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TRIPLE POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEM OF A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper we generalize the main results of Bai, Wang and Ge (*Electron J. Diff. Eqns.* 6 (2004), 1–8) by considering a general type of boundary value problem associated with a nonlinear fractional differential equation. We obtain sufficient conditions for the existence of at least three positive solution with corresponding upper and lower bounds.

1. **Introduction**

Fractional differential equations have been studied by many authors caused by its applications in solving ordinary differential equations and also many applications in physics, mechanics, chemistry, and engineering. Positive solutions for ordinary differential equations and difference equations also have been considered by many authors, e.g. [1, 6, 8]. The major tool in finding positive solutions for both fractional and ordinary differential equations have been fixed point theorems and Leray-Schauder theory. In this paper we consider a typical fractional differential equations and improve the results obtained in [4, 5] using fixed

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point theorems. We consider a nonlinear fractional differential equation of the following type:

$$\begin{cases} D^{\alpha}x(t) + q(t)f(t, x(t), x'(t)) = 0 \qquad (0 < t < 1) & (1.1) \\ x(0) = x(1) = 0, & (1.2) \end{cases}$$

where $x(t)$ and $q(t)$ are nonnegative in $(0, 1)$, $f : [0, 1] \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ is continuous, $2 \leq \alpha < 3$ is a real number, and D^{α} is the standard Riemann-Lioville fractional differentiation (see definition 2.2). Furthermore $q(t)$ does not identically vanish on any subinterval of $(0, 1)$ and $q(t)$ satisfies the following inequality:

$$0 < \int_0^1 [t(1-t)]^{\alpha-1} q(t)\, dt < \infty.$$

The motivation for considering $2 \leq \alpha < 3$ is the importance of second order differential equations, which is equivalent to $\alpha = 2$ in this case. We obtain our main result by using the fixed point theorem of a cone preserving operator on an ordered Banach space that will be defined in Section 2. First we obtain an integral representation of the solution by the corresponding Green's function.

2. **Background materials and definitions**

First we present the necessary definitions from fractional calculus theory and cone theory in Banach spaces, e.g. [2, 9].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$.

Remark 2.3. As a basic example we quote for $\lambda > -1$,

$$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$

giving in particular $D^\alpha t^{\alpha-m} = 0, \quad m = 1, 2, \dots, N.$
Where N is the smallest integer greater than or equal to α.

From Definition 2.2 and Remark 2.3, we obtain the following lemma.

Lemma 2.4. *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L_1(0,1)$, then the fractional differential equation*

$$D^\alpha u(t) = 0,$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},\ C_i \in \mathbb{R}\ \ ,\ i = 1, \dots, N$$

as unique solutions.

As $D^\alpha I^\alpha u = u$ for all $u \in C(0,1) \cap L(0,1)$, from Lemma 2.4 we deduce the following law of composition.

Lemma 2.5. *Assume that $u \in C(0,1) \cap L_1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L_1(0,1)$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + C'_1 t^{\alpha-1} + C'_2 t^{\alpha-2} + \cdots + C'_N t^{\alpha-N},$$

for some $C'_i \in \mathbb{R}\ \ ,\ i = 1, \dots, N$.

Now we find the integral representation of the solution and specify the corresponding Green's function.

Lemma 2.6. *Given $y \in C[0,1]$ and $2 \le \alpha < 3$, the unique solution of*

$$D^\alpha u(t) + y(t) = 0 \qquad (0 < t < 1) \tag{2.1}$$
$$u(0) = u(1) = 0, \tag{2.2}$$

is

$$u(t)=\int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s)=\begin{cases}\frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} & (0\le s\le t\le 1)\\ \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}. & (0\le t\le s\le 1)\end{cases}\tag{2.3}$$

Proof. We may apply Lemma 2.2 to reduce Eq.(2.1) to an equivalent equation

$$u(t)=-I^{\alpha}y(t)-C_1't^{\alpha-1}-C_2't^{\alpha-2}-C_3't^{\alpha-3},$$

for some $C_1', C_2', C_3'\in\mathbb{R}$. Consequently, the general solution of Eq.(2.1) is

