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THE FIBERING MAP APPROACH TO A QUASILINEAR DEGENERATE P(X)-LAPLACIAN EQUATION

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Abstract. By considering a degenerate $p(x)-$Laplacian equation, a generalized compact embedding in weighted variable exponent Sobolev space is presented. Multiplicity of positive solutions are discussed by applying fibering map approach for the corresponding Nehari manifold.
Keywords: Nehari Manifold, fibering map, weak solution, variable exponent Lebesgue space, variable exponent Sobolev space.
MSC(2010): Primary: 35J20; Secondary:35R01.

1. Introduction

The classes of problems dealing with variable exponent Lebesgue and Sobolev spaces have attracted steadily increased interest over the last ten years. This was mainly stimulated by development of the studies of problems in elasticity, image processing, flow in porous media, etc. (e.g. [1, 12, 18]).

Among these problems, $p(x)$-Laplacian problem, defined by $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$, is a natural generalization of the p-Laplacian operator where $p > 1$ is a positive constant.

In recent years, many problems on $p(x)$-Laplacian type have been studied by many authors using various methods specially variational technics; see [3, 4, 9, 13, 17]. The fibering map approach for description of the Nehari manifolds and seeking a solutions in an appropriate subset of the Sobolev space is introduced by Drabek and Pohozaev in [8] and is also discussed by Brown and Zhang in [7].

In variable exponent cases this method has some difficulties in comparison with the fibering method in p-Laplacian problems. This is due to nonhomogenities resulting from variable exponent p. Accordingly, it was not welcomed by mathematicians. Nevertheless, some interesting papers on the application

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of Nehari manifold method in variable exponent problem have recently been published [16, 19].

In [16], using a Nehari manifold and some variational techniques for a $p(\cdot)$-Laplacain Dirichlet problem, the existence of at least two positive solutions is proved and in [19], considering a $p(\cdot)$-Laplacian system with Neumann boundary condition and using the variational method on corresponding Nehari manifold, the existence of a positive solutions is proved.

In this paper, we generalize the results of [2], for $p(x)-$Laplacian equation by applying the idea of the fibering map approach in a quasilinear degenerate $p(x)-$Laplacian problem defined by

$$(\mathbf{P})\begin{cases}-div(a(x)|\nabla u|^{p(x)-2}\nabla u)=\lambda b(x)|u|^{q(x)-2}u+c(x)|u|^{r(x)-2}u; & in \quad \Omega\\ u\equiv 0; & on \quad \partial\Omega.\end{cases}$$

Here we refer to [6] for application of an intuitive insight on fibering map approach which is used by Brown and Wu. The organization of the paper is as follows: After reviewing preliminaries and introducing a proper Sobolev space for studying the problem and its applicable Sobolev embedding, we describe a Nehari manifold as a target set for finding weak solutions, this leads to the behavior of an Euler functional corresponding to problem (P) on it. In section 3, we analyze the fibering map related to the Euler functional and we provide the existence situation for λ in the case where the turning points set of the fibering map is empty. Finally the existence of minimizer theorems for (P) is presented in Section 4.

2. **Preliminaries**

We refer to [10] for the basic information on variable exponent Lebesgue and Sobolev spaces. We briefly mention some of the main properties of variable exponent spaces that are used in this paper.

Let Ω be an open subset of $\mathbb{R}^N$, $p\in L^\infty(\Omega)$ and

$$p^- := ess\inf_{x\in\Omega} p(x)\geq 1.$$

The variable exponent Lebesgue space $\mathbf{L}^{p(\cdot)}(\Omega)$ is defined by

$$\mathbf{L}^{p(\cdot)}(\Omega)=\{u:\ u:\Omega\longrightarrow\mathbb{R}\ \ is\ measurable\ \int_\Omega |u|^{p(x)}dx<\infty\};$$

which is considered by the norm

$$|u|_{\mathbf{L}^{p(\cdot)}(\Omega)}=\inf\{\sigma>0:\ \int_\Omega |\frac{u}{\sigma}|^{p(x)}dx\leq 1\}.$$

We summarize the main properties of $L^{p(\cdot)}(\Omega)$ in the following list:

(i) The space $(\mathbf{L}^{p(x)}(\Omega), |.|_{\mathbf{L}^{p(x)}(\Omega)})$ is a separable, uniform convex Banach space, and its conjugate space is $\mathbf{L}^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For any $u \in \mathbf{L}^{p(x)}(\Omega)$ and $v \in \mathbf{L}^{p'(x)}(\Omega)$, we have

$$|\int_{\Omega} uvdx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{\mathbf{L}^{p(x)}(\Omega)}|u|_{\mathbf{L}^{p'(x)}(\Omega)}.$$

(ii) If Ω is bounded, $p_1, p_2 \in C(\overline{\Omega})$ and $1 < p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there is a continuous embedding $\mathbf{L}^{p_2(x)}(\Omega) \hookrightarrow \mathbf{L}^{p_1(x)}(\Omega)$.

(iii)

$$\min(|u|^{p^-}_{\mathbf{L}^{p(.)}(\Omega)}, |u|^{p^+}_{\mathbf{L}^{p(.)}(\Omega)}) \leq \int_{\Omega} |u|^{p(x)}dx \leq \max(|u|^{p^-}_{\mathbf{L}^{p(.)}(\Omega)}, |u|^{p^+}_{\mathbf{L}^{p(.)}(\Omega)});$$

where $p^+ := ess\sup_{x\in\Omega} p(x)$.

