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THE JACOBSTHAL SEQUENCES IN FINITE GROUPS

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Abstract. In this paper, we study the generalized order-k Jacobsthal sequences modulo m for $k \geq 2$ and the generalized order-k Jacobsthal-Padovan sequence modulo m for $k \geq 3$. Also, we define the generalized order-k Jacobsthal orbit of a k-generator group for $k \geq 2$ and the generalized order-k Jacobsthal-Padovan orbit a k-generator group for $k \geq 3$. Furthermore, we obtain the lengths of the periods of the generalized order-3 Jacobsthal orbit and the generalized order-3 Jacobsthal-Padovan orbit of the direct product $D_{2n} \times \mathbb{Z}_{2m}$, $(n,\ m \geq 3)$ and the semidirect product $D_{2n} \times_{\varphi} \mathbb{Z}_{2m}$, $(n,\ m \geq 3)$.
Keywords: Length, Jacobsthal sequence, finite group.
MSC(2010): Primary: 11B50; Secondary: 11C20, 20F05, 20D60.

1. Introduction

It is known that the Jacobsthal sequence $\{J_n\}$ is defined recursively by the equation

$$J_n = J_{n-1} + 2J_{n-2} \tag{1.1}$$

for $n \geq 2$, where $J_0 = 0$ and $J_1 = 1$.
In [10], Koken and Bozkurt showed that the Jacobsthal numbers are also generated by a matrix

$$F=\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, F^n=\begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}.$$

Kalman [8] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

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where $c_0, c_1, \cdots, c_{k-1}$ are real constants. In [8], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In [13], Yilmaz and Bozkurt defined the k sequences of the generalized order-k Jacobsthal numbers as follows:
for $n > 0$ and $1 \leq \mathrm{i} \leq k$

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + ... + J_{n-k}^i, \tag{1.2}$$

with initial conditions

$$J_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where J_n^i is the nth term of the ith sequence. If $k = 2$ and $i = 1$, the generalized order-k Jacobsthal sequence is reduced to the conventional Jacobsthal sequence. In [13], the generalized order-k Jacobsthal matrix C had been given as:

$$C = \begin{bmatrix} 1 & 2 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.3}$$

Also, it was proved that $B_n = C \cdot B_{n-1}$ where

$$B_n = \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^k \\ \vdots & \vdots & & \vdots \\ J_{n-k+1}^1 & J_{n-k+1}^2 & \cdots & J_{n-k+1}^k \end{bmatrix}. \tag{1.4}$$

Lemma 1.1. *([13]) Let C and B_n be as is (1.3) and (1.4), respectively. Then, for all integers $n \geq 0$*

$$B_n = C^n.$$

In [3], Deveci defined the Jacobsthal-Padovan sequence $\{J(n)\}$ as follows:

$$J(n+2) = J(n) + 2J(n-1) \tag{1.5}$$

for $n \geq 0$, where $J(-1) = 0$ and $J(0) = J(1) = 1$.
In [3], the Jacobsthal-Padovan matrix G had been given as:

$$G = [g_{ij}]_{3\times 3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}. \tag{1.6}$$

Definition 1.2. ([3]) For a generating pair $(x, y) \in G$, we define the Jacobsthal-Padovan orbit $J_{x,y,y}(G) = \{x_i\}$ by

$$x_0 = x,\ x_1 = y,\ x_2 = y,\ x_{i+2} = (x_{i-1})^2 \cdot (x_i),\ \ i \geq 1.$$

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6.

Theorem 1.3. *([3])A Jacobsthal-Padovan orbit of a finite group is periodic.*

Many references may be given for Fibonacci sequence and k-step Fibonacci (k-nacci) sequence in groups and related issues; see for example, [1, 4, 5, 9, 11, 12, 14]. Deveci [3] expanded the theory to the Pell-Padovan sequence and the Jacobsthal-Padovan sequence. Now we extend the concept to the generalized order-k Jacobsthal sequence and the generalized order-k Jacobsthal-Padovan sequence.
In this paper, the usual notation p is used for a prime number.

2. The generalized order-k Jacobsthal sequences modulo m and the generalized order-k Jacobsthal-Padovan sequences modulo m

Now we define a new sequence called The generalized order-k ($k \geq 3$) Jacobsthal-Padovan sequence $\left\{JP^k(n)\right\}$, defined by

$$JP^k(n+k) = JP^k(n+k-2) + 2JP^k(n+k-3) + \cdots + JP^k(n-1) \tag{2.1}$$

for $n \geq 0$, where $J(i) = 0$ for $-1 \leq i \leq k-3$ and $J(k-2) = J(k-1) = 1$.

