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SPECIAL CONNECTIONS IN ALMOST PARACONTACT METRIC GEOMETRY

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(Communicated by Mohammad Bagher Kashani)

ABSTRACT. Two types of properties for linear connections (natural and almost paracontact metric) are discussed in almost paracontact metric geometry with respect to four linear connections: Levi-Civita, canonical (Zamkovoy), Golab and generalized dual. Their relationship is also analyzed with a special view towards their curvature. The particular case of an almost para-cosymplectic manifold gives a major simplification in computations since the paracontact form is closed.

Keywords: Almost paracontact metric manifold, natural connection, canonical connection, Golab connection, generalized dual connections.

MSC(2010): Primary: 53C15; Secondary: 53C05.

1. Introduction

The paracontact geometry appears as a natural counter-part of the almost contact geometry in [9]. Comparing with the huge literature in almost contact geometry, it seems that there are necessary new studies in almost paracontact geometry; a very interesting paper connecting these fields is [4]. The present work is another step in this direction, more precisely from the point of view of linear connections living in the almost paracontact universe; it can be considered as a continuation and generalization of [1].

Since the Levi-Civita connection is a fundamental object in (pseudo-) Riemannian geometry we add to our study a pseudo-Riemannian metric; so, we work in the so-called (*hyperbolical*) *paracontact metric geometry*, see also [6]. In this framework there already exists a *canonical connection* introduced in [11] in correspondence with the Tanaka-Webster connection of pseudo-convex CR-geometry; we study the relationship between this linear connection and our

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connections. For example, in Section 2 we consider the notions of *almost paracontact metric connection* and *natural connection* to which the canonical connection belongs.

An important feature of the canonical connection of Zamkovoy is that it is metrical but not symmetrical. We consider in Section 2 another linear connection which is metrical and not torsion-free. More precisely, a quarter-symmetric connection of Golab type [7] is introduced and its properties are analyzed. The particular case of *almost para-cosymplectic manifolds* is a special situation when the computation is more simple and we obtain a case when the Golab curvature coincides with the Levi-Civita curvature.

A last notion introduced in this paper is that of *generalized dual connections* as a generalization of Norden duality of linear connections. So, the last Section is devoted to the study of the generalized dual of the Golab connection. An important tensor field of $(1, 1)$ -type studied for various connections is the projector corresponding to the characteristic vector field (also called *the structural vector field*); a natural connection makes parallel this vector field.

2. Almost paracontact metric geometry and some adapted connections

Let M be a $(2n + 1)$ -dimensional smooth manifold, φ a tensor field of $(1, 1)$ -type called *the structural endomorphism*, ξ a vector field called *the characteristic vector field*, η a 1-form called *the paracontact form* and g a pseudo-Riemannian metric on M of signature $(n + 1, n)$. We say that (φ, ξ, η, g) defines an *almost paracontact metric structure* on M if [11, p. 38], [3]:

1. $\varphi(\xi) = 0, \eta \circ \varphi = 0,$ 2. $\eta(\xi) = 1, \varphi^2 = I - \eta \otimes \xi,$
3. φ induces on the $2n$ -dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P i.e. $P^2 = -1$ and the eigensubbundles T^+, T^- , corresponding to the eigenvalues $1, -1$ of P respectively, have equal dimensions n ; hence $\mathcal{D} = T^+ \oplus T^-$,
4. $g(\varphi \cdot, \varphi \cdot) = -g + \eta \otimes \eta.$

For a list of examples of almost paracontact metric structures see [8, p 84]. From the definition it follows that η is the g -dual of ξ i.e. $\eta(X) = g(X, \xi)$, ξ is a unitary vector field, $g(\xi, \xi) = 1$, and φ is a g -skew-symmetric operator, $g(\varphi X, Y) = -g(X, \varphi Y)$. The tensor field

$$(2.1) \quad \omega(X, Y) := g(X, \varphi Y)$$

is skew-symmetric and

$$(2.2) \quad \omega(\varphi X, Y) = -\omega(X, \varphi Y), \quad \omega(\varphi X, \varphi Y) = -\omega(X, Y).$$

Then ω is called *the fundamental form*. Remark that the canonical distribution \mathcal{D} is φ -invariant since $\mathcal{D} = \text{Im} \varphi$: if $X \in \mathcal{D}$ has the decomposition $X = X^+ + X^-$ with $X^* \in T^*$ then $\varphi X = X^+ - X^-$. Moreover, ξ is orthogonal to \mathcal{D} and

therefore the tangent bundle splits orthogonally, as

$$(2.3) \quad TM = T\mathcal{F} \oplus \langle \xi \rangle.$$

We are interested now in linear connections compatible with the almost paracontact structure. To this aim we introduce:

