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LINEAR MAPS PRESERVING OR STRONGLY PRESERVING MAJORIZATION ON MATRICES

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(Communicated by Bamdad Yahaghi)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. For $A, B \in M_{nm}$, we say that A is left matrix majorized (resp. left matrix submajorized) by B and write $A \prec_\ell B$ (resp. $A \prec_{\ell s} B$), if $A = RB$ for some $n \times n$ row stochastic (resp. row substochastic) matrix R. Moreover, we define the relation $\sim_{\ell s}$ on M_{nm} as follows: $A \sim_{\ell s} B$ if $A \prec_{\ell s} B \prec_{\ell s} A$. This paper characterizes all linear preservers and all linear strong preservers of $\prec_{\ell s}$ and $\sim_{\ell s}$ from M_{nm} to M_{nm}.
Keywords: Linear preserver, row substochastic matrix, matrix majorization.
MSC(2010): Primary: 15A04; Secondary: 15A21, 15A51.

1. Introduction

Throughout the paper, the notation M_{nm} is used for the space of all $n \times m$ real matrices. We also write $M_{nn} = M_n$ and $M_{n1} = \mathbb{R}^n$. I_n is the $n \times n$ identity matrix and $\mathcal{P}(n)$ will denote all $n \times n$ permutation matrices. An $n \times m$ matrix $R = [r_{ij}]$ is called *row stochastic* (resp. *row substochastic*) if for all i, j, $r_{ij} \geq 0$ and $\Sigma_{k=1}^{m} r_{ik}$ is equal (resp. at most equal) to 1. For $A, B \in M_{nm}$, we say that A is left matrix majorized (resp. left matrix submajorized) by B and write $A \prec_\ell B$ (resp. $A \prec_{\ell s} B$) if $A = RB$ for some $n \times n$ row stochastic (resp. row substochastic) matrix R. For a given relation $\prec$, we write $A \sim B$ if $A \prec B \prec A$. A linear operator $T\colon M_{nm} \to M_{nm}$ is said to be a linear preserver of $\prec$ if $A \prec B$ implies that $T(A) \prec T(B)$ for all $A,\ B \in M_{nm}$. It is a strong preserver of $\prec$ when $A \prec B$ if and only if $T(A) \prec T(B)$.

A.M. Hasani and M. Radjabalipour [7] characterized the structure of all linear operators $T\colon M_{nm} \to M_{nm}$ preserving $\prec_\ell$. In particular, they proved that if $T : M_n \to M_n$ strongly preserves $\prec_\ell$, then there exists a permutation

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matrix $P \in \mathcal{P}(n)$ and an invertible matrix $L \in M_n$ such that $T(X) = PXL$ for all $X \in M_n$.

A. Armandnejad and A. Salemi [2] characterized the structure of all linear preservers of $\prec_\ell$ on complex matrices. Also, M. Radjabalipour and P. Torabian [14] characterized all preservers of $\prec_\ell$ on $\mathbb{R}^n$ which are not necessarily linear.

For more information about left matrix majorization and the previous work on this subject we also refer to [3, 5, 8, 9, 10] and [13]. The structure of linear operators that preserve other types of majorization have been derived by Ando [1], Beasley, Lee and Y.H. Lee [4], Dahl [6], and Li and E. Poon [11]. Marshall and Olkin's text [12] is a standard general reference for majorization.

The present paper is organized as follows. In Section 2 we derive necessary and sufficient conditions for a linear operator T from $\mathbb{R}^n$ to $\mathbb{R}^n$ to preserve $\prec_{\ell s}$. In particular, we prove that the structure of linear preservers of $\prec_\ell$, $\prec_{\ell s}$ and $\sim_{\ell s}$ are the same for $n \geq 3$. In Section 3 we characterize a general linear preserver T from M_{nm} to M_{nm}. In particular, we give necessary and sufficient conditions for a linear operator $T\colon M_{nm} \to M_{nm}$ to strongly preserve $\prec_{\ell s}$.

We note that necessary and sufficient conditions for $T\colon \mathbb{R}^n \to \mathbb{R}^n$ to be a linear preserver of $\prec_\ell$ have been derived before and the following theorems are known.

