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THE WITNESS SET OF COEXISTENCE OF QUANTUM EFFECTS AND ITS PRESERVERS

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Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. One of unsolved problems in quantum measurement theory is to characterize coexistence of quantum effects. In this paper, applying positive operator matrix theory, we give a mathematical characterization of the witness set of coexistence of quantum effects and obtain a series of properties of coexistence. We also devote to characterizing bijective morphisms on quantum effects leaving the witness set invariant. Furthermore, applying linear maps preserving commutativity, which are characterized by Choi, Jafarian and Radjavi [Linear maps preserving commutativity, Linear Algebra Appl. 87 (1987), 227–241.], we classify multiplicative general morphisms leaving the witness set invariant on finite dimensional Hilbert space effect algebras.

Keywords: Positive operator matrices, coexistence, Hilbert space effect algebras, isomorphisms.
MSC(2010): Primary: 15A48, 15A90; Secondary: 46N50, 47N50, 47B65.

1. Introduction

Let H be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators acting on H. The operator interval in $\mathcal{B}(H)$: $\mathcal{E}(H) = \{T \in \mathcal{B}(H) \mid 0 \leq T \leq I\}$, where I is the identity operator, is called the Hilbert space effect algebra or the standard effect algebra. Elements in $\mathcal{E}(H)$ are called quantum effects. Let $\mathcal{P}(H)$ be the set of projections on H (projections are also called sharp effects). The concept of standard effect algebras plays a fundamental role in quantum measurement ([11]). For $A, B \in \mathcal{E}(H)$, A, B are coexist if there exists a $C \in \mathcal{E}(H)$ such that $A = A_1 + C$, $B = B_1 + C$ and $A_1 + B_1 + C \in \mathcal{E}(H)$. In quantum measurement theory, if two quantum effects A, B coexist, they can be measured together. An important unsolved problem

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of quantum measurement theory is to characterize coexistence of two quantum effects. Many authors pay their attentions to such a problem ([4,18] and their references). If $\dim H = 2$, Gudder [4] completely characterizes coexistence of two quantum effects by their eigenvalues and eigenvectors. However in the case of $\dim H \geq 3$, there exists no complete operational characterization of coexistence of quantum effects. In order to study coexistence more deeply, Gudder [4] introduces a set

$$\mathcal{W}(A,B) = \{C \in \mathcal{E}(H) | A + B - I \leq C \leq A, B\}.$$

$\mathcal{W}(A,B)$ is called the witness set for coexistence of A and B. The set $\mathcal{W}(A,B)$ is nonempty if and only if A, B coexist. In matrix analysis, the problem on positivity of operator matrices has been focused on by many researchers ([2, 9, 10]). Let $\mathcal{A}$ be a unital C* algebra and $A, B, C \in \mathcal{A}$. A well-known fact is that the operator matrix $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \geq 0$ if and only if $A \geq 0$, $B \geq 0$ and there is a contractive operator X such that $C = A^{\frac{1}{2}} X B^{\frac{1}{2}}$ ([2]). Hou and Gao [7] give a condition for positivity of 3×3 operator matrices. In the paper, we apply positive operator matrix techniques to study coexistence of quantum effects (see Theorem 2.1). Applying this result, we obtain more properties of coexistence of two quantum effects (see Corollary 2.3-2.6).

It is mentioned that linear preserver problems deal with the characterization of linear maps on matrix algebras with some special properties such as leaving certain functions, subsets or relations invariant ([16]). Commutativity is one of the important relations. Professors Choi, Jafarian and Radjavi [1] gave a characterization of linear maps on matrix algebras preserving commutativity, where the following conclusion is showed:

Theorem 1.1 (**Choi-Jafarian-Radjavi Theorem**)**.** *Denote by $\mathcal{H}_n$ the real linear space of all $n\times n$ Hermitian complex matrices. If $\Phi : \mathcal{H}_n \to \mathcal{H}_n$ is a linear map and preserves commutativity, i.e., $AB = BA \Rightarrow \Phi(A)\Phi(B) = \Phi(B)\Phi(A)$ for $A, B \in \mathcal{H}_n$, then either $\Phi(\mathcal{H}_n)$ is commutative, or there exist a unitary matrix U, a linear functional $f : \mathcal{H}_n \to \mathbb{R}$ and a real number t such that Φ has one of the following forms: (i) $\Phi(A) = tUAU^* + f(A)I$ for all $A \in \mathcal{H}_n$; (ii) $\Phi(A) = tUA^TU^* + f(A)I$ for all $A \in \mathcal{H}_n$, where A^T is the transpose of A.*