Definition 2.1. A linear connection ∇ is a natural connection on the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ if it satisfies

$$(2.4) \quad \nabla \eta = \nabla g = 0.$$

So, a natural connection is a g -metric connection making η a parallel 1-form. A direct consequence of the definition is:

Proposition 2.2. *If ∇ is a natural connection on the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ then ξ is a ∇ -parallel vector field: $\nabla \xi = 0$. Hence, the integral curves of ξ are autoparallel curves for ∇ .*

Proof. From the conditions (2.4) we obtain

$$g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi) = X(g(Y, \xi)) = X(\eta(Y)) = \eta(\nabla_X Y) = g(\nabla_X Y, \xi)$$

and whence $\nabla \xi = 0$. \square

The next important problem is if φ is ∇ -parallel, and then, with respect to a general linear connection ∇ , we introduce a new tensor field of $(0, 3)$ -type given by

$$(2.5) \quad F_\nabla(X, Y, Z) := g((\nabla_X \varphi)Y, Z).$$

F_∇ satisfies:

$$(2.6) \quad \begin{cases} F_\nabla(X, Y, Z) + F_\nabla(X, Z, Y) = -(\nabla g)(X, \varphi Y, Z) - (\nabla g)(X, Y, \varphi Z) \\ F_\nabla(X, \varphi Y, Z) - F_\nabla(X, Y, \varphi Z) = -\eta(Z)(\nabla_X \eta)Y - \eta(Y)g(\nabla_X \xi, Z) \\ F_\nabla(X, Y, Z) - F_\nabla(X, \varphi Y, \varphi Z) = \eta(Z)\eta((\nabla_X \varphi)Y) + \eta(Y)g(\nabla_X \xi, \varphi Z) \end{cases}$$

which yields:

Proposition 2.3. *If ∇ is a natural connection on the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ then for any $X, Y, Z \in \mathfrak{X}(M)$ its tensor field F_∇ satisfies*

$$(2.7) \quad \begin{cases} F_\nabla(X, Y, Z) = -F_\nabla(X, Z, Y) \\ F_\nabla(X, \varphi Y, Z) = F_\nabla(X, Y, \varphi Z) \\ F_\nabla(X, \varphi Y, \varphi Z) = F_\nabla(X, Y, Z) - \eta(Z)\eta((\nabla_X \varphi)Y). \end{cases}$$

The relations (2.7) say that $\Omega_X^\nabla := F_\nabla(X, \cdot, \cdot)$ is a 2-form on M with

$$(2.8) \quad \Omega_X^\nabla(\varphi Y, Z) = \Omega_X^\nabla(Y, \varphi Z), \quad \Omega_X^\nabla(\varphi Y, \varphi Z) = \Omega_X^\nabla(Y, Z) - \eta(Z)\eta((\nabla_X \varphi)Y).$$

These relations are a counter-part of equations (2.2).

Proposition 2.4. *If $\nabla\varphi = 0$ then*

$$(2.9) \quad (\nabla_X g)(\xi, Y) = 2(\nabla_X \eta)Y.$$

Proof. From hypothesis it follows $F_{\nabla} = 0$ and then from (2.6) we get

$$\eta(Z)(\nabla_X \eta)Y = -\eta(Y)g(\nabla_X \xi, Z)$$

and with $Z = \xi$ it results

$$(2.10) \quad (\nabla_X \eta)Y = -\eta(Y)\eta(\nabla_X \xi).$$

From (2.6) we obtain

$$g(\nabla_X \xi, \varphi Z) = 0$$

which with $Z \rightarrow \varphi Y$ yields

$$(2.11) \quad g(\nabla_X \xi, Y) = \eta(Y)\eta(\nabla_X \xi).$$

Adding (2.10) and (2.11) it results:

$$(2.12) \quad (\nabla_X \eta)Y = -g(\nabla_X \xi, Y)$$

which is equivalent with (2.9). □

The next step is to unify all these conditions in:

Definition 2.5. ∇ is called almost paracontact metric connection if it satisfies

$$(2.13) \quad \nabla\varphi = \nabla\eta = \nabla g = 0.$$

Therefore, ∇ is an almost paracontact metric connection if it is a natural connection with $\nabla\varphi = 0$. The characteristic vector field ξ is parallel with respect to such a linear connection. From Proposition 2.4 a metric linear connection with $\nabla\varphi = 0$ is an almost paracontact metric connection.