Proposition 2.1. *([11]) Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min(|u|^{p^-}_{\mathbf{L}^{p(.)q(.)}(\Omega)}, |u|^{p^+}_{\mathbf{L}^{p(.)q(.)}(\Omega)}) \leq ||u|^{p(x)}|_{\mathbf{L}^{q(.)}(\Omega)} \leq \max(|u|^{p^-}_{\mathbf{L}^{p(.)q(.)}(\Omega)}, |u|^{p^+}_{\mathbf{L}^{p(.)q(.)}(\Omega)}).$$

The variable exponent Sobolev space $\mathbf{W}^{1,p(.)}(\Omega)$ is defined by

$$\mathbf{W}^{1,p(.)}(\Omega) = \{u \in \mathbf{L}^{p(.)}(\Omega);\ |\nabla u| \in \mathbf{L}^{p(.)}(\Omega)\}$$

with the norm

$$\|u\|_{\mathbf{W}^{1,p(.)}(\Omega)} = |u|_{\mathbf{L}^{p(.)}(\Omega)} + |\nabla u|_{\mathbf{L}^{p(.)}(\Omega)}.$$

Define $\mathbf{W}^{1,p(.)}_0(\Omega)$ as the closure of $\mathbf{C}^\infty_0(\Omega)$ in $\mathbf{W}^{1,p(.)}(\Omega)$ and let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}; & p(x) < N \\ \infty; & p(x) \geq N. \end{cases}$$

The main properties of $\mathbf{W}^{1,p(.)}(\Omega)$ are given by the following items:

(iv) $\mathbf{W}^{1,p(.)}(\Omega)$ and $\mathbf{W}^{1,p(.)}_0(\Omega)$ are separable reflexive Banach spaces.

(v) If $p : \Omega \longrightarrow \mathbb{R}$ is Lipschitz continuous, then for $q \in L^\infty(\Omega)$ with $q^- \geq 1$ and $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $\mathbf{W}^{1,p(.)}(\Omega) \hookrightarrow \mathbf{L}^{q(.)}(\Omega)$.

(vi) Let Ω be a bounded domain in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$. Then for any $q \in L^\infty(\Omega)$ with $q^- \geq 1$ and $q \ll p^*$(i.e., ess $\inf_{x\in\overline{\Omega}}(p^*(x) - q(x)) > 0$), there is a compact embedding from $\mathbf{W}^{1,p(.)}(\Omega)$ to $\mathbf{L}^{q(.)}(\Omega)$. It is denoted by $\mathbf{W}^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^{q(.)}(\Omega)$.

(vii) There is a constant $C > 0$ such that

$$|u|_{\mathbf{L}^{p(.)}(\Omega)} \leq C|\nabla u|_{\mathbf{L}^{p(.)}(\Omega)} \quad \forall u \in \mathbf{W}^{1,p(.)}_0(\Omega).$$

We consider $\mathbf{W}^{1,p(.)}_{a(.)}(\Omega)$ as an appropriate Sobolev space for studding problem (P) on a bounded domain, which is defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm, $\|u\| = |\nabla u|_{L^{p(.)}_{a(.)}(\Omega)} + |u|_{L^{p(.)}(\Omega)}$ where

$$\mathbf{L}^{p(.)}_{a(.)}(\Omega) = \{u: \ u : \Omega \longrightarrow \mathbb{R} \ \ \textit{is measurable}, \int_\Omega a(x)|u|^{p(x)}dx < \infty\}$$

is equipped with the norm

$$|u|_{\mathbf{L}^{p(.)}_{a(.)}(\Omega)} = \inf\{\sigma > 0 : \ \int_\Omega a(x)|\frac{u}{\sigma}|^{p(x)}dx \le 1\}.$$

The Sobolev space $\mathbf{W}^{1,p(.)}_{a(.)}(\Omega)$ which is called weighted variable exponent Sobolev space, is introduced in [15], where $a(x)$ is a measurable, nonnegative real valued function for $x \in \Omega$.

Theorem 2.2. *([14]) Let $p, s \in C(\overline{\Omega})$, $1 < p(x)$, $1 < s(x)$ for all $x \in \overline{\Omega}$ and $a(x)$ be a measurable positive and a.e. finite function in $\mathbb{R}^N$ satisfying*

(a_1) $0 < a \in \mathbf{L}^1_{Loc}(\Omega)$, $a(x)^{-\frac{1}{p(x)-1}} \in \mathbf{L}^1_{Loc}(\Omega)$.
(a_2) $a(x)^{-s(x)} \in \mathbf{L}^1(\Omega)$ *where* $s(x) \in C(\overline{\Omega})$ *and*

$$s(x) > \frac{1}{p(x)-1}.$$

Then we have the following continuous embedding,

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow W^{1,p_s(x)}(\Omega);$$

where $p_s(x) = \frac{p(x)s(x)}{1+s(x)}$.

The condition (a_1) is essential; without it, the space $W^{1,p(x)}_{a(x)}(\Omega)$ is not necessarily a Banach space even though $p(x)$ is a constant [15].

Theorem 2.3. *[16]. Assume that $p \in C(\overline{\Omega})$ and $1 < p(x)$ for all $x \in \overline{\Omega}$. Suppose that*

(b_1) $0 < b \in \mathbf{L}^{\beta(x)}(\Omega)$, $1 < \beta(x) \in C(\overline{\Omega})$.
(q) $1 < q(x) < \frac{p^*(x)}{\beta'(x)}$ *for all* $x \in \overline{\Omega}$,

where $\beta'(x) = \frac{\beta(x)}{\beta(x)-1}$, *then we have the following compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega).$$

Corollary 2.4. *Suppose that all conditions in Theorems 2.2 are satisfied. Furthermore, Assume that the condition in Theorem 2.3 replacing $p(x)$ by $p_s(x)$ is satisfied. Then we have the following compact embedding,*

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega);$$

where $1 < q(x) < \frac{p_s^*(x)}{\beta'(x)}$ *in* $\overline{\Omega}$.