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\ -C_1't^{\alpha-1}-C_2't^{\alpha-2}-C_3't^{\alpha-3}.$$

By (2.2), therefore, $C_2'=C_3'=0$ and

$$C_1'=\frac{-1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)\,ds.$$

Therefore, the unique solution of the differential equation (2.1) subject to (2.2) is

$$\begin{aligned}u(t)&=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\int_0^1\frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\\&=\int_0^t\frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\int_t^1\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\\&=\int_0^1 G(t,s)y(s)\,ds.\end{aligned}$$

The proof is now completed. □

Lemma 2.7. *The function $G(t,s)$ defined by Eq.(2.3) satisfies the following conditions:*
(i) $G(t,s)>0$*, for* $t,s\in(0,1)$
(ii) *There exists a positive function* $\gamma\in C(0,1)$ *such that*

$$\min_{\frac14\le t\le\frac34}G(t,s)\ge\gamma(s)\max_{0\le t\le1}G(t,s)=\gamma(s)G(s,s),\qquad for\ \ 0<s<1.$$

Proof. From (2.3) it is clear that $G(t,s)>0$ for $s,t\in(0,1)$. In the following, we consider the existence of $\gamma(s)$. For $s\in(0,\frac{1}{4})$, $G(t,s)$ is increasing with respect to t for $t\in\left(0,\frac{s}{\left(1-(1-s)^{\frac{\alpha-1}{\alpha-2}}\right)}\right)$ and decreasing with respect to t for $t\in\left(\frac{s}{\left(1-(1-s)^{\frac{\alpha-1}{\alpha-2}}\right)},\frac{1}{4}\right)$, and for $s\in(\frac{1}{4},1)$, $G(t,s)$ is decreasing with respect to t for $s\le t$ and increasing with respect to t for $s\ge t$. If we define

$$g_1(t,s)=\frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)},\qquad g_2(t,s)=\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)},$$

then

$$\begin{aligned}
&\min_{\frac{1}{4}\le t\le\frac{3}{4}}G(t,s)\\
&=\begin{cases}\min\{g_1(0,s),g_1(\frac{1}{4},s)\}, & s\in(0,\frac{1}{4})\\ \min\{g_1(\frac{3}{4},s),g_2(\frac{1}{4},s)\}, & s\in(\frac{1}{4},\frac{3}{4})\\ g_2(\frac{1}{4},s), & s\in(\frac{3}{4},1)\end{cases}\\
&=\begin{cases}\min\{g_1(0,s),g_1(\frac{1}{4},s)\}, & s\in(0,\frac{1}{4})\\ g_1(\frac{3}{4},s), & s\in(\frac{1}{4},r)\\ g_2(\frac{1}{4},s), & s\in(r,1)\end{cases}\\
&=\begin{cases}\min\{-\frac{(-s)^{\alpha-1}}{\Gamma(\alpha)},\frac{1}{\Gamma(\alpha)}\{[\frac{1}{4}(1-s)]^{\alpha-1}-(\frac{1}{4}-s)^{\alpha-1}\}\}, & s\in(0,\frac{1}{4})\\ \frac{1}{\Gamma(\alpha)}\{[\frac{3}{4}(1-s)]^{\alpha-1}-(\frac{3}{4}-s)^{\alpha-1}\}, & s\in(\frac{1}{4},r)\\ \frac{1}{\Gamma(\alpha)}\frac{(1-s)^{\alpha-1}}{4^{\alpha-1}}, & s\in(r,1)\end{cases}
\end{aligned}$$

where $\frac{1}{4}<r<\frac{3}{4}$ is the unique solution of the equation

$$[\frac{3}{4}(1-r)]^{\alpha-1}-(\frac{3}{4}-r)^{\alpha-1}=\frac{(1-r)^{\alpha-1}}{4^{\alpha-1}}\qquad;$$

in particular $r\to0.5$ as $\alpha\to2$ and $r\to0.26$ as $\alpha\to3$.
Using the monotonicity of $G(t,s)$, we have

$$\max_{0\le t\le1}G(t,s)=G(s,s)=\frac{1}{\Gamma(\alpha)}[s(1-s)]^{\alpha-1},\quad s\in(0,1).$$

