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AUTOMATIC CONTINUITY OF ALMOST MULTIPLICATIVE MAPS BETWEEN FRÉCHET ALGEBRAS

T. GHASEMI HONARY*, M. OMIDI AND A. H. SANATPOUR

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Abstract. For Fréchet algebras $(A, (p_n))$ and $(B, (q_n))$, a linear map $T : A \to B$ is *almost multiplicative* with respect to (p_n) and (q_n), if there exists $\varepsilon \geq 0$ such that $q_n(Tab - TaTb) \leq \varepsilon p_n(a)p_n(b)$, for all $n \in \mathbb{N}$, $a, b \in A$, and it is called *weakly almost multiplicative* with respect to (p_n) and (q_n), if there exists $\varepsilon \geq 0$ such that for every $k \in \mathbb{N}$, there exists $n(k) \in \mathbb{N}$, satisfying the inequality $q_k(Tab - TaTb) \leq \varepsilon p_{n(k)}(a)p_{n(k)}(b)$, for all $a, b \in A$.

We investigate the automatic continuity of (weakly) almost multiplicative maps between certain classes of Fréchet algebras, such as Banach algebras and Fréchet Q-algebras. We also obtain some results on the automatic continuity of dense range (weakly) almost multiplicative maps between Fréchet algebras.
Keywords: Automatic continuity, Fréchet algebras, Q-algebras, almost multiplicative maps, weakly almost multiplicative maps, dense range almost multiplicative maps.
MSC(2010): Primary: 46H40, 47A10; Secondary: 46H05, 46J05, 47B33.

1. Introduction

We first recall some notions in topological algebras.

Definition 1.1. *A subset V of a complex algebra A is called m-convex (multiplicatively convex) if V is convex and idempotent, i.e., $VV \subseteq V$. A subset V of A is called balanced if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$. A topological algebra is locally convex if there is a base of neighborhoods of zero consisting of convex sets.*

Since each base of convex neighborhoods of zero consists a base of absolutely convex (convex and balanced) sets, we may always assume that a locally convex algebra has a base of neighborhoods of zero consisting of absolutely convex sets.

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A topological algebra is a locally multiplicatively convex algebra, or briefly, an lmc algebra, if there is a base of neighborhoods of the origin consisting of sets which are absolutely convex and idempotent.

An interesting class of topological algebras is the class of Fréchet algebras, defined as follows:

Definition 1.2. *A topological algebra is called a Fréchet algebra if it is a complete metrizable topological algebra which has a neighborhood basis* (V_n) *of zero such that* V_n *is absolutely convex and* $V_nV_n \subseteq V_n$ *for all* $n \in \mathbb{N}$.

The topology of a Fréchet algebra A can be generated by a sequence (p_n) of separating submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and $x, y \in A$, such that $p_n(x) \leq p_{n+1}(x)$, whenever $n \in \mathbb{N}$ and $x \in A$. If A is unital then p_n can be chosen such that $p_n(1) = 1$ for all $n \in \mathbb{N}$. The Fréchet algebra A with the above generating sequence of seminorms (p_n) is denoted by $(A, (p_n))$. Note that a sequence (x_k) in the Fréchet algebra $(A, (p_n))$ converges to $x \in A$ if and only if $p_n(x_k - x) \to 0$ for each $n \in \mathbb{N}$, as $k \to \infty$. Banach algebras are important examples of Fréchet algebras.

We now define another important class of topological algebras, called Q-algebras.

Definition 1.3. *Let* A *be an algebra. An element* $x \in A$ *is called quasi-invertible if there exists* $y \in A$ *such that*

$$x \diamond y = x + y - xy = 0 \text{ and } y \diamond x = y + x - yx = 0.$$

The set of all quasi-invertible elements of A *is denoted by q-Inv*A.