Recently, this result is applied to determine the structure of some quantum transformations in quantum information ([3,6]). Applying Choi-Jafarian-Radjavi theorem, Guo and Hou [6] gave a characterization of local quantum channels whose images are in the set of states with zero quantum discord. Dolinar and Molnár [3] described the structure of all continuous sequential endomorphisms of the effect algebra of a finite-dimensional Hilbert space. By the Choi-Jafarian-Radjavi theorem, without assumption of bijectivity, we characterize general multiplicative maps leaving the witness set invariant on finite-dimensional Hilbert space effect algebras. If a morphism $\Phi : \mathcal{E}(H) \to \mathcal{E}(H)$

satisfies

$$\Phi(\mathcal{W}(A,B)) = \mathcal{W}(\Phi(A),\Phi(B)) \text{ for } A,B \in \mathcal{E}(H)$$

or

$$\mathcal{W}(A,B) = \mathcal{W}(\Phi(A),\Phi(B)) \text{ for } A,B \in \mathcal{E}(H),$$

we call Φ a morphism that leaves the witness set of coexistence invariant. In the third section, we give a characterization of bijective morphisms that leave the witness set of coexistence invariant. Furthermore, applying the Choi-Jafarian-Radjavi theorem, we discuss the structure of multiplicative general morphisms leaving the witness set invariant on finite-dimensional Hilbert space effect algebras in section four.

Let us give some notations. For $A \in \mathcal{B}(H)$, $\mathrm{Ran}(A)$ and $\mathrm{Ker}(A)$ denote the range and kernel of the operator A respectively. $\overline{\mathrm{Ran}\ (\mathrm{A})}$ is the closure of $\mathrm{Ran}(A)$. An operator A on H is a contraction if $\|A\| \leq 1$. We denote by P_L the projection on the space L.

(1) if P is a sharp effect (a projection) and $E \in \mathcal{E}(H)$, P, E coexist, then $\mathcal{W}(P,E) = \{PE\}$. The reverse implication does not hold.

(2) A, B **are compatible (or coexist)** if and only if $\mathcal{W}(A,B) \neq \emptyset$.

(3) $W(E,I) = \{E\}$ and E coexists with all other effects if and only if $E = \lambda I$

(4) $A \leq B$ if and only if $A \in \mathcal{W}(A,B)$.

(5) $0 \in \mathcal{W}(A,B)$ if and only if $A \perp B$.

(6) $AB = BA$ if and only if $AB \in \mathcal{W}(A,B)$.

2. Characterizing the witness set $\mathcal{W}(A,B)$

In the following theorem, we give a complete characterization of the structure of the witness set $\mathcal{W}(A,B)$. Through this section, we always assume that H is arbitrary a complex Hilbert space.

Theorem 2.1. *Let $A, B \in \mathcal{E}(H)$ and A, B coexist, assume that the space decomposition $H = H_0 \oplus H_1 \oplus H_2 = (\mathrm{Ker}(A) \cap \overline{\mathrm{Ran}(B)}) \oplus (\overline{\mathrm{Ran}(A)} \cap \overline{\mathrm{Ran}(B)}) \oplus \mathrm{Ker}(B)$ and P, Q, R is the projection on H_0, H_1 and H_2 respectively. Then every $C \in \mathcal{W}(A,B)$ satisfies $C = QCQ, QCQ \geq QAQ + QBQ - Q$, and there exist contractions X, Y, X', Y', X'', Y'' such that*

$$QA^{\frac{1}{2}}QXRA^{\frac{1}{2}}R = Q(A-C)^{\frac{1}{2}}QX'RA^{\frac{1}{2}}R = -Q(C-A-B+I)^{\frac{1}{2}}QX''R(I-A)^{\frac{1}{2}}R,$$

$$PB^{\frac{1}{2}}PYQB^{\frac{1}{2}}Q = PB^{\frac{1}{2}}PY'Q(B-C)^{\frac{1}{2}}Q = -P(I-B)^{\frac{1}{2}}PY''Q(C-A-B+I)^{\frac{1}{2}}Q.$$

To prove Theorem 2.1, we need the following lemma, which is proved by Hou and Gao [7].