S. Zamkovoy [11, p. 49] defined on an almost paracontact metric manifold a connection $\tilde{\nabla}$, using the Levi-Civita connection ∇^g , as

$$(2.14) \quad \tilde{\nabla}_X Y := \nabla_X^g Y + \eta(X)\varphi Y - \eta(Y)\nabla_X^g \xi + (\nabla_X^g \eta)Y \cdot \xi$$

and called it *canonical paracontact connection*. This linear connection is a natural one according to Proposition 4.2 of [11, p. 49] and it is an almost paracontact metric connection if and only if

$$(2.15) \quad (\nabla_X^g \varphi)Y = \eta(Y)(X - hX) - g(X - hX, Y)\xi$$

where

$$(2.16) \quad h = \frac{1}{2}\mathcal{L}_\xi \varphi$$

with \mathcal{L} the Lie derivative. The tensor field h vanishes if and only if M is *K-paracontact* i.e. ξ is a Killing vector field with respect to g . If M is *K-paracontact* then the condition $\nabla^g \varphi = 0$ in (2.15) yields $\eta(Y)X = g(X, Y)\xi$ and the g -product with ξ in this last relation gives $g = \eta \otimes \eta$ an impossible

relation since it implies $g|_{\mathcal{D}} = 0$. So, in the K -paracontact case ∇^g and $\tilde{\nabla}$ can not be both almost paracontact metric connections.

By using the conventions of [11], for example, the exterior differential of η is given by:

$$(2.17) \quad 2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])$$

respectively [11, p. 39]:

Definition 2.6. $(M, \varphi, \xi, \eta, g)$ is called paracontact metric manifold if $d\eta = \omega$.

On a paracontact metric manifold we have [11, p. 41]: $\nabla_{\xi}^g \varphi = 0$ and ξ is a geodesic vector field i.e. $\nabla_{\xi}^g \xi = 0$. For the following notion we consider the product manifold $M \times \mathbb{R}$ with the tensor field as

$$(2.18) \quad J \left(X, f \frac{d}{dt} \right) = \left(\varphi X + f\xi, \eta(X) \frac{d}{dt} \right).$$

Definition 2.7. ([11, p. 39], [3]) The paracontact structure (φ, η, ξ) is called normal if J is integrable. Moreover, a normal paracontact metric manifold is called paraSasakian manifold.

An important feature of a paraSasakian manifold is that it is K -paracontact.

Let us end this section with the following remark for a linear connection ∇ :

-if ∇ is g -metric then $(\mathcal{L}_{\xi} g)(Y, Z) = (\nabla_X \eta)Y + (\nabla_Y \eta)X$,

-if ∇ is symmetric then $2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X$.

It results that if ∇^g is a natural connection then M is a K -paracontact manifold and η is closed ($d\eta = 0$) which means that M is not a paracontact metric manifold.

3. The Golab connection

In this section we search for a weak version of ∇^g and $\tilde{\nabla}$. Since the metrical condition is a common property of these two connections we look for a weak condition in terms of torsion.

Definition 3.1. The Golab connection [7] associated to the structure (φ, η, g) is the linear connection ∇^G satisfying

$$(3.1) \quad \nabla^G g = 0, \quad T^G = \varphi \otimes \eta - \eta \otimes \varphi.$$

It is known that the unique connection with these properties is given by

$$(3.2) \quad \nabla^G = \nabla^g - \eta \otimes \varphi.$$

We can express the Golab connection, by using the canonical connection (2.14).

$$(3.3) \quad \nabla_X^G Y = \tilde{\nabla}_X Y - 2\eta(X)\varphi Y + \eta(Y)\nabla_X^g \xi + (\nabla_X^g \eta)Y \cdot \xi$$

and then it results that if ∇^g is a natural connection then

$$(3.4) \quad \nabla_X^G Y = \tilde{\nabla}_X Y - 2\eta(X)\varphi Y.$$

The Golab connection is different from the Levi-Civita connection; but from (3.3) it coincides with the canonical connection if and only if

$$(3.5) \quad 2\eta(X)\varphi Y = \eta(Y)\nabla_X^g \xi + (\nabla_X^g \eta)Y \cdot \xi.$$

With $Y = \xi$ it results $\nabla_X^g \xi = -(\nabla_X^g \eta)\xi \cdot \xi$ and since $\nabla_X^g \xi$ is g -orthogonal on ξ we get $\nabla^g \xi = 0$. Returning to (3.5) it results:

$$(3.6) \quad 2\eta(X)\varphi Y = (\nabla_X^g \eta)Y \cdot \xi$$

and with $X = \xi$ we get

$$(3.7) \quad 2\varphi Y = (\nabla_\xi^g \eta)Y \cdot \xi.$$

Then we have $\nabla_\xi^g \eta \neq 0$, in particular ∇^g must not be a natural connection.