Proof. From Theorem 2.2, we obtain $W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow W^{1,p_s(x)}(\Omega)$; and by Theorem 2.3, we deduce $W^{1,p_s(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega)$ and hence

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega).$$

□

Now we state a mild generalization of a compact embedding for a weighted variable exponent Sobolev space.

Theorem 2.5. *Assume that $p \in C(\overline{\Omega})$, $1 < p(x)$ for all $x \in \overline{\Omega}$, (a_1), (b_1) are satisfied and moreover,*

(a_3) $a(x)^{-\frac{\xi(x)}{p(x)-\xi(x)}} \in \mathbf{L}^1(\Omega)$ *where $\xi(x) \in C(\overline{\Omega})$ and $1 < \xi(x) < p(x)$.*

Then we have the following compact embedding,

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega)$$

for every $q \in C(\overline{\Omega})$ and $1 < q(x) < \frac{\xi^*(x)}{\beta'(x)}$.

Proof. First, we show that $W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow W^{1,\xi(x)}(\Omega)$ continuously. Let $u \in W^{1,p(x)}_{a(x)}(\Omega)$. We have

$$\int_\Omega |\nabla u|^{\xi(x)} dx = \int_\Omega |\nabla u|^{\xi(x)} a(x)^{\frac{\xi(x)}{p(x)}} a(x)^{-\frac{\xi(x)}{p(x)}} dx$$

$$\leq C|a(x)^{-\frac{\xi(x)}{p(x)}}|_{L^{\frac{p(x)}{p(x)-\xi(x)}}(\Omega)} |a(x)^{\frac{\xi(x)}{p(x)}} |\nabla u|^{\xi(x)}|_{L^{\frac{p(x)}{\xi(x)}}(\Omega)}.$$

By (iii) of the main properties of the variable exponent spaces we deduce

$$|a(x)^{-\frac{\xi(x)}{p(x)}}|_{L^{\frac{p(x)}{p(x)-\xi(x)}}(\Omega)} \leq (\int_\Omega a(x)^{-\frac{\xi(x)}{p(x)-\xi(x)}} dx + 1)^{\frac{p^+-\xi^-}{p^-}}.$$

So, by assumption (a_3), there exists a $C > 0$ such that

$$\int_\Omega |\nabla u|^{\xi(x)} dx \leq C|a(x)^{\frac{\xi(x)}{p(x)}} |\nabla u|^{\xi(x)}|_{L^{\frac{p(x)}{\xi(x)}}(\Omega)}. \tag{2.1}$$

Without loss of generality, we can assume that $\int_\Omega |\nabla u|^{\xi(x)} > 1$. By Proposition 2.1 and (iii) when $\int_\Omega a(x)|\nabla u|^{p(x)} < 1$, from (2.1) we obtain

$$|\nabla u|_{L^{\xi(x)}(\Omega)} \leq C|\nabla u|^{\frac{p^-}{p^+}}_{L^{p(x)}_{a(x)}(\Omega)}.$$

Moreover, if $\int_\Omega a(x)|\nabla u|^{p(x)} > 1$ we deduce,

$$|\nabla u|_{L^{\xi(x)}(\Omega)} \leq C|\nabla u|^{\beta}_{L^{p(x)}_{a(x)}(\Omega)};$$

where $\beta = \frac{p^+\xi^+}{p^-\xi^-}$. So we get $\nabla u \in L^{\xi(x)}(\Omega)$. On the other hand, $L^{p(x)}(\Omega) \hookrightarrow L^{\xi(x)}(\Omega)$; hence

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow W^{1,\xi(x)}(\Omega). \tag{2.2}$$

Now by the classical Sobolev embedding (iv) we have,

$$W^{1,\xi(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega) \tag{2.3}$$

for $r(x) < \xi^*(x)$. Let $r(x) = q(x)\beta'(x)$. So if $u \in W^{1,p(x)}_{a(x)}(\Omega)$ then

$$\begin{aligned}\int_\Omega b(x)|u|^{q(x)}dx &\leq C|b|_{L^{\beta(x)}(\Omega)}||u|^{q(x)}|_{L^{\beta'(x)}(\Omega)}\\ &\leq C|b|_{L^{\beta(x)}(\Omega)}\min(|u|^{q^+}_{L^{r(x)}(\Omega)}, |u|^{q^-}_{L^{r(x)}(\Omega)});\end{aligned}$$

and since $u \in L^{r(x)}(\Omega)$, we have $u \in L^{q(x)}_{b(x)}(\Omega)$. Moreover, if $u_n \rightharpoonup 0$ in $W^{1,p(x)}_{a(x)}(\Omega)$, then by (2.2) $u_n \rightharpoonup 0$ in $W^{1,\xi(x)}(\Omega)$ and by (2.3) $u_n \longrightarrow 0$ in $L^{r(x)}(\Omega)$. Then we have

$$\int_\Omega b(x)|u_n|^{q(x)}dx \leq C|b|_{L^{\beta(x)}}||u_n|^{q(x)}|_{L^{\beta'(x)}} \longrightarrow 0,$$

which implies $|u_n|_{L^{q(x)}_{b(x)}} \longrightarrow 0$ and hence we can deduce

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega).$$

□

Corollary 2.6. *Assume that the conditions of Theorem 2.5 are satisfed, then there exist positive constants C_1 and C_2 such that*

$$\int_\Omega b(x)|u|^{q(x)}dx \leq \begin{cases} C_1\|u\|^{q^+}; & if \quad \|u\| > 1 \\ C_2\|u\|^{q^-}; & if \quad \|u\| < 1. \end{cases}$$