Lemma 2.2. *([9, Corollary 1.3]) The operator matrix*

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{12}^* & D_{22} & D_{23} \\ D_{13}^* & D_{23}^* & D_{33} \end{pmatrix} \in \mathcal{B}(H_0 \oplus H_1 \oplus H_2)$$

is positive if and only if $D_{ii} \geq 0$ *for* $i = 1, 2, 3$ *and there exist contractions* X, Y, W *such that* $D_{12} = D_{11}^{\frac{1}{2}} X D_{22}^{\frac{1}{2}}$, $D_{23} = D_{22}^{\frac{1}{2}} Y D_{33}^{\frac{1}{2}}$ *and*

$$D_{13} = D_{11}^{\frac{1}{2}} X P_{\mathrm{Ran}(D_{22})} Y D_{33}^{\frac{1}{2}} + (D_{11} - D_{11}^{\frac{1}{2}} X P_{\mathrm{Ran}(D_{22})} X^* D_{11}^{\frac{1}{2}})^{\frac{1}{2}} W (D_{33} - D_{33}^{\frac{1}{2}} Y^* P_{\mathrm{Ran}(D_{22})} Y D_{33}^{\frac{1}{2}})^{\frac{1}{2}}.$$

Proof of Theorem 2.1. Let $H_0 = \overline{\mathrm{Ran}(B)} \cap \mathrm{Ker}(A)$, $H_1 = \overline{\mathrm{Ran}(B)} \cap \overline{\mathrm{Ran}(A)}$ and $H_2 = \mathrm{Ker}(B)$. Since $A, B \geq 0$, on the space decomposition $H = H_0 \oplus H_1 \oplus H_2$, we have

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & E \\ 0 & E^* & A_2 \end{pmatrix}, B = \begin{pmatrix} B_0 & F & 0 \\ F^* & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 2.2, $A_i \geq 0$, $B_i \geq 0$ and there exist contractions X, Y such that $E = A_1^{\frac{1}{2}} X A_2^{\frac{1}{2}}$, $F = B_0^{\frac{1}{2}} Y B_1^{\frac{1}{2}}$. Let $C \in \mathcal{W}(A, B)$. We write

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12}^* & C_{22} & C_{23} \\ C_{13}^* & C_{23}^* & C_{33} \end{pmatrix}.$$

It follows from Lemma 2.2 again and $C \geq 0$ that $C_{ii} \geq 0$ for $i = 1, 2, 3$ and there exist contractions U, V, W such that $C_{12} = C_{11}^{\frac{1}{2}} U C_{22}^{\frac{1}{2}}$, $C_{23} = C_{22}^{\frac{1}{2}} V C_{33}^{\frac{1}{2}}$ and

$$C_{13} = C_{11}^{\frac{1}{2}} U P_{\mathrm{Ran}(C_{22})} V C_{33}^{\frac{1}{2}} + (C_{11} - C_{11}^{\frac{1}{2}} U P_{\mathrm{Ran}(C_{22})} U^* C_{11}^{\frac{1}{2}})^{\frac{1}{2}} W (C_{33} - C_{33}^{\frac{1}{2}} V^* P_{\mathrm{Ran}(C_{22})} V C_{33}^{\frac{1}{2}})^{\frac{1}{2}}.$$

Since $C \in \mathcal{W}(A, B)$, we have that $C \leq A, B$. It follows that $C_{11} = 0$, $C_{12} = 0$, $C_{13} = 0$, $C_{33} = 0$ and therefore, $C_{23} = 0$ by Lemma 2.2. This implies that $C = QCQ$.

Furthermore, we have

$$0 \leq A - C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 - C_{22} & E \\ 0 & E^* & A_2 \end{pmatrix}.$$

It follows from Lemma 2.2 that there is a contraction X' such that $E = (A_1 - C_{22})^{\frac{1}{2}} X' (A_2)^{\frac{1}{2}}$. Similarly we have from $B - C \geq 0$ that there exists a contraction Y' such that $F = (B_0)^{\frac{1}{2}} Y' (B_1 - C_{22})^{\frac{1}{2}}$. Since $C \in \mathcal{W}(A, B)$, $C \geq A + B - I$, we see that

$$0 \leq C - A - B + I = \begin{pmatrix} P - B_0 & -F & 0 \\ -F^* & C_{22} - A_1 - B_1 + Q & -E \\ 0 & -E^* & R - A_2 \end{pmatrix}.$$

Then $C_{22} \geq A_1 + B_1 - Q$, i.e., $QCQ \geq QAQ + QBQ - Q$.