By (2.1), we can write

$$\begin{bmatrix} JP^k(n) \\ JP^k(n+1) \\ JP^k(n+2) \\ \vdots \\ JP^k(n+k-1) \\ JP^k(n+k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} JP^k(n-1) \\ JP^k(n) \\ JP^k(n+1) \\ \vdots \\ JP^k(n+k-2) \\ JP^k(n+k-1) \end{bmatrix}$$

for the Jacobsthal-Padovan sequence. Let

$$E = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 2 & 1 & 0 \end{bmatrix}.$$

The matrix G is said to be generalized order-k Jacobsthal-Padovan matrix. Reducing the generalized order-k Jacobsthal sequence $(k \geq 2)$ and the generalized order-k $(k \geq 3)$ Jacobsthal-Padovan sequence by a modulus m, we can get the repeating sequences, respectively denoted by

$$\left\{ J_n^{k,m} \right\} = \left\{ J_{1-k}^{k,m},\ J_{2-k}^{k,m},\ \cdots,\ J_0^{k,m},\ J_1^{k,m},\ \cdots,\ J_i^{k,m},\ \cdots, \right\}$$

and

$$\begin{aligned} \left\{ JP^{k,m}(n) \right\} = &\left\{ JP^{k,m}(-1), JP^{k,m}(0), \cdots, JP^{k,m}(k-2), \right. \\ &\left. JP^{k,m}(k-1), \cdots, JP^{k,m}(i), \cdots, \right\} \end{aligned}$$

where $J_i^{k,m} \equiv J_i^k \pmod m$ and $JP^{k,m}(i) \equiv JP^k(i) \pmod m$. They have the same recurrences relation as in (1.2) and (2.1), respectively.

Theorem 2.1. *[3] The sequence* $\left\{ J^{(m)}(n) \right\}$ *is simply periodic if* m *is odd, and it is periodic if* m *is even.*

Theorem 2.2. *The sequences* $\left\{ J_n^{k,m} \right\}$ $(k \geq 2)$ *and* $\left\{ JP^{k,m}(n) \right\}$ $(k \geq 3)$ *are periodic.*

Proof. Let us consider the sequence $\left\{ J_n^{k,m} \right\}$ and put

$$U_k = \left\{ (x_1, x_2, \cdots, x_k) \middle|\, 0 \leq x_i \leq m-1 \right\}.$$

Then we have $|U_k| = m^k$ which is finite, that is, for any $a \geq 0$, there exists $b \geq a$ such that $J_{a+1}^{k,m} = J_{b+1}^{k,m}, \cdots, J_{a+k}^{k,m} = J_{b+k}^{k,m}$, respectively.
The proof for the sequence $\left\{ JP^{k,m}(n) \right\}$ $(k \geq 3)$ is similar to the above and is omitted.
Let $hJ^{k,m}$ and $hJP^{k,m}$ denote the smallest periods of $\left\{ J_n^{k,m} \right\}$ $(k \geq 2)$ and $\left\{ JP^{k,m}(n) \right\}$ $(k \geq 3)$. □

Example 2.3. We have $\left\{J_n^{3,3}\right\} = \{1, 0, 0, 1, 1, 0, 0, 1, \cdots\}$, and then repeat. So, we get $hJ^{3,3} = 4$.

Example 2.4. We have $\left\{JP^{3,2}(n)\right\} = \{0, 0, 1, 1, 1, 1, 0, 0, 1, 1, \cdots\}$, and then repeat. So, we get $hJP^{3,2} = 6$.

Given an integer matrix $A = (a_{ij})$, $A \pmod m$ means that all entries of A are modulo m, that is, $A\ (mod\ m) = (a_{ij}\ (mod\ m))$. Let $\langle C\rangle_{p^a} = \left\{C^i \pmod{p^a} \middle| i \geq 0\right\}$ and $\langle E\rangle_{p^a} = \left\{E^i \pmod{p^a} \middle| i \geq 0\right\}$ be cyclic groups for $p \neq 2$ and let $\left|\langle C\rangle_{p^a}\right|$ and $\left|\langle E\rangle_{p^a}\right|$denote the orders of $\langle C\rangle_{p^a}$ and $\langle E\rangle_{p^a}$, respectively.

Theorem 2.5. *If $p \neq 2$, then $hJ^{k,p^a} = \left|\langle C\rangle_{p^a}\right|$ and $hJP^{k,p^a} = \left|\langle E\rangle_{p^a}\right|$.*

Proof. Firstly, let us consider the case $hJ^{k,p^a} = \left|\langle C\rangle_{p^a}\right|$. It is clear that $\left|\langle C\rangle_{p^\alpha}\right|$ is divisible by hJ^{k,p^a}. Then we need only to prove that hJ^{k,p^a} is divisible by $\left|\langle C\rangle_{p^\alpha}\right|$. Let $hJ^{k,p^a} = n$. We have already seen that $B_n = C \cdot B_{n-1}$ and $B_n = C^n$ [13]. Since $B_n \equiv I \pmod{p^\alpha}$, where I is the identity matrix, we get that $C^{n+1} \equiv C \pmod{p^\alpha}$. Therefore, $C^n \equiv I \pmod{p^\alpha}$, which yields that n is divisible by $\left|\langle C\rangle_{p^\alpha}\right|$.

