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CONVERGENCE OF AN IMPLICIT ITERATION FOR AFFINE MAPPINGS IN NORMED AND BANACH SPACES

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ABSTRACT. The purpose of this paper is to study the weak and strong convergence of an implicit iteration process to a fixed point of a(n) (asymptotically) quasi-nonexpansive affine mapping in normed and Banach spaces.

1. Introduction

Let E be a real normed space, C be a nonempty subset of E and T be a self-mapping on C. T is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Mann [8], Halpern [6], Browder [2, 3], Goebel and Kirk [5], Liu [7], Wittmann [13], Reich [9], Shoji and Takahashi [11] in the settings of Hilbert spaces and uniformly convex Banach spaces. Mann [8] introduced the following iterative procedure for approximation of fixed points of a nonexpansive mapping T on a nonempty closed convex subset

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C in a Hilbert space:

$$x_1 \in C \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [10] studied this iterative procedure in a uniformly convex Banach space whose norm is Frchet differentiable. (For more recently works see e.g., [1] and the references therein).

Our goal is to consider an implicit iteration process of Mann's type for (asymptotically) quasi-nonexpansive affine mappings in normed and Banach spaces and prove the weak and strong convergence of the process to a fixed point of the mappings. This process is defined as follows:

$$x_1 = x \in C \text{ and } x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S_n(x_{n+1})$$

for every $n \in \mathbb{N}$, where $S_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. The preference of the current study is that we obtain our convergence results in more general spaces.

2. **Preliminaries**

For the sake of convenience, we recall some definitions. We assume that C is a nonempty closed convex subset of a real normed space E.

Definition 2.1. A mapping $T : C \to C$ is said to be *quasi-nonexpansive* provided $\|Tx - f\| \le \|x - f\|$ for all $x \in C$ and $f \in F(T)$.

Definition 2.2. T is called *asymptotically nonexpansive* if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - T^n y\| \le (1 + u_n)\|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$.

Definition 2.3. T is called asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - f\| \le (1 + u_n)\|x - f\|$ for all $x \in C$, $f \in F(T)$ and $n \in \mathbb{N}$.

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, but the converse does not hold (see [4, 5]).

Definition 2.4. A mapping $T : C \to C$ is said to be *affine* if $T(\alpha x + (1 - \alpha)y) = \alpha Tx + (1 - \alpha)Ty$ for all $\alpha \in [0, 1]$ and $x, y \in C$.

Definition 2.5. A mapping $T : C \to C$ is said to be *semi-compact* if for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$.

Definition 2.6. A Banach space E is said to satisfy *Opial's property* if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } \ y \in E, \ y \neq x.$$

We will use the following lemma in the sequel.

Lemma 2.7. *(see e.g., [12]). Let $\{t_n\}$ and $\{v_n\}$ be sequences of non-negative real numbers such that $t_{n+1} \leq t_n + v_n$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} v_n < \infty$. Then the limit $\lim_{n\to\infty} t_n$ exists.*

3. A Strong Convergence theorem of Mann's type

Throughout this section C is a nonempty closed convex subset of a normed or Banach space. In this section, using an implicit iterative method of Mann's type [8], we study how to find a fixed point of a quasi-nonexpansive or more generally asymptotically quasi-nonexpansive affine mapping. Consider the following iteration scheme:

$$x_1 = x \in C \text{ and } x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n(x_{n+1}) \qquad (3.1)$$

for every $n \in \mathbb{N}$, where $S_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. If T is quasi-nonexpansive and $0 < \alpha_n \leq 1$ for all n, then for any $f \in F(T)$ it is easy to prove

$$\|x_{n+1} - f\| \leq \|x_n - f\| \qquad (3.2)$$

for every $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} \|x_n - f\|$ exists. It is worth mentioning that if T is nonexpansive and E is a Banach space then the existence of the iterative sequence $\{x_n\}$ follows by the *Banach contraction principal.* The following lemma is essential.

Lemma 3.1. *Let C be a nonempty bounded convex subset of a normed space E and $T : C \to C$ be an affine mapping. Then*

$$\lim_{n\to\infty} \|S_n(x) - TS_n(x)\| = 0$$

uniformly in $x \in C$.