A topological algebra A *is called a* Q*-algebra if q-Inv*A *is open, or equivalently, if q-Inv*A *has an interior point in* A *[19, Lemma E2].*

Definition 1.4. *A topological algebra* A *is advertibly complete if every Cauchy net* (x_α) *in* A *converges in* A *whenever for some* $x \in A$, *both* $x_\alpha \diamond x$ *and* $x \diamond x_\alpha$ *converge to zero.*

It is clear that every complete topological algebra is advertibly complete. Moreover, every Q-algebra is advertibly complete, see, for example, [18, I. Theorem 6.4]

Definition 1.5. *For a unital algebra* A *with the unit* 1, *the spectrum of an element* $x \in A$, *denoted by* $\sigma_A(x)$, *is the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda 1 - x$ *is not invertible in* A. *For a non-unital algebra* A, *the spectrum of* $x \in A$ *is* $\sigma_A(x) = \{0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \frac{x}{\lambda} \notin q\text{-}InvA\}$. *The spectral radius of an element* $x \in A$ *is* $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$ *and* $\mathfrak{Q}(A) = \{x \in A : r_A(x) = 0\}$.

The (Jacobson) radical rad$A$ *of an algebra* A *is the intersection of all maximal left (right) ideals in* A. *The algebra* A *is called semisimple if rad*$A = \{0\}$.

In the case that A is a Banach algebra we have $r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

It is known that for any algebra A

$$radA = \{x \in A : r_A(xy) = 0, \text{ for every } y \in A\}.$$

If A is a commutative Fréchet algebra, then $radA = \bigcap_{\varphi\in M(A)} ker\varphi$, where $M(A)$ is the continuous character space of A, i.e. the space of all continuous non-zero multiplicative linear functionals on A. See, for example, [6, Proposition 8.1.2].

Remark 1.6. *It is interesting to note that Fréchet algebras are lmc algebras. Moreover, an lmc algebra A is a Q-algebra if and only if the spectral radius r_A is continuous at zero and it is uniformly continuous on A if A is also commutative. A Fréchet algebra A is a Q-algebra if and only if the spectral radius $r_A(x)$ is finite for all $x \in A$. See, for example, [4, Theorem 6.18], [18, III. Proposition 6.2], or [19, Theorem 13.6].*

The automatic continuity of homomorphisms between different classes of topological algebras, including Fréchet algebras, Q-algebras and Banach algebras, have been widely studied by many authors. The monographs of Husain [9], Dales [2], Fragoulopoulou [4], Jarosz [10], Kaniuth [14], Mallios [18] and the interesting article of Michael [19], contain many results on this subject.

It is well-known that every homomorphism $T : A \to B$ is automatically continuous, when A and B are Banach algebras and B is commutative and semisimple. See, for example, [2, Theorem 2.3.3]. Hence every commutative semisimple Banach algebra has a unique complete norm. It was a major open question for many years whether every semisimple Banach algebra has a unique complete norm, even if it is not commutative. This was eventually proved in 1967 by B. E. Johnson and as a consequence of this result, it was shown that if $T : A \to B$ is a surjective homomorphism between a Banach algebra A and a semisimple Banach algebra B, then T is automatically continuous. However, in the case that $T : A \to B$ is a dense range homomorphism between a Banach algebra A and a semisimple Banach algebra B, the continuity of T is, in fact, an open question for more than 40 years.

Regarding wide study of automatic continuity problems for the homomorphisms between Banach algebras, many authors have also investigated the automatic continuity of homomorphisms between Fréchet algebras, and one can find many open questions in this area. For example, in 1952, E. A. Michael posed the question as whether each multiplicative linear functional on a commutative Fréchet algebra is automatically continuous [19]. This question, known as the Michael's problem, has been intensively studied, but only partial answers have been obtained so far. In fact, since Michael posed the above question in 1952, no one has been able to present an example of a multiplicative linear functional on a Fréchet algebra, which is not continuous. For further results on automatic continuity of homomorphisms between certain classes of Fréchet algebras, or partial answers to Michael's problem, one may refer, for example, to [5, 7] and the references therein.

Recall that a (complex) topological algebra A is called *functionally continuous* if every complex homomorphism on A is continuous. Banach algebras and Q-algebras are the most well-known classes of functionally continuous algebras. However, there are functionally continuous topological algebras, which are not Q-algebras. For example, the algebra of continuous complex-valued functions on $\mathbb{R}$, denoted by $C(\mathbb{R})$, with the compact open topology, or equivalently, with the sequence of algebra seminorms $p_n(f) = sup\{|f(x)| : x \in [-n, n]\}$, is a functionally continuous Fréchet algebra [2, Corollary 4.10.14, or page 589], which is not a Q-algebra, see, for example, [2, page 187].