Returning to the general case and computing $T^G(\varphi \cdot, \varphi \cdot) = 0$ we get that ∇^G is symmetrical on $Im\varphi = \mathcal{D}$ and therefore it coincides with ∇^g on \mathcal{D} . The main properties of the Golab connection are stated in the next proposition.

Proposition 3.2. *The Golab connection of an almost paracontact metric manifold satisfies*

$$(3.8) \quad \nabla^G \varphi = \nabla^g \varphi, \quad \nabla^G \eta = \nabla^g \eta, \quad \nabla^G \xi = \nabla^g \xi.$$

Proof. By a direct computation we get $\nabla_X^G \varphi Y = \nabla_X^g \varphi Y - \eta(X)\varphi^2 Y$ and

$$\varphi(\nabla_X^G Y) = \varphi(\nabla_X^g Y) - \eta(X)\varphi^2 Y$$

respectively

$$(\nabla_X^G \eta)Y = \nabla_X^G \eta(Y) - \eta(\nabla_X^G Y) = X(\eta(Y)) - \eta(\nabla_X^g Y) + \eta(X)\eta \circ \varphi Y = (\nabla_X^g \eta)Y.$$

□

A natural problem is to determine the necessary and sufficient condition for the Golab connection of an almost paracontact metric manifold to be a natural connection. We obtain:

Theorem 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. Then its Golab connection ∇^G is a natural connection if and only if the Levi-Civita connection ∇^g is a natural connection. This last condition reduces to $\nabla^g \eta = 0$. Moreover, ∇^G is an almost paracontact metric connection if and only if ∇^g is an almost paracontact metric connection.*

A long but straightforward computation gives also:

Theorem 3.4. *The curvature of the Golab connection is*

$$(3.9) \quad R_{XYZ}^G = R_{XYZ}^g - 2d\eta(X, Y)\varphi Z + \eta(X)(\nabla_Y^g \varphi)Z - \eta(Y)(\nabla_X^g \varphi)Z.$$

So, if ∇^g is almost paracontact metric connection then $R^G = R^g$.

If the 1-form η and the 2-form ω are closed we say that $(M, \varphi, \xi, \eta, g)$ is an *almost para-cosymplectic manifold* after [5, p. 562].

Proposition 3.5. *Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold. Then its curvature satisfies*

$$(3.10) \quad R_{XYZ}^G = R_{XYZ}^g + \eta(X)(\nabla_Y^g \varphi)Z - \eta(Y)(\nabla_X^g \varphi)Z.$$

Let us point out an application of the formulae (3.8). Let P_0 be the projector corresponding to $\langle \xi \rangle$ in the decomposition (2.3); namely, if $X \in \mathfrak{X}(M)$ has the decomposition

$$(3.11) \quad X = X^+ + X^- + \eta(X)\xi$$

then $P_0(X) = \eta(X)\xi$. For a general linear connection ∇ we have

$$(3.12) \quad (\nabla_X P_0)Y = \nabla_X(\eta(Y)\xi) - \eta(\nabla_X Y)\xi = (\nabla_X \eta)(Y) \cdot \xi + \eta(Y)\nabla_X \xi$$

and then (3.8) yields

$$(3.13) \quad \nabla^G P_0 = \nabla^g P_0.$$

If ∇^g is a natural connection we get that P_0 is covariant constant with respect to both ∇^g and ∇^G . Since the canonical connection $\tilde{\nabla}$ is natural we already have that P_0 is covariant constant with respect to $\tilde{\nabla}$. Another interesting fact is that the P_0 -Golab connection i.e. with φ of (3.1) replaced by P_0 , it is in fact ∇^g since $P_0 \otimes \eta - \eta \otimes P_0 = 0$.

