 0$ for all $a_1, a_2, \dots, a_{n-1} \in A$, then $a \in Rad(A)$.*
(2) *given $a \in A$ satisfies $\rho_A(a a_1 a_2 \cdots a_{n-1}) = 0$ for all $a_1, a_2, \dots, a_{n-1} \in A$, then $a \in Rad(A)$.*

Proof. (1) Suppose that $\rho_A(a_1 a_2 \cdots a_{n-1} a) = 0$ for all $a_1, a_2, \dots, a_{n-1} \in A$. By Lemma 2.1, $a_2 \cdots a_{n-1} a \in Rad(A)$. Since the radical of any normed algebra is a topologically nil ideal ([9, Theorem 2.3.4]), $\rho_A(a_2 \cdots a_{n-1}) = 0$ for all $a_2, \dots, a_{n-1} \in A$. Repeating the argument we get $\rho_A(a_{n-1} a) = 0$ for all $a_{n-1} \in A$, which, by Lemma 2.1, assures that $a \in Rad(A)$. Part (2) needs a similar argument. □

Let $T : A \longrightarrow B$ be a linear mapping between Banach algebras. The separating space of T is defined by

$$\mathfrak{S}(T) = \{b \in B : \text{there exists } (a_n) \subseteq A \text{ such that } a_n \to 0 \text{ and } T(a_n) \longrightarrow b\}.$$

We know that $\mathfrak{S}(T)$ is a closed linear subspace of B. By the closed graph theorem, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$ ([10, Lemma 1.2]). The proof of the following lemma is clear and lefts to the reader.

Lemma 2.3. *Let $\theta : A \longrightarrow B$ be an n-homomorphism between Banach algebras. The following statements hold:*

(1) *Given $b_1, \dots, b_{n-1}$ in $\theta(A)$ and $b \in \mathfrak{S}(\theta)$, the product*
$$b_1 \dots b_{i-1} b b_{i+1} \dots b_{n-1}$$
lies in $\mathfrak{S}(\theta)$.
(2) *When θ has dense range and $b \in \mathfrak{S}(\theta)$, then*
$$b_1 \dots b_{i-1} b b_{i+1} \dots b_{n-1} \in \mathfrak{S}(\theta),$$
for $b_1, \dots, b_{n-1}$ in B.
(3) *When θ has dense range, then*
$$b_1 \dots b_{n-1} b,\ b b_1 \dots b_{n-1} \in \mathfrak{S}(\theta),$$
for $b_1, \dots, b_{n-1} \in \mathfrak{S}(\theta)$ and $b \in B$.

Now, we consider our main result. Note that the first part of the proof is taken from [4, Theorem 2.7] see also [8], and for completeness we include the proof.

Theorem 2.4. *Let A and B be Banach algebras (non-unital) which B is semisimple. Then every surjective n-homomorphism $\theta : A \longrightarrow B$ is automatically continuous.*

Proof. Suppose that $(a_m) \subseteq A$ such that $a_m \to 0$ and $\theta(a_m) \longrightarrow b$ in B. Our aim is showing that $b = 0$. Since θ is surjective, there exists $a \in A$ such that $\theta(a) = b$. For $m \geq 1$, we define

$$P_m(z) = z\theta(a_m) + (\theta(a) - \theta(a_m)) \qquad (z \in \mathbb{C}).$$

Then for every $z \in \mathbb{C}$, we have

$$\rho_B(P_m(z)) \leq \|P_m(z)\| \leq |z|\|\theta(a_m)\| + \|\theta(a) - \theta(a_m)\|.$$

In light of [4, Lemma 2.6], we have

$$\begin{aligned} \rho_B(P_m(z)^{n-1}) &\leq \rho_A((za_m + (a - a_m))^{n-1}) \leq \|(za_m + (a - a_m))^{n-1}\| \\ &\leq (|z|\|a_m\| + \|a - a_m\|)^{n-1}. \end{aligned} \tag{2.1}$$

By [8, Lemma 2], we have

$$\rho_B(b)^2 \leq (R\|a_m\| + \|a - a_m\|)(R^{-1}\|\theta(a_m)\| + \|\theta(a) - \theta(a_m)\|) \longrightarrow 0, \tag{2.2}$$

as $m \longrightarrow \infty$ and $R \longrightarrow \infty$. This implies that $\rho_B(b) = 0$. Choose nonzero elements $b_1, b_2, \ldots, b_{n-1}$ in B. There are $a_1, a_2, \ldots, a_{n-1} \in A$ such that $\theta(a_1) = b_1$, $\theta(a_2) = b_2, \ldots, \theta(a_{n-1}) = b_{n-1}$. By Lemma 2.3 (3), $b_1 \ldots b_{n-1} b \in \mathfrak{G}(\theta)$ and by the first part of the proof, $\rho_B(b_1 b_2 \cdots b_{n-1} b) = 0$, and Lemma 2.2 implies that $b \in Rad(B)$. Since B is semisimple, we get $b = 0$. □

The next result is devoted to the automatic continuity of $n-$Jordan homomorphisms.

Corollary 2.5. *Let A be a Banach algebra and B be a semisimple Banach algebra. Then every surjective n-Jordan homomorphism $\theta : A \longrightarrow B$ that satisfies $\partial(sp_B(\theta(a)^{n-1})) \subseteq sp_A(a^{n-1}) \cup \{0\}$, for all $a \in A$, is automatically continuous.*

Proof. Similar to the proof of Theorem 2.4, suppose that $(a_m) \subseteq A$ such that $a_m \to 0$ and $\theta(a_m) \longrightarrow b$ in B. As well as, there exists $a \in A$ such that $\theta(a) = b$. Since $\partial(sp_B(\theta(a)^{n-1})) \subseteq sp_A(a^{n-1}) \cup \{0\}$, for $a \in A$, the relations (2.1) and (2.2) hold. Therefore $\rho_B(b) = 0$. Clearly, $\underbrace{a_m a_m \ldots a_m}_{\text{n times}} \longrightarrow 0$ and $\theta(a_m a_m \ldots a_m) = \theta(a_m)^n = b^n$. This follows that $b^n \in \mathfrak{G}(\theta)$ and $\rho_B(\underbrace{bb \ldots b}_{\text{n times}}) = 0$. By Lemma 2.2, $b \in Rad(B)$. Then $b = 0$ and this completes the proof. □

By a similar argument as Theorem 2.4, we have the following result for n-anti-homomorphisms.

Theorem 2.6. *Let A and B be Banach algebras (non-unital) which B is semisimple. Then every surjective n-anti-homomorphism $\theta : A \longrightarrow B$ is automatically continuous.*

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