In 1985, K. Jarosz introduced the concept of *almost multiplicative maps* between Banach algebras [10]. For the Banach algebras A and B, a linear map $T : A \to B$ is called *almost multiplicative* if there exists $\varepsilon \geq 0$ such that $\|Tab - TaTb\| \leq \varepsilon\|a\|\|b\|$, for every $a, b \in A$. K. Jarosz investigated the problem of automatic continuity for almost multiplicative linear maps between Banach algebras. Later, in 1986, B. E. Johnson obtained some results on the continuity of *approximately* (*almost*) *multiplicative functionals* [12] and then in 1988, he extended his own results to *approximately* (*almost*) *multiplicative maps* between Banach algebras [13].

Since then, many authors investigated almost multiplicative maps between different classes of Banach algebras. See, for example, [11, 15–17, 20].

In this paper, we study the automatic continuity of (weakly) almost multiplicative maps between certain classes of Fréchet algebras.

2. Automatic continuity of almost multiplicative maps between Fréchet algebras

We now introduce the concept of *almost multiplicative maps* and *weakly almost multiplicative maps* between Fréchet algebras.

Definition 2.1. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras and ε be non-negative. A linear map $T : (A, (p_n)) \to (B, (q_n))$ is multiplicative if $Tab = TaTb$ for every $a, b \in A$, it is ε-*multiplicative *with respect to (p_n) and (q_n) if*

$$q_n(Tab - TaTb) \leq \varepsilon p_n(a)p_n(b),$$

for all $n \in \mathbb{N}$, $a, b \in A$, and it is weakly ε-multiplicative with respect to (p_n) and (q_n), if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(Tab - TaTb) \leq \varepsilon p_{n(k)}(a)p_{n(k)}(b),$$

for every $a, b \in A$. A linear map $T : A \to B$ is almost multiplicative or weakly almost multiplicative if it is ε-multiplicative, or weakly ε-multiplicative, respectively, for some $\varepsilon \geq 0$.

Remark 2.2. *Since $p_k \leq p_{k+1}$ for all k, we may take a subsequence (p_{n_k}) instead of $p_{n(k)}$, in the definition of weakly ε-multiplicative maps.*

Since (q_n) is a separating sequence of seminorms on B, it is clear that, ε-multiplicative and weakly ε-multiplicative maps turn out to be multiplicative, whenever $\varepsilon = 0$. Moreover, any multiplicative map is ε-multiplicative for every $\varepsilon \geq 0$.

We first extend the result of [7, Theorem 2.1] to (weakly) almost multiplicative maps.

Theorem 2.3. *Let $(A, (p_n))$ be a Fréchet algebra, $(B, (q_n))$ be a semisimple Fréchet algebra and B_k be the completion of $B/kerq_k$, with respect to the norm $q'_k(y + kerq_k) = q_k(y)$ for all $y \in B$. If $T : A \to B$ is a surjective weakly almost multiplicative map such that $r_{B_k}(Tx + kerq_k) \leq p_k(x)$ for infinitely many $k \in \mathbb{N}$ and for all $x \in A$, then T is continuous.*

Proof. Let T be weakly ε-multiplicative for some $\varepsilon \geq 0$. For the continuity of T, by the Closed Graph Theorem, it is enough to show that for any sequence (a_n) in A, if $a_n \to 0$ in A and $Ta_n \to b$ in B, then $b = 0$. By the surjectivity of T, there exists $a \in A$ such that $Ta = b$. Since B_k is a Banach algebra with the norm $q'_k(y + kerq_k) = q_k(y)$ and $q_k(y) \leq q_{k+1}(y)$ for all $y \in B$, $k \in \mathbb{N}$, we have

$$\begin{aligned} r_{B_k}(b + kerq_k) &= \lim_{n\to\infty} (q'_k(b + kerq_k)^n)^{1/n} = \lim_{n\to\infty} (q'_k(b^n + kerq_k))^{1/n} \\ &= \lim_{n\to\infty} (q_k(b^n))^{1/n} \leq \lim_{n\to\infty} (q_{k+1}(b^n))^{1/n} \\ &= \lim_{n\to\infty} (q'_{k+1}(b + kerq_{k+1})^n)^{1/n} \\ &= r_{B_{k+1}}(b + kerq_{k+1}), \end{aligned}$$

for all $k \in \mathbb{N}$. Therefore, $(r_{B_k}(b + kerq_k))$ is an increasing sequence of real numbers and hence by [19, Corollary 5.3] we have