Since $A, B, A-C, B-C, C-A-B+I$ are all positive operators, we have that there exist contractions X'', Y'' such that $E = -(C_{22}-A_1-B_1+Q)^{\frac{1}{2}}X''(R-A_2)^{\frac{1}{2}}$ and $F = -(P-B_0)^{\frac{1}{2}}Y''(C_{22}-A_1-B_1+Q)^{\frac{1}{2}}$. Therefore,

$$E = A_1^{\frac{1}{2}}XA_2^{\frac{1}{2}} = (A_1-C_{22})^{\frac{1}{2}}X'A_2^{\frac{1}{2}} = -(C_{22}-A_1-B_1+Q)^{\frac{1}{2}}X''(R-A_2)^{\frac{1}{2}},$$

$$F = B_0^{\frac{1}{2}}YB_1^{\frac{1}{2}} = B_0^{\frac{1}{2}}Y'(B_1-C_{22})^{\frac{1}{2}} = -(P-B_0)^{\frac{1}{2}}Y''(C_{22}-A_1-B_1+Q)^{\frac{1}{2}}.$$

This completes the proof. □

Next applying Theorem 2.1, we will explore properties of coexisting quantum effects. In the following corollary, we characterize coexistence witness sets of two orthogonal quantum effects.

Corollary 2.3. *Let $A, B \in \mathcal{E}(H)$, $AB = 0$. Then $\mathcal{W}(A, B) = \{0\}$.*

Proof. If $AB = 0$, then $\overline{\mathrm{Ran}(A)} \cap \overline{\mathrm{Ran}(B)} = \{0\}$. It follows from Theorem 2.1 that $\mathcal{W}(A, B) = \{0\}$. This completes the proof. □

In the following corollary, we characterize coexistence of sharp effects and general ones.

Corollary 2.4. *Let $A \in \mathcal{E}(H)$ and $P \in \mathcal{P}(H)$, and A, P coexist. Then $AP = PA$ and $\mathcal{W}(A, P) = \{AP\}$.*

Proof. Let $H_0 = \overline{\mathrm{Ran}(P)} \cap \mathrm{Ker}(A)$, $H_1 = \overline{\mathrm{Ran}(P)} \cap \overline{\mathrm{Ran}(A)}$ and $H_2 = \mathrm{Ker}(P)$, based on the space decomposition $H = H_0 \oplus H_1 \oplus H_2$, we have that

$$P = \begin{pmatrix} I_0 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & D \\ 0 & D^* & A_2 \end{pmatrix}.$$

Let $C \in \mathcal{W}(A, P)$, by Theorem 2.1,

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $A_1 = A_1 + I_1 - I_1 \leq C_{22} \leq A_1$. It follows that $C_{22} = A_1$. Thus,

$$\mathcal{W}(A, P) = \{T : T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\}.$$

It follows from $T \in \mathcal{W}(A, P)$ that $T \leq A$, i.e.,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & D \\ 0 & D^* & A_2 \end{pmatrix}.$$

This implies that $D = 0$. Thus, A has to be of the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix}.$$

So $AP = PA$. Therefore,

$$\mathcal{W}(A,P) = \{T : T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\} = \{PAP\} = \{AP\}.$$

This completes the proof. □

From Corollary 2.4, we have the following corollary since an effect that commutes with all sharp effects is a multiple of the identity.

Corollary 2.5. *The effect coexists with all sharp effects is a multiple of the identity.*

Corollary 2.6. *For $A \in \mathcal{E}(H)$ and $P \in \mathcal{P}(H)$ with rank 1, $\mathcal{W}(A,B) = \{\lambda_0 P\}$, where $\lambda_0 = \mathrm{tr}(AP)$.*

Proof. This follows from Corollary 2.4. □

In the following corollary, we give a necessary and sufficient condition of equivalence of quantum effects associated with their witness set of coexistence.

Corollary 2.7. *For $A, B \in \mathcal{E}(H)$ and $dimH = n < \infty$, the following are equivalent:*
(i) $A = B$.
(ii) $W(A,P) = W(B,P)$ for all rank one projections P that coexist with A.

Proof. (i)$\Rightarrow$ (ii) is obvious. Next we claim that (ii)$\Rightarrow$(i).