Theorem 1.1. [7, Theorem 2.3] *Let $n \geq 3$. Then $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_\ell$ if and only if T has the form $T(X) = aPX$, for all $X \in R^n$, for some some $a \in \mathbb{R}$ and some $P \in \mathcal{P}(n)$.*

Theorem 1.2. [7, Theorem 2.3] *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. Then, T is a linear preserver of $\prec_\ell$ if and only if T has the form $T(X) = (aI + bP)X$ for all $X \in \mathbb{R}^2$, where P is a 2×2 permutation matrix not equal I_2, and $ab \leq 0$.*

The following theorem states necessary and sufficient conditions for a linear operator $T\colon M_{nm} \to M_{nm}$ to be a linear preserver of $\prec_\ell$.

Theorem 1.3. [7, Theorem 3.1] *Let $T\colon M_{nm} \to M_{nm}$ be a linear operator. Then T preserves $\prec_\ell$ if and only if $T(X) = (aI + bP)XL$ for all $X \in M_{nm}$, where $L \in M_m$, P is an $n \times n$ permutation matrix, $P \neq I$, a and b are real numbers such that $ab \leq 0$, and, if $n \neq 2$, $ab = 0$. Moreover, if $n \neq 2$, then $aI + bP = cQ$ for some $c \in \mathbb{R}$ and, hence, $T(X) = QXK$ for some $K \in M_m$.*

2. Linear preservers of $\prec_{\ell s}$ on $\mathbb{R}^n$

In what follows, $[T] = [t_{ij}]$ will denote the matrix representation of an operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis $\{e_1, e_2, \dots, e_n\}$ of $\mathbb{R}^n$. Also, $e = \Sigma_{j=1}^n e_j \in \mathbb{R}^n$ and

$$\begin{aligned} \mathbf{a} : &= \max\{t_{ij} \mid 1 \leq i, j \leq n\}, \\ \mathbf{b} : &= \min\{t_{ij} \mid 1 \leq i, j \leq n\}. \end{aligned} \tag{2.1}$$

By Theorem 1.2, the matrix representation of a linear preserver of $\prec_\ell$ with respect to the standard basis of $\mathbb{R}^2$ is as follows:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for some real numbers a, b satisfying $ab \leq 0$.

All linear operators $T\colon \mathbb{R} \to \mathbb{R}$ are preservers of $\prec_{\ell s}$ ($T(rx) \prec_{\ell s} T(x)$ for all $x \in \mathbb{R}$ and for all $r \in [0,1]$). Also, $T = 0$ is a linear preserver of $\prec_{\ell s}$. Hence, throughout the paper, for a linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ we shall assume that $T \neq 0$ and $n \geq 2$.

$T\colon M_{nm} \to M_{nm}$ is a linear preserver of $\prec_{\ell s}$ if and only if αT is a linear preserver of $\prec_{\ell s}$ for all nonzero real numbers α. Hence without loss of generality we shall assume that $\mathbf{a} > 0$ and $\mid \mathbf{b} \mid \leq \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are as in (2.1).

Throughout the paper, for a given vector $x \in \mathbb{R}^n$, $\max x$ and $\min x$ denote the maximum and minimum values of components of x, respectively. Also, we write $x_M = \max x$ and $x_m = \min x$.

The following important lemmas are easy consequences of the definitions of $\prec_{\ell s}$ and $\sim_{\ell s}$.

Lemma 2.1. *Let $x, y \in \mathbb{R}^n$. If $x \prec_{\ell s} y$ then the following assertions are true.*
(a) $x_i \in \mathrm{Conv}(\{y_1, \ldots, y_n\} \cup \{0\})$, for all i $(1 \leq i \leq n)$.
(b) If $y_m \geq 0$, then $x_m \geq 0$.
(c) If $y_M \leq 0$, then $x_M \leq 0$.
(d) If $y_m \leq 0$ and $y_M \geq 0$, then $y_m \leq x_m \leq x_M \leq y_M$.