$$r_B(b) = \sup_{k\in\mathbb{N}} r_{B_k}(b + kerq_k) = \sup_{k\in I} r_{B_k}(b + kerq_k),$$

where I is an infinite subset of $\mathbb{N}$ satisfying the hypothesis of the theorem. Using a similar method as in [7, Theorem 2.1], we conclude that $r_B(b) = 0$. Now, let d be an arbitrary element of B. By the surjectivity of T, there exists $c \in A$ such that $Tc = d$. Since T is ε-weakly almost multiplicative for some $\varepsilon \geq 0$, there exists a subsequence (p_{m_k}) such that for each n we have

$$\begin{aligned} q_k(Tca_n - TcTa) &= q_k(Tca_n - TcTa_n + TcTa_n - TcTa) \\ &\leq q_k(Tca_n - TcTa_n) + q_k(TcTa_n - TcTa) \\ &\leq \varepsilon p_{m_k}(c) p_{m_k}(a_n) + q_k(Tc) q_k(Ta_n - Ta). \end{aligned} \tag{2.1}$$

Since $a_n \to 0$ in A, it follows that $p_{m_k}(a_n) \to 0$ for each k as $n \to \infty$, and since $Ta_n \to Ta$ in B, we have $q_k(Ta_n - Ta) \to 0$ for each k as $n \to \infty$. Hence (2.1) implies that $q_k(Tca_n - TcTa) \to 0$, for each $k \in \mathbb{N}$, as $n \to \infty$. Consequently,

$Tca_n \to TcTa = db$ as $n \to \infty$. Since $ca_n \to 0$ in A, by the same argument mentioned at the beginning of the proof, we have $r_B(db) = 0$. Since $d \in B$ was arbitrary, we conclude that $b \in radB = \{0\}$, and this completes the proof. □

Corollary 2.4. *Let $(A, (p_n))$ be a Fréchet Q-algebra and $(B, (q_n))$ be a semisimple Fréchet algebra (not necessarily commutative). Let the surjective map $T : A \to B$ be weakly almost multiplicative such that $r_B(Tx) \leq r_A(x)$ for every $x \in A$. Then T is automatically continuous.*

Proof. Since A is a Q-algebra, it is known that $r_A(x) \leq p_m(x)$, for some $m \in \mathbb{N}$ and for all $x \in A$. See, for example, [4, Theorem 6.18]. Since $p_k(x) \leq p_{k+1}(x)$ for all $x \in A$ and $k \in \mathbb{N}$, we have

$$r_{B_k}(Tx + kerq_k) \leq r_B(Tx) \leq r_A(x) \leq p_m(x) \leq p_k(x),$$

for all $k \geq m$ and $x \in A$. Therefore, all conditions of Theorem 2.3 are satisfied and hence T is continuous. □

Remark 2.5. *In the above corollary, whenever T is multiplicative we have $\sigma_B(Tx) \subseteq \sigma_A(x)$ and hence the inequality $r_B(Tx) \leq r_A(x)$ holds for all $x \in A$. Therefore, Corollary 2.4 is an extension of [7, Corollary 2.2]. It is now natural to ask whether the above corollary is valid without the assumption $r_B(Tx) \leq r_A(x)$ for all $x \in A$. We do not yet know the answer to this question.*

We now recall a property of an advertibly complete algebra, which will be used in the proof of Theorem 2.9.

Theorem 2.6. *[18, III. Corollary 6.5] If A is an lmc algebra which is commutative and advertibly complete, then $r_A(x) = \sup_{\varphi \in M(A)} |\varphi(x)|$ for all $x \in A$.*

In the case that both algebras A and B are Fréchet Q-algebras we have the following result.

Theorem 2.7. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet Q-algebras such that B is commutative and semisimple. If the linear map $T : A \to B$ satisfies $r_B(Tx) \leq r_A(x)$ for all $x \in A$, then T is continuous.*

Proof. Since A and B are Fréchet algebras, by the Closed Graph Theorem, it is enough to prove that if $a_n \to 0$ in A and $Ta_n \to b$ in B, then $b = 0$.