Proof. By Theorem 2.5, we can deduce

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q^+}_{b(x)}(\Omega)$$

and

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q^-}_{b(x)}(\Omega).$$

So that there exist positive constants c_1, c_2 such that

$$(\int_\Omega b(x)|u|^{q^+}dx)^{\frac{1}{q^+}} = |u|_{L^{q^+}_{b(x)}(\Omega)} \leq c_1\|u\|$$

and

$$(\int_\Omega b(x)|u|^{q^-}dx)^{\frac{1}{q^-}} = |u|_{L^{q^-}_{b(x)}(\Omega)} \leq c_2\|u\|.$$

And hence,

$$\int_\Omega b(x)|u|^{q(x)}dx \le \int_\Omega b(x)|u|^{q^+}dx + \int_\Omega b(x)|u|^{q^-}dx \le c_1^{q^+}\|u\|^{q^+} + c_2^{q^-}\|u\|^{q^-}$$

$$\le \begin{cases} C_1\|u\|^{q^+}; & if \;\; \|u\| > 1 \\ C_2\|u\|^{q^-}; & if \;\; \|u\| < 1. \end{cases}$$

□

We consider problem $(\mathbf{P})$ with $(a_1),(b_1),(a_3)$ defined previously. Let

(c_1) $0 < c(x) \in \mathbf{L}^{\gamma(x)}(\Omega)$, $1 < \gamma(x) \in C(\overline{\Omega})$;

(q_1) $q \in C(\overline{\Omega})$ and $1 < q(x) \le q^+ < \min(\frac{\xi^*(x)}{\beta'(x)}, p^-)$.

(r_1) $r \in C(\overline{\Omega})$ and $p^+ < r^- \le r(x) < \frac{\xi^*(x)}{\gamma'(x)}$.

Hence, by Corollary 2.6, there exist positive constants C_3, C_4 such that the following inequalities hold

$$\int_\Omega c(x)|u|^{r(x)}dx \le \begin{cases} C_3\|u\|^{r^+}; & if \;\; \|u\| > 1 \\ C_4\|u\|^{r^-}; & if \;\; \|u\| < 1. \end{cases}$$

Now we are ready to study the behavior of an Euler functional on the corresponding Nehari manifold. The Euler functional associated with problem $(\mathbf{P})$ is

$$E_\lambda(u) = \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}dx - \lambda\int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega \frac{c(x)}{r(x)}|u|^{r(x)}dx.$$

It is well known that the weak solutions of $(\mathbf{P})$ corresponds to critical points of E_λ on $X = \mathbf{W}^{1,p(x)}_{a(x)}(\Omega)$.

In many problems, such as $(\mathbf{P})$, E_λ is not bounded below on X, but it is bounded below on the corresponding Nehari manifold which is defined by

$$M(\lambda) = \{u \in X \setminus \{0\}; \langle E'_\lambda(u), u\rangle = 0\},$$

where $\langle .,. \rangle$ denotes the usual duality between X and X^*.

Now we see that the corresponding Euler functional to problem $(\mathbf{P})$ would be unbounded from below on X. Indeed,

$$E_\lambda(u) \ge \tfrac{1}{p^+}\int_\Omega a(x)|\nabla u|^{p(x)}dx - \tfrac{\lambda}{q^-}\int_\Omega b(x)|u|^{q(x)}dx - \tfrac{1}{r^-}\int_\Omega c(x)|u|^{r(x)}dx$$

$$\ge \tfrac{1}{p^+}\|u\|^{p^-} - \tfrac{\lambda}{q^-}C_1\|u\|^{q^+} - \tfrac{C_3}{r^-}\|u\|^{r^+}.$$

Since $r^+ > p^- > q^+$, this shows that E_λ is not bounded on the whole X. However, we see that it is bounded on the corresponding Nehari Manifold

(2.4)

$$M(\lambda) = \{u \in X\setminus\{0\}; \int_\Omega a(x)|\nabla u|^{p(x)}dx - \lambda\int_\Omega b(x)|u|^{q(x)}dx = \int_\Omega c(x)|u|^{r(x)}dx\}.$$

Clearly, $M(\lambda)$ is a much smaller set than X and E_λ is much better behaved on $M(\lambda)$.

From now on, for the problem $(\mathbf{P})$ we suppose that the conditions $(a_1), (b_1), (q_1)$, (c_1), and (r_1) are satisfied

Theorem 2.7. *E_λ is coercive and bounded below on $M(\lambda)$.*

Proof. Let $u \in M(\lambda)$ and $\|u\| > 1$. By applying (2.4) and Theorem 2.5, we have

$$E_\lambda(u) \geq (\tfrac{1}{p^+} - \tfrac{1}{r^-}) \int_\Omega a(x)|\nabla u|^{p(x)}dx - \lambda(\tfrac{1}{q^-} - \tfrac{1}{r^-}) \int_\Omega b(x)|u|^{q(x)}dx$$

$$\geq (\tfrac{1}{p^+} - \tfrac{1}{r^-})\|u\|^{p^-} - \lambda(\tfrac{1}{q^-} - \tfrac{1}{r^-})C_1\|u\|^{q^+}.$$

Since $p^- > q^+$, $E_\lambda(u) \to \infty$ as $\|u\| \to \infty$. This implies that E_λ is coercive and bounded below. □

For $u \in X$ the corresponding fibering map to (P) is defined by

$$\phi_{\lambda,u}(t) = \int_\Omega \tfrac{a(x)}{p(x)} t^{p(x)}|\nabla u|^{p(x)}dx - \lambda \int_\Omega \tfrac{b(x)}{q(x)} t^{q(x)}|u|^{q(x)}dx - \int_\Omega \tfrac{c(x)}{r(x)} t^{r(x)}|u|^{r(x)}dx.$$

And we have

$$\phi'_{\lambda,u}(t) = \int_\Omega a(x)t^{p(x)-1}|\nabla u|^{p(x)}dx - \lambda\int_\Omega b(x)t^{q(x)-1}|u|^{q(x)}dx - \int_\Omega c(x)t^{r(x)-1}|u|^{r(x)}dx.$$

$$\phi''_{\lambda,u}(t) = \int_\Omega a(x)(p(x)-1)t^{p(x)-2}|\nabla u|^{p(x)}dx - \lambda \int_\Omega b(x)(q(x)-1)t^{q(x)-2}|u|^{q(x)}dx - \int_\Omega c(x)(r(x)-1)t^{r(x)-2}|u|^{r(x)}dx.$$

It is easy to see that if u is a local minimizer of E_λ, then $\phi_{\lambda,u}$ has a local minimizer at $t = 1$.