Lemma 2.2. *Let x, y be nonzero vectors in $\mathbb{R}^n$. If $x \sim_{\ell s} y$, then exactly one of the following occurs:*
(a) x, y are entrywise nonnegative and $x_M = y_M$.
(b) x, y are entrywise nonpositive and $x_m = y_m$.
(c) $x_m = y_m \leq 0$ and $x_M = y_M \geq 0$.
Furthermore, if $x, y \in \mathbb{R}^n$ and at least one of the conditions (a), (b) and (c) holds, then $x \sim_{\ell s} y$.

Theorem 2.3 presents some necessary conditions for a nonzero operator $T\colon \mathbb{R}^n \to \mathbb{R}^n, n \geq 2$, to be a linear preserver of $\sim_{\ell s}$.

Theorem 2.3. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a nonzero linear preserver of $\sim_{\ell s}$, and assume that $n \geq 2$, and $\mathbf{a}$ and $\mathbf{b}$ are as in (2.1). Then the following assertions are true*

(a) *For each $j \in \{1, 2, \ldots, n\}$, $\max T(e_j) = \mathbf{a}$. In particular, every column of $[T]$ contains at least one entry equal to $\mathbf{a}$.*
(b) $\max T(e) = \mathbf{a}$; *moreover, if a row of $[T]$ contains an entry equal to $\mathbf{a}$, then all other nonnegative entries of that row are zero.*
(c) $\mathbf{b} = 0$.

Proof. **(a).** Without loss of generality, we can assume that $t_{11} = \mathbf{a}$ and $\mathbf{a} > 0$. $t_{11} = \mathbf{a}$ implies that $\max T(e_1) = \mathbf{a}$. Let $j \in \{1, 2, \ldots, n\}$ be fixed. Since $e_j \sim_{\ell s} e_1$ and T preserves $\sim_{\ell s}$, hence $T(e_j) \sim_{\ell s} T(e_1)$. By Lemma 2.2, $\max T(e_j) = \max T(e_1) = \mathbf{a}$. Since $j \in \{1, 2, \ldots, n\}$ is arbitrary, $\max T(e_j) = \mathbf{a}$, for all j $(1 \leq j \leq n)$, therefore, every column of $[T]$ has at least one entry equal to $\mathbf{a}$.
(b). By Lemma 2.2, $\Sigma_{j\in J} e_j \sim_{\ell s} e_1$, for all $J \subseteq \{1, \ldots, n\}$ and hence $\Sigma_{j\in J} T(e_j) \sim_{\ell s} T(e_1)$. Lemma 2.2 implies that $\max \Sigma_{j\in J} T(e_j) = \mathbf{a}$, for all $J \subseteq \{1, 2, \ldots, n\}$. Therefore, for all $J \subseteq \{1, \ldots, n\}$, $\max \Sigma_{j\in J} t_{ij} = \mathbf{a}$ where the maximum is taken over i $(1 \leq i \leq n)$. Thus, if a row of $[T]$ contains an entry equal to $\mathbf{a}$, then all nonnegative entries of that row are zero. In particular, $\max T(e) = \mathbf{a}$.
(c). From (a), it follows that every column of $[T]$ has at least one entry equal to $\mathbf{a}$. Also, (b) implies that every row of $[T]$ has at most one entry equal to $\mathbf{a}$. Since $[T]$ is $n \times n$, every row of $[T]$ has exactly one entry equal to $\mathbf{a}$. Hence by (b), all other nonnegative entries of rows of $[T]$ must be zero. Therefore $\mathbf{b} \leq 0$. If $\mathbf{b} < 0$, without loss of generality, we may write $t_{11} = \mathbf{b}$. So, $\max T(e_1) = \mathbf{a} > 0$ and $\min T(e_1) = \mathbf{b} < 0$. Let $k \in \{1, \ldots, n\}$ be fixed, since $e_1 \sim_{\ell s} e_k$ and T preserves $\sim_{\ell s}$, then $T(e_1) \sim_{\ell s} T(e_k)$. Hence by Lemma 2.2, $\max T(e_k) = \max T(e_1) = \mathbf{a}$ and $\min T(e_k) = \min T(e_1) = \mathbf{b}$. Since k is arbitrary, each column of $[T]$ has at least one entry equal to $\mathbf{b}$. Let $J \subseteq \{1, \ldots, n\}$. Since $\Sigma_{j\in J} e_j \sim_{\ell s} e_1$, $\Sigma_{j\in J} T(e_j) \sim_{\ell s} T(e_1)$, by Lemma 2.2, $\min \Sigma_{j\in J} T(e_j) = \mathbf{b}$, for all $J \subseteq \{1, \ldots, n\}$. Thus, if a row of $[T]$ has one entry equal to $\mathbf{b}$, then all its other nonpositive entries of it must be zero. Thus, at most one entry of each row of $[T]$ equals to $\mathbf{b}$. Since $[T]$ is $n \times n$, each row of $[T]$ has one entry equal to $\mathbf{b}$ and other nonpositive entries are zero. But one entry of each row of $[T]$ is equal to $\mathbf{a}$, which is a contradiction, hence $\mathbf{b} = 0$. □