Since A is a Q-algebra, r_A is continuous at 0 by Remark 1.6. By the hypothesis it follows that

$$r_B(Ta_n) \to 0, \quad \text{as } n \to \infty.$$

On the other hand, since B is a commutative Q-algebra, by Remark 1.6 r_B is continuous on B. Hence, $Ta_n \to b$ in B implies that

$$r_B(Ta_n) \to r_B(b), \quad \text{as } n \to \infty.$$

Therefore, $r_B(b) = 0$ and since B is a commutative semisimple Fréchet algebra, it follows from Theorem 2.6 that $b = 0$. □

We now present the following result for a map $T : A \to B$, which may not be linear.

Theorem 2.8. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras and let $T : A \to B$ be a map. Consider the following statements:*

(i) $r_{B_k}(Tx + kerq_k) \leq p_k(x)$ for all $x \in A$ and for infinitely many $k \in \mathbb{N}$.
(ii) $r_B(Tx) \leq r_A(x)$ for all $x \in A$.

If T is multiplicative (not necessarily linear) then (i) implies (ii). If A is a Q-algebra, then (ii) implies (i). Consequently, if T is multiplicative and A is a Q-algebra, then (i) and (ii) are equivalent

Proof. Let T be multiplicative, $x \in A$ and $k \in I$, where I is the infinite subset of $\mathbb{N}$ satisfying the hypothesis of the theorem. Then, by applying (i) for x^n, we have

$$r_{B_k}(Tx^n + kerq_k) \leq p_k(x^n) \quad (n \in \mathbb{N}, k \in I). \tag{2.2}$$

Since T is multiplicative, by [4, Theorem 4.6 (5)], for each $n, k \in \mathbb{N}$ we have

$$r_{B_k}(Tx^n + kerq_k) = r^n_{B_k}(Tx + kerq_k).$$

Therefore, (2.2) implies that

$$r_{B_k}(Tx + kerq_k) \leq \lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}},$$

for each $k \in I$, and hence

$$\sup_{k\in I} r_{B_k}(Tx + kerq_k) \leq \sup_{k\in I} \lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}}. \tag{2.3}$$

Since for each $a \in A$ and $b \in B$ the sequences

$$\left(\lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}}\right)_k \ and \ \left(r_{B_k}(b + kerq_k)\right)_k$$

are increasing, by (2.3) and [4, Theorem 4.6 (5)], or [19, Corollary 5.3], we conclude that

$$\begin{aligned} r_B(Tx) &= \sup_{k\in\mathbb{N}} r_{B_k}(Tx + kerq_k) = \sup_{k\in I} r_{B_k}(Tx + kerq_k) \\ &\leq \sup_{k\in I} \lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}} = \sup_{k\in\mathbb{N}} \lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}} = r_A(x). \end{aligned}$$

Now, let A be a Q-algebra and $r_B(Tx) \leq r_A(x)$ for all $x \in A$. Then, by following the same argument as in Corollary 2.4, statement (i) holds. □

One of the main assumptions of Theorem 2.3 is the surjectivity of the map $T : A \to B$. By relaxing the surjectivity assumption on $T : A \to B$ and adding the commutativity assumption on B we obtain the following result.

Theorem 2.9. *Let $(A,(p_n))$ be a Fréchet algebra, $(B,(q_n))$ be a semisimple commutative Fréchet algebra and B_k be the completion of $B/kerq_k$, with respect to the norm $\acute{q_k}$, defined by $\acute{q_k}(y+kerq_k)=q_k(y)$ for all $y\in B$. Let $T:A\to B$ be a linear map such that $r_{B_k}(Tx+kerq_k)\leq p_k(x)$ for infinitely many $k\in\mathbb{N}$ and for all $x\in A$. Then T is continuous.*

Proof. For each $k\in\mathbb{N}$, consider the natural (continuous) embedding $\pi_k:(B,(q_n))\to B_k$ with $\pi_k(b)=b+kerq_k$ for every $b\in B$. Let

$$T_k=\pi_k\circ T:(A,(p_n))\to B_k,\ S_k=r_{B_k}\circ T_k:(A,(p_n))\to\mathbb{C}.$$