For $P \in \mathcal{P}(H)$, since P coexists with A, we have $PA = AP$ by Corollary 2.4. Taking unit eigenvectors x_i of A, $i = 1, 2, \ldots, n$, and $Ax_i = \lambda_i$. With respect to the orthonormal basis $\{x_i\}_{i=1}^n$, we write

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \bar{b}_{12} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{b}_{1n} & \bar{b}_{2n} & \cdots & b_{nn} \end{pmatrix}.$$

By (II), $W(A,P_i) = W(B,P_i)$ for all sharp effects $P_i = x_i \otimes x_i$. So by Corollary 2.4,

$$\{AP_i\} = W(A,P_i) = W(B,P_i) = \{BP_i\}.$$

Thus $AP_i = BP_i$ for all P_i. A short computation show that $\lambda_i = b_{ii}$ and $b_{ij} = 0$ when $i \neq j$. So $A = B$. This completes the proof. □

3. Bijective morphisms leaving the coexistence witness set invariant

In this section, we give a characterization of two kinds of morphisms on quantum effects that leave the witness set of coexistence invariant.

Theorem 3.1. *Let $\mathcal{E}(H)$ be arbitrary a Hilbert space effect algebra on a complex Hilbert space H. Then the bijective morphism $\Phi : \mathcal{E}(H) \to \mathcal{E}(H)$ satisfies*

$$\Phi(\mathcal{W}(A, B)) = \mathcal{W}(\Phi(A), \Phi(B))$$

for $A, B \in \mathcal{E}(H)$ if and only if there is a unitary or anti-unitary operator U such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{E}(H)$.

Theorem 3.2. *Let $\mathcal{E}(H)$ be a Hilbert space effect algebra on a complex Hilbert space H. Then the bijective map $\Phi : \mathcal{E}(H) \to \mathcal{E}(H)$ satisfies*

$$\mathcal{W}(A, B) = \mathcal{W}(\Phi(A), \Phi(B))$$

for $A, B \in \mathcal{E}(H)$ if and only if $\Phi(A) = A$ for every $A \in \mathcal{E}(H)$.

Proof. Checking the "if" part is straightforward, so we will only deal with the "only if" part.

Since $\Phi(\mathcal{W}(A, B)) = \mathcal{W}(\Phi(A), \Phi(B))$ for $A, B \in \mathcal{E}(H)$, we have $A \in \mathcal{W}(A, B)$ $\Leftrightarrow$ $\Phi(A) \in \mathcal{W}(\Phi(A), \Phi(B))$. It is well known that $A \in \mathcal{W}(A, B)$ if and only if $A \leq B$ by [4, Lemma 2.4]. Thus, $A \leq B$ if and only if $\Phi(A) \leq \Phi(B)$, i.e., Φ preserves the order of effects in both directions. Since $\Phi(\mathcal{W}(A, B)) = \mathcal{W}(\Phi(A), \Phi(B))$ and Φ is bijective, we have $\mathcal{W}(A, B) \neq \emptyset$ if and only if $\mathcal{W}(\Phi(A), \Phi(B)) \neq \emptyset$. If follows that A, B coexist if and only if $\Phi(A), \Phi(B)$ coexist. So the bijective morphism Φ preserves the order and coexistence of quantum effects in both directions. Such a map is characterized by L. Molnár in [12, Corollary 2.7.5], as follows: there exist a unitary or anti-unitary operator U on H such that

$$\Phi(A) = UAU^*$$

for $A \in \mathcal{E}(H)$. This completes the proof. ∎

Proof of Theorem 3.2. It is easy to check the "if" part, so we still only deal with the "only if" part.

Since $\mathcal{W}(A, B) = \mathcal{W}(\Phi(A), \Phi(B))$, so A, B coexist if and only if $\Phi(A), \Phi(B)$ coexist. As we know the effect A coexists with all other effects if and only if $A = \mu I$ for some positive scalar $\mu \leq 1$. Since I coexists with all other effects and Φ is bijective and preserves coexistence of effects in both directions, it follows that $\Phi(I) = \lambda I$.