Theorem 2.8. *Let $u \in X \setminus \{0\}$ and $t > 0$. Then $tu \in M(\lambda)$ if and only if $\phi'_{\lambda,u}(t) = 0$*

By the above theorem, we see that $u \in M(\lambda)$ if and only if $\phi'_{\lambda,u}(1) = 0$. Thus it is natural to divide $M(\lambda)$ into tree subset $M^+(\lambda)$, $M^-(\lambda)$ and $M^0(\lambda)$ corresponding to local minima, local maxima and points of inflection of fibering maps. Hence, we define,

$$M^+(\lambda) = \{u \in M(\lambda); \phi''_{\lambda,u}(1) > 0\};$$

$$M^-(\lambda) = \{u \in M(\lambda); \phi''_{\lambda,u}(1) < 0\};$$

$$M^0(\lambda) = \{u \in M(\lambda); \phi''_{\lambda,u}(1) = 0\}.$$

Note that if $u \in M(\lambda)$ then

$$\phi''_{\lambda,u}(1) = \int_\Omega a(x)(p(x)-1)|\nabla u|^{p(x)}dx - \lambda \int_\Omega b(x)(q(x)-1)|u|^{q(x)}dx - \int_\Omega c(x)(r(x)-1)|u|^{r(x)}dx. \tag{2.5}$$

Also, as proved in [5] or in [7], we have the following Lemma.

Lemma 2.9. *Suppose that u_0 is a local minimizer of E_λ on $M(\lambda)$ and $u_0 \notin M^0(\lambda)$, then u_0 is a critical point of E_λ.*

Proof. If u_0 is a local minimizer of E_λ on $M(\lambda)$, by Lagrange multipliers (see [21]), there exists an $\alpha \in \mathbb{R}$ such that $E'_\lambda(u_0) = \alpha J'_\lambda(u_0)$ where

$$J_\lambda(u_0) = \int_\Omega a(x)|\nabla u_0|^{p(x)}dx - \lambda \int_\Omega b(x)|u_0|^{q(x)}dx - \int_\Omega c(x)|u_0|^{r(x)}dx.$$

Thus, for any $v \in X$ we have

$$\langle E'_\lambda(u_0), v\rangle = \alpha \langle J'_\lambda(u_0), v\rangle \tag{2.6}$$

where

$$\begin{aligned}\langle E'_\lambda(u_0), v\rangle &= \int_\Omega a(x)|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v dx - \lambda \int_\Omega b(x)|u_0|^{q(x)-2}u_0 v dx \\ &\quad - \int_\Omega c(x)|u_0|^{r(x)-2}u_0 v dx\end{aligned}$$

and

$$\begin{aligned}\langle J'_\lambda(u_0), v\rangle &= \int_\Omega a(x)p(x)|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v dx - \lambda \int_\Omega b(x)q(x)|u_0|^{q(x)-2}u_0 v dx \\ &\quad - \int_\Omega c(x)r(x)|u_0|^{r(x)-2}u_0 v dx.\end{aligned}$$

Since $u_0 \in M(\lambda)$, we have

$$\langle E'_\lambda(u_0), u_0\rangle = J_\lambda(u_0) = 0. \tag{2.7}$$

Thus $\langle J'_\lambda(u_0), u_0\rangle = \phi''_{\lambda,u_0}(1) + J_\lambda(u_0) = \phi''_{\lambda,u_0}(1)$. Since $u_0 \notin M^0(\lambda)$ we obtain $\langle J'_\lambda(u_0), u_0\rangle \neq 0$. So by (2.6) and (2.7) we deduce $\alpha = 0$. Hence the proof is complete. □

3. Analysis of the fibering maps

We shall now describe the nature of the fibering maps for all possible situations. Let $A_u := \int_\Omega a(x)|\nabla u|^{p(x)}dx$, $B_u := \int_\Omega b(x)|u|^{q(x)}dx$ and $C_u := \int_\Omega c(x)|u|^{r(x)}dx$. Hence,

$$\begin{aligned}
&(t^{p^-}A_u - \lambda t^{q^+}B_u - t^{r^+}C_u)\chi_{[1,+\infty)}(t) + (t^{p^+}A_u - \lambda t^{q^-}B_u - t^{r^-}C_u)\chi_{(0,1)}(t)\\
&\le \phi'_{\lambda,u}(t) \le \qquad (3.1)\\
&(t^{p^+}A_u - \lambda t^{q^-}B_u - t^{r^-}C_u)\chi_{[1,+\infty)}(t) + (t^{p^-}A_u - \lambda t^{q^+}B_u - t^{r^+}C_u)\chi_{(0,1)}(t).
\end{aligned}$$

For $\gamma > \alpha > \beta > 0$ and $A, B, C, \lambda > 0$ let $f_\lambda(t) = t^\alpha A - t^\gamma C - t^\beta B$. Then for $t \neq 0$ we have,

$$f_\lambda(t) = 0 \Longleftrightarrow t^{-\beta} f_\lambda(t) = 0 \Longleftrightarrow t^{\alpha-\beta}A - t^{\gamma-\beta}C = \lambda B \qquad (3.2)$$

Set $g(t) = t^{\alpha-\beta}A - t^{\gamma-\beta}C$ and $h(t) = \lambda B$. The three possible situations for $g(t)$ and $h(t)$ related to each other are shown in the following graphs.