Theorem 2.4. *If T is such that $T(x) = aPx$, for all $x \in \mathbb{R}^n$, for a real number a and a permutation matrix $P \in \mathcal{P}(n)$, the operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$ is a linear preserver of $\prec_{\ell s}$.*

Proof. Let $x \in \mathbb{R}^n$ and R be a row substochastic matrix in M_n. Since $PR = R'P$ for some row substochastic matrix R', $T(Rx) = aPRx = R'aPx = R'(T(x))$. Therefore, T is a linear preserver of $\prec_{\ell s}$. □

The following theorem follows from Theorem 2.2 and Theorem 2.4.

Theorem 2.5. *Let $n \geq 2$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then the following assertions are equivalent:*

(a) *T preserves $\prec_{\ell s}$,*
(b) *T preserves $\sim_{\ell s}$,*
(c) *$T(x) = aPx$, for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$.*

Theorem 1.1 and Theorem 2.2 imply the following corollary.

Corollary 2.6. *Let $n \geq 3$. Then $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_\ell$ if and only if T is a linear preserver of $\prec_{\ell s}$.*

The following example shows that, the Corollary 2.6 is not true for $n = 2$.

Example 2.7. The linear operator whose matrix representation is

$$[T] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

is a linear preserver of $\prec_\ell$ but not a linear preserver of $\prec_{\ell s}$.

3. Linear Preservers of $\prec_{\ell s}$ on M_{nm}

For each i $(1 \leq i \leq m)$, define the linear operators $E_i\colon \mathbb{R}^n \to M_{nm}$ by $E_i(x) = xe_i^t$ for all $x \in \mathbb{R}^n$ and $E^i\colon M_{nm} \to \mathbb{R}^n$ by $E^i(X) = Xe_i$ for all $X \in M_{nm}$, where $\{e_1, \ldots, e_m\}$ denotes the standard basis for $\mathbb{R}^m$ [7].

Lemma 3.1. *Let $T\colon M_{nm} \to M_{nm}$ be a linear preserver of $\prec_{\ell s}$. Then the linear operators $T_{ij} = E^j \circ T \circ E_i$ preserve $\prec_{\ell s}$ for all $i, j = 1, 2, \ldots, m$.*

Proof. Let $x \in \mathbb{R}^n$ and R be a row substochastic matrix in M_n. $Rx \prec_{\ell s} x$ implies that $E_i(Rx) \prec_{\ell s} E_i(x)$. Since T is a linear preserver of $\prec_{\ell s}$, for every i $(1 \leq i \leq m)$, $T(E_i(Rx)) \prec_{\ell s} T(E_i(x))$. Therefore $E^j(T(E_i(Rx))) \prec_{\ell s} E^j(T(E_i(x)))$, for all $i, j = 1, 2, \ldots, m$. □

Theorem 3.2. *Let $T\colon M_{nm} \to M_{nm}$ be a linear operator. If T preserves $\sim_{\ell s}$, then $T(X) = PXA$, for all $X \in M_{nm}$, for some $A \in M_n$ and some $n \times n$ permutation matrix P.*

Proof. For each $X = [x_1, x_2, \ldots, x_m] \in M_{nm}$, it is easily seen that

$$T(X) = T([x_1, x_2, \ldots, x_m]) = [\Sigma_{i=1}^m T_{i1}(x_i), \ldots, \Sigma_{i=1}^m T_{im}(x_i)].$$

It follows from Lemma 3.1 that every T_{ij} is a linear preserver of $\sim_{\ell s}$. Hence, by Theorem 2.4, $T_{ij}(x) = a_{ij}P_{ij}x$ for some permutation matrices P_{ij} and some real numbers a_{ij}, where $i, j = 1, 2, \ldots, m$. Since $T \neq 0$, $a_{ij} \neq 0$, for some i, j $(1 \leq i, j \leq m)$. Without loss of generality, let $i = j = 1$ and $P = P_{11}$.