Next we show $\lambda = 1$. Indeed, in $\mathcal{W}(A, B) = \mathcal{W}(\Phi(A), \Phi(B))$, taking $A = B = I$, we have that

$$\{I\} = \mathcal{W}(I, I) = W(\Phi(I), \Phi(I)) = W(\lambda I, \lambda I) = [(2\lambda - 1)I, \lambda I].$$

This implies that $\lambda = 1$. Taking $B = I$ only, we have from Corollary 2.4 that

$$\{A\} = \mathcal{W}(A, I) = W(\Phi(A), \Phi(I)) = W(\Phi(A), I) = \{\Phi(A)\}.$$

So $\Phi(A) = A$. We complete the proof. □

4. Multiplicative general morphisms preserving the coexistence witness set

In the section, without assumption of bijectivity, we discuss the classification of multiplicative general morphisms leaving the witness set invariant on finite dimensional Hilbert space effect algebras. Let $\mathcal{E}^{-1}(H)$ be the set of all invertible effects.

Theorem 4.1. *Let $\mathcal{E}(H)$ be a Hilbert space effect algebra on a complex Hilbert space H and $\dim H = n < \infty$. If a multiplicative map $\Phi : \mathcal{E}(H) \to \mathcal{E}(H)$ satisfies that $\Phi(\mathbb{C}I\backslash\{0\}) \subseteq \mathbb{C}I\backslash\{0\}$ and*

$$\Phi(\mathcal{W}(A,B)) \subseteq \mathcal{W}(\Phi(A),\Phi(B))$$

for $A, B \in \mathcal{E}(H)$, then one of the following statements holds:

(I) there exists a unitary or an anti-unitary operator U on H and a nonzero real constant d such that $\Phi(A) = \exp(\mathrm{tr}(S\ln A))UA^dU^*$ for all invertible $A \in \mathcal{E}(H)$;

(II) $\Phi(\mathcal{E}^{-1}(H))$ is commutative.

Proof. For $A, B \in \mathcal{E}(H)$, if $AB = BA$, then $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$ since Φ is multiplicative. So Φ preserves commutativity. Furthermore, if $A \leq B$, by [4, Lemma 2.4 (a)], $A \in \mathcal{W}(A,B)$, so $\Phi(A) \in \mathcal{W}(\Phi(A),\Phi(B))$. By [4, Lemma 2.4 (d)] again, $\Phi(A) \leq \Phi(B)$. Next we claim that Φ maps projections to projections. For $P \in \mathcal{P}(H)$, $P^2 = P$, and hence $\Phi(P)^2 = \Phi(P)$. So $\Phi(P) \in \mathcal{P}(H)$. So $\Phi(I) = I$ since $\Phi(\mathbb{C}I\backslash\{0\}) \subseteq \mathbb{C}I\backslash\{0\}$. Finally, we show that Φ maps invertible effects to themselves. If A is invertible, then there is a nonzero $\lambda_0 \leq 1$ such that $\lambda_0 I \leq A$, so $\Phi(\lambda_0 I) \leq \Phi(A)$. Since $\Phi(\mathbb{C}I\backslash\{0\}) \subseteq \mathbb{C}I\backslash\{0\}$, $\Phi(A) \geq \mu_0 I$ for some scalar μ_0. Thus, $\Phi(A)$ is invertible.

Now similar to the proof of [3, Proposition 2], define a transformation Λ on the cone $\mathcal{B}^+(H)$ of all the positive semi-definite operators on H by

$$\Lambda(D) = -\ln\Phi(\exp(-D)), (D \in \mathcal{B}^+(H)).$$

We can extend Λ to a linear map $\widetilde{\Psi}$ on the space $\mathcal{S}_a(H)$ of all self-adjoint operators on H by a standard process (see the [3, proof of Theorem 2]) and Ψ preserves commutativity. By [1, Theorem 2], there are two cases to consider.
(a) the range of Ψ is commutative;
(b) Ψ is of the standard form $\Psi(T) = dUTU^* + g(T)$ for all $T \in \mathcal{S}_a(H)$, where d is a nonzero real number, U is a unitary or an anti-unitary operator on H, and g is a linear functional on $\mathcal{S}_a(H)$.

In case (a), since the range of Ψ is commutative, $\Phi(\mathcal{E}^{-1}(H))$ is commutative. If (b) occurs, there is $S \in \mathcal{S}_a(H)$ such that $g(T) = \mathrm{tr}(ST)$ for all $T \in \mathcal{S}_a(H)$. Thus,

$$\Phi(A) = \exp(\mathrm{tr}(S\ln A))UA^dU^* \text{ for invertible } A \in \mathcal{E}^{-1}(H).$$

This completes the proof. □

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