Consequently, the desired function γ is

$$\gamma(s)=\begin{cases}\min\{-\frac{(-s)^{\alpha-1}}{[s(1-s)]^{\alpha-1}},\frac{[\frac{1}{4}(1-s)]^{\alpha-1}-(\frac{1}{4}-s)^{\alpha-1}}{[s(1-s)]^{\alpha-1}}\}, & s\in(0,\frac{1}{4})\\ \frac{[\frac{3}{4}(1-s)]^{\alpha-1}-(\frac{3}{4}-s)^{\alpha-1}}{[s(1-s)]^{\alpha-1}}, & s\in(\frac{1}{4},r)\\ \frac{1}{(4s)^{\alpha-1}} \quad . & s\in(r,1)\end{cases}$$

This completes the proof. □

Definition 2.8. Let E be a real Banach space over $\mathbb{R}$. A nonempty convex closed set $P\subset E$ is said to be a *cone* provided that
(i) $au\in P$ for all $u\in P$ and all $a\geq 0$ and
(ii) $u,-u\in P$ implies $u=0$.
Note that every cone $P\subset E$ induces an ordering in E given by $x\leq y$ if $y-x\in P$.

Let β and θ be nonnegative continuous convex functionals on P,φ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a,b,c and d, we define the following convex sets:

$$P(\beta,d)=\{x\in P\mid \beta(x)<d\},$$
$$P(\beta,\varphi,b,d)=\{x\in P\mid b\leq\varphi(x),\ \beta(x)\leq d\},$$
$$P(\beta,\theta,\varphi,b,c,d)=\{x\in P\mid b\leq\varphi(x),\ \theta(x)\leq c,\ \beta(x)\leq d\},$$

and a closed set

$$R(\beta,\psi,a,d)=\{x\in P\mid a\leq\psi(x),\ \beta(x)\leq d\}.$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proof of our main results.

Theorem 2.9. [2] *Let P be a cone in a real Banach space E. Let β and θ be nonnegative continuous convex functionals on P, φ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x)\leq\lambda\psi(x)$ for $0\leq\lambda\leq 1$ such that for some positive numbers M and d,*

$$\varphi(x)\leq\psi(x)\quad\textit{and}\quad \|x\|\leq M\beta(x) \tag{2.4}$$

for all $x\in\overline{P(\beta,d)}$. Suppose that $T:\overline{P(\beta,d)}\to\overline{P(\beta,d)}$ is completely continuous and there exist positive numbers $a,b,$ and c with $a<b$ such

that
(*s*1) $\{x \in P(\beta,\theta,\varphi,b,c,d) \mid \varphi(x) > b\} \neq \emptyset$, *and* $\varphi(Tx) > b$ *for* $x \in P(\beta,\theta,\varphi,b,c,d)$;
(*s*2) $\varphi(Tx) > b$ *for* $x \in P(\beta,\varphi,b,d)$ *with* $\theta(Tx) > c$;
(*s*3) $0 \notin R(\beta,\psi,a,d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\beta,\psi,a,d)$ *with* $\psi(x) = a$.
Then T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\beta,d)}$ *such that*

$$\begin{gathered}\beta(x_i) \le d \quad \textit{for} \quad i = 1,2,3;\\ b < \varphi(x_1);\\ a < \psi(x_2) \quad \textit{with} \quad \varphi(x_2) < b;\\ \psi(x_3) < a.\end{gathered}$$