Lemma 3.2. *Let C be a nonempty bounded closed convex subset of a normed space E and $T : C \to C$ be an affine mapping. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Suppose that the sequence $\{x_n\}$ defined implicitly by (3.1) exists. Then*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Proof. Fix $\varepsilon > 0$ and set $M_0 = \sup\{\|z\| : z \in C\}$. From Lemma 3.1, there exists $M \in \mathbb{N}$ such that $\|S_n(y) - TS_n(y)\| < \epsilon$ for every $n \geq M$ and $y \in C$. Thus for every $n \geq M$,

$$S_n(x_{n+1}) \in F_\epsilon(T), \qquad (3.3)$$

where $F_\varepsilon(T) = \{x \in C : \|x - Tx\| \leq \varepsilon\}$. We have for each $k \in \mathbb{N}$,

$$\begin{aligned} x_{M+k} &= \alpha_{M+k-1}x_{M+k-1} + (1-\alpha_{M+k-1})S_{M+k-1}(x_{M+k}) \\ &= \alpha_{M+k-1}(\alpha_{M+k-2}x_{M+k-2} + (1-\alpha_{M+k-2})S_{M+k-2}(x_{M+k-1})) \\ &\quad +(1-\alpha_{M+k-1})S_{M+k-1}(x_{M+k}) = \dots \\ &= (\prod_{i=M}^{M+k-1}\alpha_i)x_M + \sum_{j=M}^{M+k-2}((\prod_{i=j+1}^{M+k-1}\alpha_i)(1-\alpha_j)S_j(x_{j+1})) \\ &\quad +(1-\alpha_{M+k-1})S_{M+k-1}(x_{M+k}). \end{aligned}$$

Thus

$$x_{M+k} = (\prod_{i=M}^{M+k-1}\alpha_i)x_M + (1-\prod_{i=M}^{M+k-1}\alpha_i)y_k, \qquad (3.4)$$

where

$$\begin{aligned} y_k = \frac{1}{1-\prod_{i=M}^{M+k-1}\alpha_i}&(\sum_{j=M}^{M+k-2}((\prod_{i=j+1}^{M+k-1}\alpha_i)(1-\alpha_j)S_j(x_{j+1})) \\ &+(1-\alpha_{M+k-1})S_{M+k-1}(x_{M+k})). \end{aligned}$$

Now from

$$\sum_{j=M}^{M+k-2}((\prod_{i=j+1}^{M+k-1}\alpha_i)(1-\alpha_j)) + (1-\alpha_{M+k-1}) = 1 - \prod_{i=M}^{M+k-1}\alpha_i,$$

it follows that $y_k \in \operatorname{co}\{S_n(x_{n+1}) : n \geq M\}$ and hence $y_k \in F_\varepsilon(T)$ for each $k \in \mathbb{N}$ by (3.3). From the Abel-Dini theorem and $\sum_{i=M}^{\infty}(1-\alpha_i) = \infty$, there exists $p \in \mathbb{N}$ such that $\prod_{i=M}^{M+k-1} \alpha_i < \frac{\varepsilon}{2M_0}$ for all $k \geq p$. From (3.4) we obtain

$$\|x_{M+k} - y_k\| = \prod_{i=M}^{M+k-1} \alpha_i \|x_M - y_k\| < \frac{\varepsilon}{2M_0} 2M_0 = \varepsilon$$

for each $k \geq p$. By another application of (3.4) and the affiness of T we have

$$\|Tx_{M+k} - Ty_k\| = \prod_{i=M}^{M+k-1} \alpha_i \|Tx_M - Ty_k\| < \varepsilon$$

for each $k \geq p$. Hence

$$\|Tx_{M+k} - x_{M+k}\| \leq \|Tx_{M+k} - Ty_k\| + \|Ty_k - y_k\| + \|y_k - x_{M+k}\| \leq 3\varepsilon$$

for every $k \geq p$. Therefore $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. □

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a normed space E and $T : C \to C$ be a quasi-nonexpansive continuous semi-compact affine mapping. Let $\{\alpha_n\}$ be a sequence in $(0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of T*

Proof. Since T is semi-compact and continuous, using Lemma 3.2, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in F(T)$. Therefore $\{x_n\}$ converges to $x^* \in F(T)$ by (3.2).