We claim that $P_{ij} = P$, for all $i, j = 1, 2, \ldots, m$. Let $r, s \in \{1, \ldots, m\}$, α, β be scalars and $(X)_i$ denote the i^{th} column of the matrix $X \in M_{nm}$. Fix $k \in \{1, \ldots, n\}$ and define $X, Y \in M_{nm}$ by $(X)_r = \alpha e$, $(Y)_r = \alpha e_k$, $(X)_s = \beta e$, $(Y)_s = \beta e_k$ and $(X)_i = (Y)_i = 0$, if $i \neq r$, $i \neq s$. $X \sim_{\ell s} Y$ implies that $T(X) \sim_{\ell s} T(Y)$, and hence,

$$[(T(X))_r, (T(X))_s] \sim_{\ell s} [(T(Y))_r, (T(Y))_s].$$

Therefore,

$$[\alpha a_{rr}e + \beta a_{sr}e, \alpha a_{rs}e + \beta a_{ss}e] \sim_{\ell s} [\alpha a_{rr}P_{rr}e_k + \beta a_{sr}P_{sr}e_k, \alpha a_{rs}P_{rs}e_k + \beta a_{ss}P_{ss}e_k].$$

If $a_{rr}a_{rs} \neq 0$, we prove that $P_{rr} = P_{rs}$. Let $\alpha = 1$ and $\beta = 0$. We have $e = RP_{rr}e_k = RP_{rs}e_k$, for some row substochastic matrix R. Since R has at most one column equal to e and k is arbitrary, $P_{rr} = P_{rs}$.

Now, suppose $a_{rr}a_{sr} \neq 0$. We prove that $P_{rr} = P_{sr}$. Let α, β be such that $(\alpha a_{rr})(\beta a_{sr}) > 0$. We know that

$$\alpha a_{rr}e + \beta a_{sr}e \sim_{\ell s} \alpha a_{rr}P_{rr}e_k + \beta a_{sr}P_{sr}e_k$$

If $P_{rr} \neq P_{sr}$, then $\alpha a_{rr} + \beta a_{sr} \in \mathrm{Conv}(\{\alpha \mathrm{a_{rr}}, \beta \mathrm{a_{sr}}\} \cup \{0\})$, which is a contradiction. Therefore, $P_{rr} = P_{sr}$.

Now suppose that $a_{rr}a_{ss} \neq 0$, but $a_{rs} = a_{sr} = 0$. Thus,

$$[\alpha a_{rr}e, \beta a_{ss}e] \sim_{\ell s} [\alpha a_{rr}P_{rr}e_k, \beta a_{ss}P_{ss}e_k].$$

Let $\alpha = \beta = 1$. Then $e = RP_{rr}e_k = RP_{ss}e_k$. Since k is arbitrary and R has at most one column equal to e, we get $P_{rr} = P_{ss}$.

We conclude that $P_{ij} = P$ for all $i, j \in \{1, \ldots, m\}$. Therefore,

$$\begin{aligned} T(X) &= [\Sigma_{i=1}^m a_{i1}P_{i1}X_i, \ldots, \Sigma_{i=1}^m a_{im}P_{im}X_i] \\ &= P[\Sigma_{i=1}^m a_{i1}X_i, \ldots, \Sigma_{i=1}^m a_{im}X_i] \\ &= PXA, \end{aligned}$$

where $A = [a_{ij}]$.