3. **Main results**

In this section, we impose growth conditions on f which allow us to apply Theorem 2.1 to establish the existence of triple positive solutions of the boundary value problem (1.1) with initial conditions (1.2).
Let $X = C^1[0,1]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0,1]$, and the maximum norm

$$\|x\| = \max\{\max_{0\le t\le 1} |x(t)|, \ \max_{0\le t\le 1} |x'(t)|\}.$$

Define the cone $P \subset X$ by

$$P = \{x \in X \mid x(t) \ge 0 \text{ on } [0,1], \text{ and } x(0) = x(1) = 0\}.$$

Let the nonnegative continuous concave functional φ, the nonnegative continuous convex functionals θ, β, and the nonnegative continuous functional ψ be defined on the cone P by

$$\beta(x) = \max_{0\le t\le 1} |x'(t)|, \quad \psi(x) = \theta(x) = \max_{0\le t\le 1} |x(t)|, \quad \varphi(x) = \min_{\frac{1}{4}\le t\le \frac{3}{4}} |x(t)|.$$

The following is well-known.

Lemma 3.1. [5]. *If* $x \in P$, *then* $\max_{0\le t\le 1} |x(t)| \le \frac{1}{2} \max_{0\le t\le 1} |x'(t)|$.

By Lemma 3.1, the functionals defined above satisfy

$$\varphi(x) \le \theta(x) = \psi(x), \ \|x\| = max\{\theta(x), \beta(x)\} = \beta(x) \tag{3.1}$$

for all $x \in \overline{P(\beta, d)} \subset P$. Therefore, condition (2.4) is satisfied.

Lemma 3.2. *Let $T : P \to P$ be the operator defined by*

$$Tx(t) = \int_0^1 G(t,s)q(s)f(s,x(s),x'(s))\,ds$$

Then $T : P \to P$ is completely continuous.

Proof. The operator $T : P \to P$ is continuous in view of nonnegativeness and continuity of $G(t,s)$ and $q(t)f(t,x,x')$. Let $\Omega \subset P$ be bounded, i.e., there exists a positive constant $M > 0$ such that $\|x\| \leq M$ for all $x \in \Omega$.
Let $L = \max\limits_{0 \leq t \leq 1,\ 0 \leq x \leq M} |q(t)f(t,x(t),x'(t))| + 1$. Then for $x \in \Omega$, we have

$$|Tx(t)| \leq \int_0^1 |G(t,s)q(s)f(s,x(s),x')|\,ds \leq L \int_0^1 G(s,s)\,ds.$$

Hence, $T(\Omega)$ is bounded. On the other hand, given $\epsilon > 0$, we set

$$\delta = \frac{1}{2}\ \left(\frac{\Gamma(\alpha)\epsilon}{L}\right)^{\frac{1}{\alpha-1}}.$$

Then, for each $x \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and $t_2 - t_1 < \delta$ one has $|Tx(t_2) - Tx(t_1)| < \epsilon$. That is to say, $T(\Omega)$ is equicontinuous.

In fact, we have

$$
\begin{aligned}
&|Tx(t_2) - Tx(t_1)| \\
&= \Big| \int_0^1 G(t_2, s) q(s) f(s, x(s), x'(s))\, ds \\
&- \int_0^1 G(t_1, s) q(s) f(s, x(s), x'(s))\, ds \Big| \\
&= \int_0^{t_1} [G(t_2, s) - G(t_1, s)] q(s) f(s, x(s), x'(s))\, ds \\
&+ \int_{t_2}^1 [G(t_2, s) - G(t_1, s)] q(s) f(s, x(s), x'(s))\, ds \\
&+ \int_{t_1}^{t_2} [G(t_2, s) - G(t_1, s)] q(s) f(s, x(s), x'(s))\, ds \\
&< \frac{L}{\Gamma(\alpha)} \Big[\int_0^{t_1} (1-s)^{\alpha-1} (t_2^{\alpha-1} - t_1^{\alpha-1})\, ds \\
&+ \int_{t_2}^1 (1-s)^{\alpha-1} (t_2^{\alpha-1} - t_1^{\alpha-1})\, ds \\
&+ \int_{t_1}^{t_2} (1-s)^{\alpha-1} (t_2^{\alpha-1} - t_1^{\alpha-1})\, ds \Big] < \frac{L}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}
$$