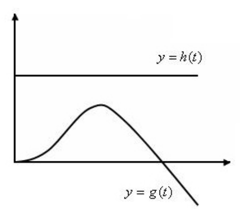

Figure 2-1

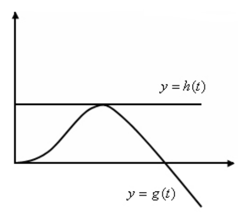

Figure 2-2

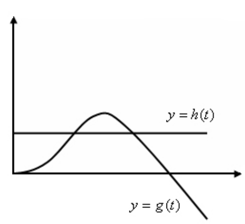

Figure 2-3

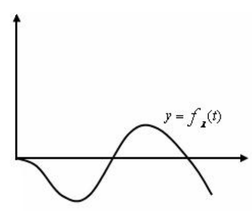

Figure 2-4

If λ is sufficiently large, (3.2) has no solution (see Figure 2-1) and so f_λ has no root. If λ is exactly of appropriate value, (3.2) has only one solution (see Figure 2-2), and f_λ takes only nonpositive values. On the other hand, if $\lambda > 0$ is sufficiently small, there are exactly two solutions $t_1(u) < t_2(u)$ for (3.2), (see Figure 2-3) with $f'_\lambda(t_1(u)) > 0$, and $f'_\lambda(t_2(u)) < 0$. It follows that f_λ has a graph as shown in the Figure 2-4.

By inequalities (3.1), we see that the graph of $\phi'_{\lambda,u}(t)$ is between two graphs $\mu_{\lambda,u}(t) = t^{p^-}A_u - \lambda t^{q^+}B_u - t^{r^+}C_u$ and $\nu_{\lambda,u}(t) = t^{p^+}A_u - \lambda t^{q^-}B_u - t^{r^-}C_u$. Hence, by the above argument about the form of f_λ, for λ sufficiently small, the graphs of $\mu_{\lambda,u}$ and $\nu_{\lambda,u}$ for $u \in M^+(\lambda)$ and $u \in M^-(\lambda)$ are shown in Figure 2-5 and Figure 2-6, respectively, and so $\phi'_{\lambda,u}$ would be placed in the gray space between them.

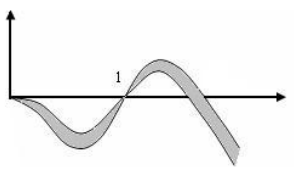

Figure 2-5

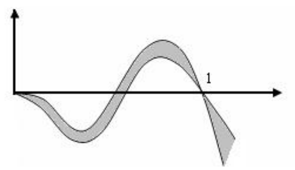

Figure 2-6

It follows that $\phi_{\lambda,u}$ has at least two critical points; a local minimum at $t_1 = t_1(u)$ and a local maximum at $t_2 = t_2(u)$ which for $u \in M^+(\lambda)$, $t_1 = 1 < t_2$ and $t_2 u \in M^-(\lambda)$ and for $u \in M^-(\lambda)$, $t_1 < t_2 = 1$ and $t_1 u \in M^+(\lambda)$.

Moreover, $\phi_{\lambda,u}$ is decreasing on $(0, t_1)$, increasing on (t_1, t_2) and deceasing on $(t_2, +\infty)$. It follows from the last argument that there exist $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$ we see that if $\phi'_{\lambda,u}(t) = 0$, i.e., $tu \in M(\lambda)$, then $tu \notin M^0(\lambda)$ and so we have the following lemma.

Lemma 3.1. *There exists a $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$, we have $M^0(\lambda) = \emptyset$.*

Proof. If $u \in M^0(\lambda)$, form (2.5), we obtain

$$p^+A_u - \lambda q^- B_u - r^- C_u \geq 0.$$

Since $u \in M(\lambda)$, $A_u - \lambda B_u = C_u$ and so

$$p^+A_u - \lambda q^- B_u - r^-(A_u - \lambda B_u) = (p^+ - r^-)A_u + \lambda(r^- - q^-)B_u \geq 0.$$

Suppose $\|u\| > 1$, using Proposition 2.6, we deduce

(3.3)

$$(p^+ - r^-)\|u\|^{p^+} + \lambda(r^- - q^-)C_1|u\|^{q^+} < 0 \quad \text{and hence} \quad |u\|^{p^+ - q^+} \leq \lambda \frac{(r^- - q^-)}{(r^- - p^+)} C_1.$$

Similarly, from 2.5 we obtain $p^- A_u - \lambda q^+ B_u - r^+ C_u < 0$ and since $u \in M(\lambda)$ we derive

$$p^- A_u - q^+(A_u - C_u) - r^+ C_u = (p^- - q^+)A_u - (r^+ - q^+)C_u \leq 0.$$

By Proposition 2.6, we have

$$(3.4)\quad (p^- - q^+)\|u\|^{p^-} - (r^+ - q^+)C_3|u\|^{r^+} \leq 0 \quad \text{and} \quad |u\|^{r^+ - p^-} \geq \frac{(p^- - q^+)}{(r^+ - q^+)C_3}.$$

From (3.3) and (3.4) we get,

$$(\frac{(p^- - q^+)}{(r^+ - q^+)C_3})^{\frac{p^+ - q^+}{r^+ - p^-}} \leq \lambda \frac{(r^- - q^-)}{(r^- - p^+)} C_1.$$