Then, for every $a,b\in A$, we have

$$\begin{aligned}|S_k(a)-S_k(b)|&=|r_{B_k}(Ta+kerq_k)-r_{B_k}(Tb+kerq_k)|\\&\leq r_{B_k}(Ta-Tb+kerq_k)\\&\leq r_{B_k}(T(a-b)+kerq_k)\leq p_k(a-b),\end{aligned}$$

where the last inequality is only valid for each $k\in I$, in which I is the infinite subset of $\mathbb{N}$ satisfying the hypothesis of the theorem. Therefore, $S_k=r_{B_k}\circ T_k:(A,(p_n))\to\mathbb{C}$ is continuous for each $k\in I$. Now, let $a_n\to 0$ in A and $Ta_n\to b$ in B. Using the continuity of $S_k=r_{B_k}\circ T_k$, we have $S_k(a_n)=(r_{B_k}\circ T_k)(a_n)\to 0$ as $n\to\infty$ for every $k\in I$. On the other hand, by the continuity of $\pi_k:(B,(q_n))\to B_k$ and $r_{B_k}:B_k\to\mathbb{C}$, we have

$$S_k(a_n)=(r_{B_k}\circ T_k)(a_n)=(r_{B_k}\circ\pi_k)(Ta_n)\to(r_{B_k}\circ\pi_k)(b)=r_{B_k}(b+kerq_k),$$

as $n\to\infty$. Therefore, $r_{B_k}(b+kerq_k)=0$ for every $k\in I$. Since the sequence $(r_{B_k}(b+kerq_k))$ is increasing, it follows that

$$r_B(b)=\sup_{k\in I}r_{B_k}(b+kerq_k)=0.$$

On the other hand, since B is a commutative Fréchet algebra, it is a commutative advertibly complete *lmc* algebra. Therefore, by Theorem 2.6, $r_B(b)=\sup_{\varphi\in M(B)}|\varphi(b)|=0$. Since $radB=\bigcap_{\varphi\in M(B)}ker\varphi$, and B is semisimple, it follows that $b=0$ and consequently, by the Closed Graph Theorem, T is continuous. □

We now present some examples of almost multiplicative maps which are not multiplicative and satisfy the hypotheses of the theorems of this section.

Example 2.10. *Let $(A,(p_n))$ be a commutative semisimple Fréchet Q-algebra. For $0\leq\varepsilon\leq 1$ take $\lambda=\frac{1-\sqrt{1+4\varepsilon}}{2}$ and define $T:A\longrightarrow A$ by $Ta=\lambda a$ for $a\in A$. Clearly T is a linear map, which is not multiplicative, but it is almost multiplicative, since*

$$p_n(Tab-TaTb)=p_n(\lambda ab-\lambda^2ab)\leq|\lambda-\lambda^2|p_n(ab)=\varepsilon p_n(a)p_n(b),$$

for all $a, b \in A$ and $n \in \mathbb{N}$. Since A is a Q-algebra, there exists $k \in \mathbb{N}$ such that $r_A(a) = \lim_{n\to\infty} p_k(a^n)^{\frac{1}{n}}$, for all $a \in A$. Hence,

$$r_A(Ta) = r_A(\lambda a) = \lim_{n\to\infty} p_k((\lambda a)^n)^{\frac{1}{n}} = |\lambda| \lim_{n\to\infty} p_k(a^n)^{\frac{1}{n}} \leq r_A(a).$$

We now follow the same argument as in the proof of Corollary 2.4. Since A is a Q-algebra, we have $r_A(x) \leq p_m(x)$, for some $m \in \mathbb{N}$ and for all $x \in A$ and since $p_n(x) \leq p_{n+1}(x)$ for all $x \in A$ and $n \in \mathbb{N}$, we have

$$r_{A_n}(Tx + kerp_n) \leq r_A(Tx) \leq r_A(x) \leq p_m(x) \leq p_n(x),$$

for all $n \geq m$ and $x \in A$. Therefore, all requirements of the previous theorems are satisfied. Moreover, it is clear from the definition of T, or for example, from Theorem 2.7, that T is continuous.

However, for an example of a commutative semisimple Fréchet Q-algebra we may consider $C^\infty[0,1]$ with the sequence of algebra seminorms

$$p_n(f) = \sum_{k=0}^{n-1} \frac{1}{k!} sup\{|f^{(k)}(t)| : 0 \leq t \leq 1\}.$$

See, for example, [4, Examples 2.4(1) and 6.23(3)]. Note that $C^\infty[0,1]$ is not a Banach algebra under any norm.