□

Theorem 3.3. *Let $T\colon M_{nm} \to M_{nm}$ be a linear operator. Then the following assertions are equivalent:*

(a) *T preserves $\prec_{\ell s}$,*

(b) *T preserves $\sim_{\ell s}$,*

(c) *$T(X) = PXA$, for all $X \in M_{nm}$, some $A \in M_m$, and some $n \times n$ permutation matrix P.*

Proof. By Theorem 3.2, it is sufficient to prove that (c) implies (a). Let $T(X) = PXA$ and R be a row substochastic matrix. Since $PR = R'P$ for some row substochastic matrix R', $T(RX) = PRXA = R'PXA = R'(T(X))$. Hence $T(RX) \prec_{\ell s} T(X)$. □

Corollary 3.4. *A linear operator $T : M_{nm} \to M_{nm}$ strongly preserves the majorization relation $\prec_{\ell s}$ if and only if there exists $P \in \mathcal{P}(n)$ and an invertible matrix L in M_m such that $T(X) = PXL$ for all $X \in M_{nm}$.*

Proof. By Theorem 3.2, there exists $P \in \mathcal{P}(n)$, $L \in M_m$ and a nonzero real number a such that $T(X) = aPXL$ for all $X \in M_{nm}$. Choose $X \in M_{nm}$ such that $XL = 0$. Thus, $T(X) = aPXL = 0 \prec_{\ell s} 0 = T(0)$ and therefore, $X \prec_{\ell s} 0$. Hence, $X = 0$ which implies that L is invertible. Replacing L by $a^{-1}L$ yields $T(X) = PXL$ for all $X \in M_{nm}$, for some $P \in \mathcal{P}(n)$ and an invertible matrix $L \in M_m$.

Let $T(X) \prec_{\ell s} T(Y)$ for $X, Y \in M_{nm}$. Then $PXL = RPYL$ for some row substochastic matrix R. Since L is invertible $PX = RPY$, then $X = RY$ and hence $X \prec_{\ell s} Y$. $\square$

References

[1] T. Ando, Majorization, doubly stochastic matrices, and comparison of eigenvalues, *Linear Algebra Appl.* **118** (1989) 163–248.
[2] A. Armandnejad and A. Salemi, The structure of linear preservers of gs-majorization, *Bull. Iranian Math. Soc.* **32** (2006), no. 2, 31–42.
[3] A. Armandnejad and H. Heydari, Linear preserving gd-majorization functions from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$, *Bull. Iranian Math. Soc.* **37** (2011), no. 1, 215–224.
[4] L. B. Beasley, S.-G. Lee and Y.-H. Lee, Linear operators preserving multivariate majorization, *Linear Algebra Appl.* **304** (2000), no. 1-3, 141–159.
[5] L. B. Beasley, S.-G. Lee and Y.-H. Lee, A characterization of strong preservers of matrix majorization, *Linear Algebra Appl.* **367** (2003) 341–346.
[6] G. Dahl, Matrix majorization, *Linear Algebra Appl.* **288** (1999), no. 1-3, 53–73.
[7] A. M. Hasani and M. Radjabalipour, Linear preserver of matrix majorization, *Int. J. Pure Appl. Math.* **32** (2006), no. 4, 475–482.
[8] F. Khalooei and A. Salemi, The structure of linear preservers of left matrix majorization on $\mathbb{R}^p$, *Electron. J. Linear Algebra* **18** (2009) 88–97.
[9] F. Khalooei and A. Salemi, Linear preservers of majorization, *Iranian J. Math. Sci. Inform.* **6** (2011), no. 2, 43–50.
[10] F. Khalooei, Linear preservers of two-sided matrix majorization, *Wavelet and linear Algebra* **1** (2014) 33–38.
[11] C. K. Li and E. Poon, Linear operators preserving directional majorization, *Linear Algebra Appl.* **325** (2001), no. 1-3, 141–146.
[12] A. W. Marshall and I. Olkin and C. B Arnold, Inequalities, Theory of Majorization and its Applications, Springer, New York, 2011.
[13] F. D. Martínez Pería, P. G. Massey and L. E. Silvestre, Weak matrix majorization, *Linear Algebra Appl.* **403** (2005) 343–368.
[14] M. Radjabalipour and P. Torabian, On nonlinear preservers of weak matrix majorization, *Bull. Iranian Math. Soc.* **32** (2006), no. 2, 21–30.

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