Lemma 3.4. *Let C be a nonempty bounded convex subset of a normed space E and $T : C \to C$ be an asymptotically quasi-nonexpansive mapping, i.e., $\|T^n x - f\| \leq (1 + u_n)\|x - f\|$, $n = 1, 2, \ldots,$ for all $x \in C$ and $f \in F(T)$. Suppose that $\sum_{n=1}^{+\infty} u_n < \infty$, $\{\alpha_n\}$ is a sequence in $(0, 1]$ such that $\sum_{n=1}^{\infty} \frac{1-\alpha_n}{n} < \infty$, $x_1 \in C$ and $\{x_n\}$ is defined by (3.1). Then for all $f \in F(T)$, $\lim_{n\to\infty} \|x_n - f\|$ exists.*

Proof. Suppose $f \in F(T)$. Put $t_n = \|x_n - f\|$ for each n, and $M_0 = \sup_n \|x_n - f\|$. Then, by letting $u_0 = 0$, we have

$$t_{n+1} \leq \alpha_n t_n + (1 - \alpha_n)\|S_n x_{n+1} - f\|$$

$$\leq \alpha_n t_n + (1 - \alpha_n)\frac{1}{n}\sum_{i=0}^{n-1}(1 + u_i)t_{n+1} = \alpha_n t_n + (1 - \alpha_n)(1 + \frac{1}{n}\sum_{i=0}^{n-1} u_i)t_{n+1},$$

and thus

$$t_{n+1} \leq t_n + \frac{1-\alpha_n}{n\alpha_n} \sum_{i=0}^{n-1} u_i t_{n+1}. \quad (3.5)$$

But $\alpha_n \to 1$, as $n \to \infty$, since $\sum_{n=1}^{\infty} \frac{1-\alpha_n}{n} < \infty$. So $\min_n \alpha_n > 0$. Now put $\alpha = \min_n \alpha_n$ and $u = \sum_{i=0}^{\infty} u_i$. Then, from (3.5) we have $t_{n+1} \leq t_n + \frac{1-\alpha_n}{n} \times \frac{uM_0}{\alpha}$. Now, using Lemma 2.7 and the assumption $\sum_{n=1}^{\infty} \frac{1-\alpha_n}{n} < \infty$, we conclude that the limit $\lim_{n\to\infty} t_n$ exists. This completes the proof.

By considering Lemma 3.4, we may prove the following similar to Theorem 3.3.

Theorem 3.5. *Let C be a nonempty bounded closed convex subset of a normed space E and $T : C \to C$ be an asymptotically quasi-nonexpansive continuous semi-compact affine mapping. Suppose that $\sum_{n=1}^{+\infty} u_n < \infty$ and that $\{\alpha_n\}$ is a sequence in $(0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$, $\sum_{n=1}^{\infty} \frac{1-\alpha_n}{n} < \infty$ and $x_1 \in C$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of T*

Corollary 3.6. *Let E, C, T and $\{x_n\}$ be either as in Theorem 3.5 or as in Theorem 3.3 with the condition $\alpha_n \to 1$. Then the limit*

$$\lim_{n\to\infty} \sum_{i=0}^{m} \alpha_n (1 - \alpha_n)^i S_n^i x_n$$

converges to a fixed point of T uniformly in m.

Proof. By the fact that T is affine, we have

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n(x_{n+1})$$

$$= \alpha_n x_n + (1 - \alpha_n) S_n(\alpha_n x_n + (1 - \alpha_n) S_n x_{n+1})$$

$$= \alpha_n x_n + (1 - \alpha_n)\alpha_n S_n x_n + (1 - \alpha_n)^2 S_n^2 x_{n+1}$$

$$= \cdots = \sum_{i=0}^{m} (1 - \alpha_n)^i \alpha_n S_n^i x_n + (1 - \alpha_n)^{m+1} S_n^m x_{n+1}.$$