Example 2.11. *Let $A = C(X)$ for some compact Hausdorff space X, μ be a regular Borel measure on X such that $\mu(X) = 1$ and T be the linear functional on A represented by μ, that is, $Tf = \int_X f d\mu$ for all $f \in A$. Then, for all $f, g \in A$ we have,*

$$\begin{aligned} |Tfg - TfTg| &= |\int_X fgd\mu - \int_X fd\mu \int_X gd\mu| \\ &\leq \|fg\|_X \mu(X) + \|f\|_X \|g\|_X \mu(X)^2 = 2\|f\|_X \|g\|_X. \end{aligned}$$

It is clear that T is not multiplicative but it is 2-multiplicative. Moreover, $|Tf| = |\int_X fd\mu| \leq \|f\|_X = r_A(f)$. Thus T is an almost multiplicative map satisfying $r(Tf) \leq r_A(f)$ for all $f \in A$. Indeed, this example also fulfils all requirements of Theorem 2.7, as well as the other theorems and corollaries of this section.

3. Automatic continuity of dense range almost multiplicative maps between Fréchet algebras

In this section we extend some results of the previous section.

Definition 3.1. *For Fréchet algebras A and B, the separating space of a linear operator $T : A \to B$ is*

$$\mathfrak{S}(T) = \{y \in B : \text{ there exists } (x_n) \text{ in } A \ \ s.t. \ \ x_n \to 0 \text{ and } Tx_n \to y\}.$$

We first bring the following two known results for easy reference, where the second one is due to B. Aupetit.

Theorem 3.2. *[3, Proposition 2.2.3] Let A be a unital Banach algebra. Then,*

(*i*) *$radA \subseteq \mathfrak{Q}(A)$.*
(*ii*) *If I is a left ideal of A with $I \subseteq \mathfrak{Q}(A)$ then $I \subseteq radA$.*
(*iii*) *In the case where A is commutative, $radA = \mathfrak{Q}(A)$.*

Theorem 3.3. *[2, Theorem 5.1.9] Let A and B be Banach algebras and $T : A \to B$ be a linear map such that $r_B(Tx) \leq r_A(x)$ for every $x \in A$. Then,*

(*i*) *If $y \in \mathfrak{S}(T)$, then $(r_B(Tx))^2 \leq r_A(x) r_B(Tx - y)$ for every $x \in A$.*
(*ii*) *$T(A) \bigcap \mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$.*

If A and B are Banach algebras and $T : A \to B$ is a dense range homomorphism, then $\mathfrak{S}(T)$ is a closed (two sided) ideal in B. See, for example, [2, Proposition 5.1.3]. In the following we extend this result for Fréchet algebras.

Theorem 3.4. *Let $T : A \to B$ be a dense range weakly almost multiplicative operator between Fréchet algebras $(A, (p_n))$ and $(B, (q_n))$. Then the separating space $\mathfrak{S}(T)$ is a closed (two sided) ideal in B.*

Proof. Clearly $\mathfrak{S}(T)$ is a closed linear subspace of B. To show that $\mathfrak{S}(T)$ is an ideal in B, let $y \in \mathfrak{S}(T)$ and $z \in T(A)$. Then there exist a sequence (x_n) and an element a in A such that $z = Ta$ and

$$p_k(ax_n) \leq p_k(a)p_k(x_n) \to 0, \ q_k(Tx_n - y) \to 0$$

for all k. Since T is weakly ε-multiplicative for some $\varepsilon \geq 0$, there exists a subsequence (p_{m_k}) such that for each n we have

$$\begin{aligned} q_k(Tax_n - zy) &\leq q_k(Tax_n - TaTx_n) + q_k(zTx_n - zy) \\ &\leq \varepsilon p_{m_k}(a)p_{m_k}(x_n) + q_k(z)q_k(Tx_n - y) \to 0, \end{aligned}$$

for all k as $n \to \infty$. Consequently, $zy \in \mathfrak{S}(T)$. Similarity, one can see that $yz \in \mathfrak{S}(T)$. We now show that $\mathfrak{S}(T)$ is, in fact, an ideal in $\overline{T(A)} = B$. Let $y \in \mathfrak{S}(T)$ and $z \in \overline{T(A)} = B$. Then there exists a sequence (z_n) in $T(A)$ such that $q_k(z_n - z) \to 0$ for all k. By the above argument $z_n y$ and yz_n are both in $\mathfrak{S}(T)$ and

$$q_k(z_n y - zy) \leq q_k(z_n - z)q_k(y) \to 0,$$

for all k as $n \to \infty$, which implies that zy and, similarly, yz are both elements of $\overline{\mathfrak{S}(T)} = \mathfrak{S}(T)$. Therefore, $\mathfrak{S}(T)$ is a two sided ideal in B. □

We now extend the results of [7, Theorem 3.2] and Corollary 2.4.