In the following, we divide the proof into two cases.
CASE 1. $\delta \le t_1 < t_2 < 1$.

$$
|Tx(t_2) - Tx(t_1)| < \frac{L}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}) \le \frac{L}{\Gamma(\alpha)} \frac{\alpha-1}{\delta^{2-\alpha}} (t_2 - t_1)
$$

$$
\le \frac{L}{\Gamma(\alpha)} (\alpha - 1) \delta^{\alpha-1} \le \epsilon.
$$

CASE 2. $0 \le t_1 < \delta,\ t_2 < 2\delta$.

$$
|Tx(t_2) - Tx(t_1)| < \frac{L}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}) \le \frac{L}{\Gamma(\alpha)} t_2^{\alpha-1} \le \frac{L}{\Gamma(\alpha)} (2\delta)^{\alpha-1} \le \epsilon.
$$

By the means of the Arzela-Ascoli theorem, we have $T : P \to P$ is completely continuous. The proof is now completed. $\square$

Let

$$\begin{aligned} M &= \int_0^1 \frac{\alpha-1}{\Gamma(\alpha)}(1-s)^{\alpha-1}s^{\alpha-2}q(s)\,ds, \\ N &= \int_0^1 G(s,s)q(s)\,ds, \\ \delta &= \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)G(s,s)q(s)\,ds. \end{aligned}$$

We assume there exist constants $0 < a < b \leq \frac{d}{8}$ such that
(A1) $f(t,u,v) \leq \frac{d}{M}$, for $(t,u,v) \in [0,1] \times [0,\frac{d}{2}] \times [-d,d]$;
(A2) $f(t,u,v) > \frac{b}{\delta}$, for $(t,u,v) \in [\frac{1}{4},\frac{3}{4}] \times [b,4b] \times [-d,d]$;
(A3) $f(t,u,v) < \frac{a}{N}$, for $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$.

Theorem 3.3. *Under the assumptions (A1)-(A3), the boundary-value problem* (1.1)-(1.2) *has at least three positive solutions* $x_1,\ x_2,$ *and* x_3 *satisfying*

$$\begin{gathered} \max_{0\leq t\leq 1}|x_i'(t)| \leq d, \qquad for\ i=1,2,3; \\ b < \min_{\frac{1}{4}\leq t\leq \frac{3}{4}}|x_1(t)|; \qquad (3.2) \\ a < \max_{0\leq t\leq 1}|x_2(t)|, \quad with \quad \min_{\frac{1}{4}\leq t\leq \frac{3}{4}}|x_2(t)| < b; \\ \max_{0\leq t\leq 1}|x_3(t)| < a. \end{gathered}$$

Proof. The boundary value problem (1.1) with initial conditions (1.2) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = Tx(t) = \int_0^1 G(t,s)q(s)f(s,x(s),x'(s))\,ds$$

where $T : P \to P$, is completely continuous (Lemma 3.2). We now show that all the conditions of Theorem 2.1 are satisfied.
If $x \in \overline{P(\beta,d)}$, then $\beta(x) = \max_{0\leq t\leq 1}|x'(t)| \leq d$. By Lemma 3.1 and $\max_{0\leq t\leq 1}|x(t)| \leq \frac{d}{2}$, the assumption (A1) implies $f(t,x(t),x'(t)) \leq \frac{d}{M}$.