Thus,

$$\lambda \geq (\frac{(p^- - q^+)}{(r^+ - q^+)C_3})^{\frac{p^+ - q^+}{r^+ - p^-}} . \frac{(r^- - p^+)}{(r^- - q^-)C_1} =: \lambda'.$$

By a similar argument, if $u \in M^0(\lambda)$ and $\|u\| < 1$, we have

$$\lambda \geq (\frac{(p^- - q^+)}{(r^+ - q^+)C_4})^{\frac{p^- - q^-}{r^- - p^+}} . \frac{(r^- - p^+)}{(r^- - q^-)C_2} =: \lambda''.$$

Therefore, we conclude that for $\lambda \leq \lambda_1$ where $\lambda_1 := \min(\lambda', \lambda'')$, $M^0(\lambda) = \emptyset$. □

4. **Existence of Minimizer**

Theorem 4.1. *If $\lambda < \lambda_1$, then there exists a minimizer of E_λ on $M^+(\lambda)$.*

Proof. Since E_λ is bounded below on $M(\lambda)$ and so on $M^+(\lambda)$, there exists a minimizing sequence $\{u_n\} \subseteq M^+(\lambda)$ such that $\lim_{n\to\infty} E_\lambda(u_n) = \inf_{u \in M^+(\lambda)} E_\lambda(u)$. Since E_λ is coercive, $\{u_n\}$ is bounded in X. Thus, we may assume that, without loos of generality, $u_n \rightharpoonup u_0$ in X and by the compact embedding, we have $u_n \to u_0$ in $L^{r(x)}_{c(x)}(\Omega)$ and in $L^{q(x)}_{b(x)}(\Omega)$. Now, we will prove $u_n \to u_0$ in X. Otherwise, suppose $u_n \not\to u_0$ in X, then

$$\int_\Omega a(x)|\nabla u_0|^{p(x)}dx < \liminf_{n\to\infty} \int_\Omega a(x)|\nabla u_n|^{p(x)}dx. \tag{4.1}$$

$$\begin{aligned}\phi'_{\lambda,u_n}(t) &= \int_\Omega a(x)t^{p(x)-1}|\nabla u_n|^{p(x)}dx - \lambda \int_\Omega b(x)t^{q(x)-1}|u_n|^{q(x)}dx \\ &\quad - \int_\Omega c(x)t^{r(x)-1}|u_n|^{r(x)}dx;\end{aligned}$$

and

$$\begin{aligned}\phi'_{\lambda,u_0}(t) &= \int_\Omega a(x)t^{p(x)-1}|\nabla u_0|^{p(x)}dx - \lambda \int_\Omega b(x)t^{q(x)-1}|u_0|^{q(x)}dx \\ &\quad - \int_\Omega c(x)t^{r(x)-1}|u_0|^{r(x)}dx.\end{aligned}$$

By the previous section, there exists $t_0 = t_0(u_0)$ such that $t_0 u_0 \in M^+(\lambda)$, and hence, $\phi'_{\lambda,u_0}(t_0) = 0$ and by (4.2), we deduce,

$\lim_{n\to\infty} \phi'_{\lambda,u_n}(t_0)$

$= \lim_{n\to\infty}(\int_\Omega a(x)t_0^{p(x)-1}|\nabla u_n|^{p(x)}dx - \lambda\int_\Omega b(x)t^{q(x)-1}|u_n|^{q(x)}dx$

$- \int_\Omega c(x)t^{r(x)-1}|u_n|^{r(x)}dx)$

$= \lim_{n\to\infty}(\int_\Omega a(x)t_0^{p(x)-1}|\nabla u_n|^{p(x)}dx) - \int_\Omega a(x)t_0^{p(x)-1}|\nabla u_0|^{p(x)}dx > 0.$

Hence, $\phi'_{\lambda,u_n}(t_0) > 0$, for sufficiently large n. Since $\{u_n\} \subseteq M^+(\lambda)$, by considering possible maps $\phi'_{\lambda,u}$ for $u \in M^+(\lambda)$, as is shown in Figure 2-5, it is easy to see that $\phi'_{\lambda,u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{\lambda,u_n}(1) = 0$; for all n. Thus, we must have $t_0 > 1$. But by considering the possible forms of the fibering maps, we deduce,

$$\phi_{\lambda,t_0u_0}(1) < \phi_{\lambda,t_0u_0}(t); \quad t < 1.$$

Let $t = \frac{1}{t_0}$, hence $E_\lambda(t_0u_0) = \phi_{\lambda,t_0u_0}(1) < \phi_{\lambda,t_0u_0}(\frac{1}{t_0}) = E_\lambda(u_0)$.

So $E_\lambda(t_0u_0) < E_\lambda(u_0) < \lim_{n\to\infty} E_\lambda(u_n) = \inf_{u\in M^+(\lambda)} E_\lambda(u)$, which contradicts $t_0u_0 \in M^+(\lambda)$. Hence, $u_n \longrightarrow u_0$ in X and

$$E_\lambda(u_0) = \lim_{n\to\infty} E_\lambda(u_n) = \inf_{u\in M^+(\lambda)} E_\lambda(u).$$

Since $u_n \longrightarrow u_0$ in X, $u_n \subset M^+(\lambda)$ and $X \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}, L^{r(x)}_{c(x)}$ hence

$$\int_\Omega a(x)|\nabla u_0|^{p(x)}dx - \lambda\int_\Omega b(x)|u_0|^{q(x)}dx = \int_\Omega c(x)|u_0|^{r(x)}dx.$$

Moreover, by boundedness of p, q, r on Ω we have $W^{1,p(x)}_{a(x)}(\Omega) = W^{1,p(x)}_{p(x)a(x)}(\Omega)$, $L^{q(x)}_{b(x)}(\Omega) = L^{q(x)}_{q(x)b(x)}(\Omega)$ and $L^{r(x)}_{c(x)}(\Omega) = L^{r(x)}_{r(x)c(x)}(\Omega)$ and since $M^0(\lambda) = \emptyset$ we obtain

$$\int_\Omega a(x)p(x)|\nabla u_o|^{p(x)}dx > \lambda\int_\Omega b(x)q(x)|u_0|^{q(x)}dx - \int_\Omega c(x)r(x)|u_0|^{r(x)}dx.$$

Thus $u_0 \neq 0$.