Theorem 3.5. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras, A be a Q-algebra, $T : A \to B$ be a weakly almost multiplicative map such that r_B is continuous on $\mathfrak{S}(T)$, and $r_B(Tx) \leq r_A(x)$ for every $x \in A$. Then*

(i) $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$.
(ii) *If T is dense range and B is semisimple, then T is automatically continuous.*

Proof. (i) For $y \in \mathfrak{S}(T)$ there exists a sequence (x_n) in A such that $x_n \to 0$ in A and $Tx_n \to y$ in B. Since A is a Q-algebra, r_A is continuous at zero and hence $r_A(x_n) \to 0$. By the continuity of r_B on $\mathfrak{S}(T)$, we conclude that $r_B(Tx_n) \to r_B(y)$. Since $r_B(Tx_n) \leq r_A(x_n)$ for all n, it follows that $r_B(Tx_n) \to 0$ and hence $r_B(y) = 0$. Therefore, $y \in \mathfrak{Q}(B)$ and so $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$.

(ii) Since $\overline{T(A)} = B$, by the previous theorem, $\mathfrak{S}(T)$ is an ideal in B. Consequently, for every $z \in B$ and $y \in \mathfrak{S}(T)$ we have $yz \in \mathfrak{S}(T)$. By the above argument it follows that $r_B(yz) = 0$. Since

$$radB = \{y \in B : r_B(yz) = 0 \text{ for every } z \in B\},$$

we conclude that $y \in radB$ and hence $\mathfrak{S}(T) \subseteq radB$. Since B is semisimple, $radB = \{0\}$, implying that $\mathfrak{S}(T) = \{0\}$ and so, by the Closed Graph Theorem, T is continuous. □

Since in commutative Fréchet Q-algebras, the spectral radius is uniformly continuous, we get the following corollary.

Corollary 3.6. *Let A and B be Fréchet Q-algebras and let B be semisimple and commutative. If $T : A \to B$ is a dense range weakly almost multiplicative map such that $r_B(Tx) \leq r_A(x)$ for every $x \in A$, then T is automatically continuous.*

The following result is a special case of Corollary 2.3 and the above corollary, which is proved by a different method.

Theorem 3.7. *Let A and B be Banach algebras such that B is semisimple. Let $T : A \to B$ be a surjective weakly almost multiplicative map such that $r_B(Tx) \leq r_A(x)$ for every $x \in A$. Then T is automatically continuous.*

Proof. By applying Theorem 3.3, we have $T(A) \bigcap \mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$. By the surjectivity of T, we have $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$ and since $\mathfrak{S}(T)$ is a closed ideal in B, we conclude that $\mathfrak{S}(T) \subseteq radB$, by Theorem 3.2. Since B is semisimple, $radB = \{0\}$ and so $\mathfrak{S}(T) = \{0\}$. Therefore, by the Closed Graph Theorem, T is continuous. □

Remark 3.8. *Note that the topology of a Fréchet algebra A, generated by the sequence (p_k), coincides with the topology of A, generated by the subsequence (p_{n_k}). Hence, in the definition of weakly ε-multiplicative maps, if we only consider the topology of a Fréchet algebra A, regardless of a particular sequence of seminorms (p_n) for A, then we may assume, without loss of generality, that the inequality $q_k(Tab - TaTb) \leq \varepsilon p_k(a)p_k(b)$, holds for all k. Therefore, the notion of weakly ε-multiplicative is, in fact, the same as ε-multiplicative in the above sense.*

Finally, we recall the following result, which concerns an interesting property of almost multiplicative linear functionals.

Theorem 3.9. *[8, Theorem 2.6] Let $(A, (p_n))$ be a Fréchet algebra and $T : A \to \mathbb{C}$ be an almost multiplicative linear functional. Then either T is multiplicative, or it is continuous.*

Remark 3.10. *In order to check whether all hypotheses of a particular theorem in this paper are essential, we have to present examples which may not satisfy one of the conditions of the theorem, while T fails to be continuous. However, considering the theorem above, it is impossible to do so, otherwise we would have been able to solve the Michael's problem!*

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