Define

$$H(t,s)=\begin{cases} \frac{(\alpha-1)}{\Gamma(\alpha)}[(1-s)^{\alpha-1}t^{\alpha-2}-(t-s)^{\alpha-2}], & (0\le s\le t\le 1)\\ \frac{(\alpha-1)}{\Gamma(\alpha)}(1-s)^{\alpha-1}t^{\alpha-2}, & (0\le t\le s\le 1)\end{cases}$$

where $2\le\alpha<3$. For given $s\in(0,1)$, $H(t,s)$ is decreasing with respect to t for $s\le t$ and increasing with respect to t for $t\le s$. Using monotonicity of $H(t,s)$, we have

$$\max_{0\le t\le 1} H(t,s)=H(s,s)=\frac{(\alpha-1)}{\Gamma(\alpha)}(1-s)^{\alpha-1}s^{\alpha-2}.$$

Consequently,

$$\begin{aligned}
&\beta(Tx)=\max_{0\le t\le 1}|(Tx)'(t)|\\
=&\max_{0\le t\le 1}\left|\int_0^t\frac{(\alpha-1)}{\Gamma(\alpha)}[(1-s)^{\alpha-1}t^{\alpha-2}-(t-s)^{\alpha-2}]q(s)f(s,x(s),x'(s))\,ds\right.\\
&\left.+\int_t^1\frac{(\alpha-1)}{\Gamma(\alpha)}(1-s)^{\alpha-1}t^{\alpha-2}q(s)f(s,x(s),x'(s))\,ds\right|\\
=&\max_{0\le t\le 1}\left|\int_0^1 H(t,s)q(s)f(s,x(s),x'(s))\,ds\right|\\
=&\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{\alpha-2}q(s)f(s,x(s),x'(s))\,ds\\
\le&\frac{d}{M}\,\frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{\alpha-2}q(s)\,ds=\frac{d}{M}\,M=d.
\end{aligned}$$

Hence, $T:\overline{P(\beta,d)}\to\overline{P(\beta,d)}$.

To check the condition (s1) of Theorem 2.9, we choose $x(t)=4b,\ 0\le t\le 1$. It is easy to see that $x(t)=4b\in P(\beta,\theta,\varphi,b,4b,d)$ and $\varphi(x)=\varphi(4b)>b$, and so $x\in\{P(\beta,\theta,\varphi,b,4b,d)\mid\varphi(x)>b\}\neq\emptyset$. Hence, if $x\in P(\beta,\theta,\varphi,b,4b,d)$, then $b\le x(t)\le 4b,\ |x'(t)|\le d$ for $\frac14\le t\le\frac34$. From assumption (A2), we have $f(t,x(t),x'(t))\ge\frac{b}{\delta}$ for $\frac14\le t\le\frac34$, and by Lemma 2.4

$$\begin{aligned}
\varphi(Tx)&=\min_{\frac14\le t\le\frac34}|(Tx)(t)|\ge\int_0^1\gamma(s)G(s,s)q(s)f(s,x(s),x'(s))\,ds\\
&\ge\frac{b}{\delta}\int_{\frac14}^{\frac34}\gamma(s)G(s,s)q(s)\,ds=\frac{b}{\delta}\,\delta=b.
\end{aligned}$$

Thus,

$$\varphi(Tx) > b, \text{ for all } x \in P(\beta, \theta, \varphi, b, 4b, d).$$

This shows that the condition (s1) of Theorem 2.9 is satisfied.

Notice that the condition $(s1)$ of Theorem 2.9 implies the condition $(s2)$ of Theorem 2.9. Clearly, as $\psi(0) = 0 < a$, there holds that $0 \notin R(\beta, \psi, a, d)$.
Suppose that $x \in R(\beta, \psi, a, d)$ with $\psi(x) = a$. Then, by the assumption (A3),

$$\begin{aligned} \psi(Tx) &= \max_{0\le t\le 1} |(Tx)(t)| = \max_{0\le t\le 1} \int_0^1 G(t,s)q(s)f(s,x(s),x'(s))\,ds \\ &< \frac{a}{N} \max_{0\le t\le 1} \int_0^1 G(t,s)q(s)\,ds = \frac{a}{N} \int_0^1 G(s,s)q(s)\,ds = a. \end{aligned}$$