On the other hand $\|(1-\alpha_n)^{m+1} S_n^m x_{n+1}\| \leq (1-\alpha_n) M_0 \to 0$, as $n \to \infty$, where $M_0 = \sup\{\|z\| : z \in C\}$. Moreover, $\lim_{n\to\infty} x_n = f \in F(T)$, either by Theorem 3.5 or by Theorem 3.3 with the condition $\alpha_n \to 1$. Therefore

$$\|\sum_{i=0}^{m}(1-\alpha_n)^i \alpha_n S_n^i x_n - f\| \le \|x_{n+1} - f\| + (1-\alpha_n)M_0 \to 0,$$

as $n \to \infty$, uniformly in m.

Lemma 3.7. *(Demiclosedness Principle). Assume that C is a closed convex subset of a normed space E and $T : C \to E$ is a continuous affine mapping. Then $I-T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$.*

Proof. Let $\{x_n\}$ be a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y. By considering the mapping $T_y z = Tz + y$ for all $z \in C$, without the loss of generality, we may assume that $y = 0$. Then $\|x_n - Tx_n\| \to 0$, as $n \to \infty$. Since T is continuous and affine, $F_\epsilon(T) = \{x \in C : \|x - Tx\| \le \epsilon\}$ is closed and convex for all $\epsilon > 0$. Therefore $x \in F_\epsilon(T)$ for each $\epsilon > 0$, and then $x \in F(T)$. That is, $(I-T)x = 0$.

Theorem 3.8. *Let C be a nonempty weakly compact convex subset of a Banach space E satisfying Opial's condition and $T : C \to C$ be a continuous quasi-nonexpansive affine mapping. Let $\{\alpha_n\}$ be a sequence in $(0,1]$ such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ defined by (3.1) converges weakly to a fixed point of T.*

Proof. Let $\{x_{k_n}\}$ and $\{x_{l_n}\}$ be subsequences of $\{x_n\}$ converging weakly to f_1 and f_2, respectively, and $f_1 \ne f_2$. By Lemmas 3.2, 3.7, it follows that $f_1, f_2 \in F(T)$. Also using the Opial's condition and (3.2), we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - f_1\| &= \lim_{n\to\infty}\|x_{k_n} - f_1\| < \lim_{n\to\infty}\|x_{k_n} - f_2\| = \lim_{n\to\infty}\|x_n - f_2\| \\ &= \lim_{n\to\infty}\|x_{l_n} - f_2\| < \lim_{n\to\infty}\|x_{l_n} - f_1\| = \lim_{n\to\infty}\|x_n - f_1\|,\end{aligned}$$

which is a contradiction. Thus, $f_1 = f_2$. This leads to the desired conclusion. Using Lemmas 3.4, by a proof as above, we have the following theorem.

Theorem 3.9. *Let C be a nonempty weakly compact convex subset of a Banach space E satisfying Opial's condition and let $T : C \to C$ be* *an asymptotically quasi-nonexpansive continuous affine mapping. Suppose that* $\sum_{n=1}^{+\infty} u_n < \infty$ *and that* $\{\alpha_n\}$ *is a sequence in* $(0,1]$ *such that* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$*,* $\sum_{n=1}^{\infty} \frac{1-\alpha_n}{n} < \infty$ *and* $x_1 \in C$*. Then the sequence* $\{x_n\}$ *defined by (3.1) converges weakly to a fixed point of* T*.*

Remark 3.10. If we impose the condition $F(T) \neq \emptyset$ in Theorems 3.3 and 3.8, then we can remove the boundedness condition on C. In fact, it is enough to consider $D = \{y \in C : \|y - z\| \leq \|x_1 - z\|\}$, where z is an arbitrary element of $F(T)$, and note that $x_i, z, T^j x_i \in D$, for all $i, j \in \mathbb{N}$, $T(D) \subset D$ and D is a bounded closed, convex subset of C. So by replacing C with D we can repeat the proof of the theorems.

Remark 3.11. The proofs of the current paper can be repeated with some insignificant changes to obtain the convergence results for the following iteration process:

$$x_1 = x \in C \text{ and } x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_n(x_n), \ \forall n \in \mathbb{N}.$$

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