The projector P_0 can be used to obtain a more simple formula for the canonical connection $\tilde{\nabla}$. Plugging (3.12) in (2.14) gives

$$(3.14) \quad \tilde{\nabla}_X Y = \nabla_X^g Y + \eta(X)\varphi Y - 2\eta(Y)\nabla_X^g \xi + (\nabla_X^g P_0)Y$$

and then, for ∇^g a natural connection we get

$$(3.15) \quad \tilde{\nabla} = \nabla^g + \eta \otimes \varphi$$

yielding a (convex) relationship between all the linear connections studied until now:

$$(3.16) \quad \tilde{\nabla} + \nabla^G = 2\nabla^g.$$

4. Generalized duality for linear connections

Let now ∇ and ∇' be two linear connections on M . We adopt the following notion of generalized conjugation of linear connections from [2, p. 28]

Definition 4.1. We say that ∇ and ∇' are generalized dual connections with respect to the pair (g, η) if, for any $X, Y, Z \in \mathfrak{X}(M)$

$$(4.1) \quad X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla'_X Z) - \eta(X)g(Y, Z)$$

or equivalently

$$(4.2) \quad g(\nabla'_X Z - \nabla_X Z - \eta(X)Z, Y) = \nabla g(X, Y, \xi).$$

Without the last term, the relation (4.1) reduces to the usual *Norden duality* of linear connections from [10].

We shall discuss the behavior of the generalized dual connection ∇' of ∇ if we impose certain conditions on ∇ . Let us remark the following relations:

$$(4.3) \quad \begin{cases} \eta(\nabla'_X Y) = \eta(\nabla_X Y) + \eta(X)\eta(Y) + \nabla g(X, Y, \xi) \\ (\nabla'_X \eta)Y = (\nabla_X \eta)Y - \eta(X)\eta(Y) - \nabla g(X, Y, \xi) \\ g((\nabla'_X \varphi)Y, Z) = -g((\nabla_X \varphi)Z, Y). \end{cases}$$

Now, if we require the following conditions:

conditions on $\nabla\varphi$:

1) $\nabla\varphi = 0$ implies $\nabla'\varphi = 0$; 2) $\nabla\varphi = \pm\eta \otimes \varphi$ implies $\nabla'\varphi = \pm\eta \otimes \varphi$;

conditions on $\nabla\eta$:

3) $\nabla\eta = 0$ implies $\nabla'\eta = -\eta \otimes \eta - \nabla g(\cdot, \cdot, \xi)$;

4) $\nabla\eta = \eta \otimes \eta$ implies $\nabla'\eta = -\nabla g(\cdot, \cdot, \xi)$;

conditions on ∇g :

5) $\nabla g = 0$ implies $\nabla' = \nabla + \eta \otimes I$; 6) $\nabla g = \eta \otimes g$ implies $\nabla' = \nabla + 2\eta \otimes I$;

7) $\nabla g = -\eta \otimes g$ implies $\nabla' = \nabla$.

Remark 4.2. If ∇ satisfies 5) and 6) then its generalized dual connection is equal to ∇ on \mathcal{D} . Also remark that if ∇ is g -metric then $\nabla_\xi \xi \in \Gamma(\mathcal{D})$ while $g(\nabla'_\xi \xi, \xi) = 1$ and $\nabla'_\xi X - \nabla_\xi X = X$ for any $X \in \mathfrak{X}(M)$.

The generalized dual connection of the Golab connection has the following properties:

Proposition 4.3. *On the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ the generalized dual connection $(\nabla^G)'$ of the Golab connection ∇^G is a quarter-symmetric connection which satisfies:*

$$(4.4) \quad g(X, (\nabla^G)'_Y \xi) = g((\nabla^G)'_X \xi, Y).$$

In the almost para-cosymplectic case $(\nabla^G)'$ has the same curvature as ∇^G .

Proof. Fix $X, Y, Z \in \mathfrak{X}(M)$; the equality (4.4) is a direct consequence of (4.1) and:

-the torsion of $(\nabla^G)'$ is $T^{G'} = \psi \otimes \eta - \eta \otimes \psi$ with $\psi := \varphi - I$.

-the curvature of $(\nabla^G)'$ is $R^{G'}(X, Y, Z) = R^G(X, Y, Z) + d\eta(X, Y)Z$. \square

A straightforward computation similar to that of the end of previous Section gives: $\nabla^{G'}P_0 = \nabla^G P_0 (= \nabla^g P_0)$ and then a natural ∇^g yields the parallelism of P_0 with respect to all three linear connections ∇^g, ∇^G and $\nabla^{G'}$.

Definition 4.4. The linear connection ∇ is called ξ -metric if: $\nabla g(\cdot, \cdot, \xi) = 0$.

Of course, a metric linear connection is ξ -metric. Similar to the calculus of Section 3 we get that for a ξ -metric connection ∇ the curvature of the generalized dual connection ∇' is:

$$(4.5) \quad R'(X, Y, Z) = R(X, Y, Z) + 2d\eta(X, Y)Z.$$

So, in the para-cosymplectic case a ξ -metric connection has the same curvature as its generalized dual connection.

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