So, the condition (s3) of Theorem 2.9 is satisfied. Therefore, an application of Theorem 2.4 implies that the boundary value problem (1.1) with initial conditions (1.2) has at least three positive solutions x_1, x_2 and x_3 satisfying (3.4). □

Example 3.4. Consider the boundary-value problem

$$D^{\frac{5}{2}}x(t) + f(t, x(t), x'(t)) = 0 \qquad 0 < t < 1, \tag{3.3}$$

$$x(0) = x(1) = 0, \tag{3.4}$$

where

$$f(t,u,v) = \begin{cases} e^t + \frac{27}{2}u^3 + (\frac{v}{6000})^3 & \text{for} \quad u \le 12, \\ e^t + 23328 + (\frac{v}{6000})^3 & \text{for} \quad u \ge 12. \end{cases}$$

Here we have $\alpha = 5/2,\ q(t) = 1$. By definition of M, N, and δ we have

$$\begin{aligned} M &= \int_0^1 \frac{\alpha - 1}{\Gamma(\alpha)}(1-s)^{\alpha-1}s^{\alpha-2}q(s)\,ds, \\ &= \frac{(\alpha-1)\Gamma(\alpha-1)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(2\alpha-1)} = \sqrt{\pi}/8 \approx 0.22. \end{aligned}$$

Similarly we have $N = \sqrt{\pi}/32 \approx 0.05$. Using Lemma 2.4, r is the root of the equation

$$[\frac{3}{4}(1-r)]^{\alpha-1} - (\frac{3}{4} - r)^{\alpha-1} = \frac{(1-r)^{\alpha-1}}{4^{\alpha-1}},$$

which is $r \approx 0.53$ in this example. Substituting $\gamma(s)$ given by Lemma 2.4 and observing that $G(s,s) = \frac{1}{\Gamma(\alpha)}[s(1-s)]^{\alpha-1}$, we find

$$\delta = \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)G(s,s)q(s)\,ds \approx 0.01.$$

Choosing $a = 1,\ b = 3,\ d = 6000$, we have

$$\begin{aligned}
f(t,u,v) &< \frac{a}{N} \approx 20 \quad \text{for } 0 \le t \le 1, \quad 0 \le u \le 1, \quad -6000 \le v \le 6000;\\
f(t,u,v) &> \frac{b}{\delta} \approx 300 \quad \text{for } \frac{1}{4} \le t \le \frac{3}{4}, \quad 3 \le u \le 12, \quad 6000 \le v \le 6000;\\
f(t,u,v) &< \frac{d}{M} \approx 27272 \text{ for } 0 \le t \le 1,\ 0 \le u \le 3000,\ -6000 \le v \le 6000.
\end{aligned}$$

Then all assumptions of Theorem 3.1 hold. Thus, by Theorem 3.1, the boundary value problem (3.3) with initial conditions (3.4) has at least three positive solutions x_1, x_2 and x_3 such that

$$\max_{0\le t\le 1} |x_i'(t)| \le 6000, \qquad for \quad i = 1,2,3\ ;$$

$$3 < \min_{\frac{1}{4}\le t\le \frac{3}{4}} |x_1(t)|\ ;$$

$$1 < \max_{0\le t\le 1} |x_2(t)|, \quad with \quad \min_{\frac{1}{4}\le t\le \frac{3}{4}} |x_2(t)| < 3\ ;$$

$$\max_{0\le t\le 1} |x_3(t)| < 1.$$

Concluding Remark. As we mentioned in the abstract, we considered a general boundary values problem for a nonlinear fractional differential equation of the form (1.1). The particular case $\alpha = 2$ has been studied by [5] for an ordinary differential equation of the form $x'' + q(t)f(t,x(t),x'(t)) = 0$ with the same boundary conditions as (1.2). Our paper in fact generalizes the main results of [5] for a more general boundary value problem.

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