□

Theorem 4.2. *If $\lambda < \lambda_1$, then there exists a minimizer of E_λ on $M^-(\lambda)$.*

Proof. As in the previous proof, we get a minimizing sequence $\{v_n\} \subseteq M^-(\lambda)$ such that $\lim_{n\to\infty} E_\lambda(v_n) = \inf_{v\in M^-(\lambda)} E_\lambda(v)$ where $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^{r(x)}_{c(x)}(\Omega)$ and $L^{q(x)}_{b(x)}(\Omega)$. Suppose that v_n is not strongly convergent to v_0 in X, hence,

$$\int_\Omega a(x)|\nabla v_0|^{p(x)}dx < \liminf_{n\to\infty}\int_\Omega a(x)|\nabla v_n|^{p(x)}dx.$$

Moreover, by the same argument in the previous theorem, there exists $t_1 = t_1(v_0)$ such that $t_1 v_0 \in M^-(\lambda)$, and so

$$\phi'_{\lambda,v_0}(t_1) = \int_\Omega a(x) t_1^{p(x)-1} |\nabla v_0|^{p(x)} dx - \lambda \int_\Omega b(x) t_1^{q(x)-1} |v_0|^{q(x)} dx$$
$$- \int_\Omega c(x) t_1^{r(x)-1} |v_0|^{r(x)} dx = 0.$$

Hence,

$$\lim_{n\to\infty} \phi'_{\lambda,v_n}(t_1)$$
$$= \lim_{n\to\infty} (\int_\Omega a(x) t_1^{p(x)-1} |\nabla v_n|^{p(x)} dx) - \int_\Omega a(x) t_1^{p(x)-1} |\nabla v_0|^{p(x)} dx > 0.$$

And so $\phi'_{\lambda,v_n}(t_1) > 0$, for sufficiently large n. Now, by considering the possible maps $\phi'_{\lambda,v}$ for $v \in M^-(\lambda)$, as is shown in Figure 2-6, it can be seen that $\phi'_{\lambda,v_n}(t) < 0$ for $t > 1$ and $\phi'_{\lambda,v_n}(1) = 0$; for all n. Hence, we must have $t_1 < 1$ and so

$$E_\lambda(t_1 v_0) < \liminf E_\lambda(t_1 v_n) \tag{4.2}$$
$$< \lim_{n\to\infty} E_\lambda(v_n) = \inf_{v\in M^-(\lambda)} E_\lambda(v) \tag{4.3}$$

which contradicts by $t_1 v_0 \in M^-(\lambda)$.
Thus, $v_n \longrightarrow v_0$ in X and the proof can be completed as in the previous theorem.

□

Corollary 4.3. *Problem* (P) *have at least two positive solutions for* $0 < \lambda < \lambda_1$.

Proof. By Theorems 4.1 and 4.2, there exist $u_0 \in M^+(\lambda)$ and $v_0 \in M^-(\lambda)$ such that

$$E_\lambda(u_0) = \inf_{u\in M^+(\lambda)} E_\lambda(u)$$

and

$$E_\lambda(v_0) = \inf_{v\in M^-(\lambda)} E_\lambda(v).$$

Moreover, $E_\lambda(u) = E_\lambda(|u|)$ and $|u_0| \in M^+(\lambda)$ and similarly $|v_0| \in M^-(\lambda)$, so we may assume $u_0, v_0 > 0$. By Lemma 2.9 u_0, v_0 are critical points of E on $W^{1,p(x)}_{a(x)}(\Omega)$ and hence, weak solutions (and so by the standard regularity result classical solutions of (P)). Finally, by Harnack inequality [20, 22], we obtain that u_0, v_0 are positive solutions of (P). □

Example 4.4. *Let $\Omega \subset \mathbb{R}^2$, then $u = u(x,y)$ and $a(x) := b(x) := c(x) := 1$. In this case the problem (P) has the following complicated form:*

$$
\begin{aligned}
(u_x^2 + u_y^2)^{\frac{p(x,y)-2}{2}} [& \frac{(p_x(x,y) + p_y(x,y))}{2} ln(u_x^2 + u_y^2) \\
& + (p(x,y) - 2)(\frac{u_x u_{xx} + u_y u_{xy} + u_x u_{xy} + u_y u_{yy}}{u_x^2 + u_y^2}) + u_{xx} + u_{yy}] \\
& = \lambda |u|^{q(x,y)-2} u + |u|^{r(x,y)-2} u
\end{aligned}
$$

Even if $\Omega := (1,2) \subset \mathbb{R}$, $p(x) := 2x + 2$, $q(x) := 2$, $r(x) := 7$ and $a(x) := b(x) := c(x) := 1$ we have

$$u'^{2x}[2lnu' + x\frac{u''}{u'} + u''] = \lambda u + |u|^5 u$$

with $u(1) = u(2) = 0$.

So we are faced with a two dimentional nonlinear PDE and a nonlinear ODE, respectively, where it is not easy to find any analytic solution for them. Whereas by using Corollary 4.3 we know for some appropriate λ these problems have at least two positive solutions.

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