Secondly, let us consider the case $hJP^{k,p^a} = \left|\langle E\rangle_{p^a}\right|$. It is clear that $\left|\langle E\rangle_{p^\alpha}\right|$ is divisible by hJP^{k,p^a}. Then we need only to prove that hJP^{k,p^a} is divisible by $\left|\langle E\rangle_{p^\alpha}\right|$. Let $hJP^{k,p^a} = n$. Thus

$$E^n = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1k+1} \\ m_{21} & m_{22} & \cdots & m_{2k+1} \\ \vdots & \vdots & & \vdots \\ m_{k+11} & m_{k+12} & \cdots & m_{k+1k+1} \end{bmatrix}.$$

The elements of the matrix E^n are in the following forms:

$$\begin{aligned} m_{12} &= JP^k(n-k+1),\ m_{22} = JP^k(n-k+2),\ \cdots, \\ m_{k2} &= JP^k(n),\ m_{k+12} = JP^k(n+1), \end{aligned}$$

$$\begin{aligned} m_{11} + m_{21} &= JP^k(n-k+2),\ m_{21} + m_{31} = JP^k(n-k+3),\ \cdots, \\ m_{k1} + m_{k+11} &= JP^k(n+1), \end{aligned}$$

$$m_{ii} = \beta_1 JP^k(n-1) + \beta_2 JP^k(n) + \cdots + \beta_k JP^k(n+k-2) + 1$$

for $1 \leq i \leq k+1$ and $\beta_1,\ \beta_2,\ \cdots,\ \beta_k \geq 0$
and

$$m_{ij} = \eta_1 JP^k(n-1) + \eta_2 JP^k(n) + \cdots + \eta_k JP^k(n+k-2)$$

for $i \neq j$, $1 \leq i, j \leq k+1$ and $\eta_1,\ \eta_2,\ \cdots,\ \eta_k \geq 0$.

We thus obtain that
$m_{ii} \equiv 1 \pmod{p^a}$ for $1 \leq i, j \leq k+1$
and
$m_{ij} \equiv 0 \pmod{p^a}$ for $1 \leq i, j \leq k+1$ such that $i \neq j$.
So we get that $E^n \equiv I \pmod{p^a}$, which yields that n is divisible by $\left|\langle E \rangle_{p^a}\right|$. □

Theorem 2.6. *Let $p \neq 2$ and let t be the largest positive integer such that $hJ^{k,p} = hP^{k,p^t}$. Then $hJ^{k,p^\alpha} = p^{\alpha - t} \cdot hJ^p$ for every $\alpha \geq t$.*

Proof. Let q be a positive integer. Since $C^{hJ^{k,p^{q+1}}} \equiv I \left(\bmod p^{q+1}\right)$, that is, $C^{hJ^{k,p^{q+1}}} \equiv I \pmod{p^q}$, we get that hJ^{k,p^q} divides $hJ^{k,p^{q+1}}$. On the other hand, writing $C^{hJ^{k,p^q}} = I + \left(a_{ij}^{(q)} \cdot p^q\right)$, we have

$$C^{hJ^{k,p^q} \cdot p} = \left(I + \left(a_{ij}^{(q)} \cdot p^q\right)\right)^p = \sum_{i=0}^{p} \begin{pmatrix} p \\ i \end{pmatrix} \left(a_{ij}^{(q)} \cdot p^q\right)^i \equiv I \left(\bmod p^{q+1}\right),$$

which yields that $hJ^{k,p^{q+1}}$ divides $hJ^{k,p^q} \cdot p$. Therefore, $hJ^{k,p^{q+1}} = hJ^{k,p^q}$ or $hJ^{k,p^{q+1}} = hJ^{k,p^q} \cdot p$, and the latter holds if, and only if, there is a $a_{ij}^{(q)}$ which is not divisible by p. Since $hJ^{k,p^t} \neq hJ^{k,p^{t+1}}$, there is an $a_{ij}^{(t+1)}$ which is not divisible by p, thus, $hJ^{k,p^{t+1}} \neq hJ^{k,p^{t+2}}$. The proof is finished by induction on t . □

Theorem 2.7. *Let $p \neq 2$ and let t be the largest positive integer such that $hJP^{k,p} = hJP^{k,p^t}$. Then $hJP^{k,p^\alpha} = p^{\alpha-t} \cdot hJP^{k,p}$ for every $\alpha \geq t$.*

Proof. The proof is smilar to the above and is omitted. □

Theorem 2.8. *If $m = \prod_{i=1}^{t} p_i^{e_i}, (t \geq 1)$ where p_i's are distinct primes, then $hJ^{k,m} =$lcm$\left[hJ^{k,p_i^{e_i}}\right]$ (where the least common multiple of $hJ^{k,p_1^{e_1}}, hJ^{k,p_2^{e_2}}, \cdots, hJ^{k,p_t^{e_t}}$ is denoted by lcm$\left[hJ^{k,p_i^{e_i}}\right]$) and $hJP^{k,m} =$lcm $\left[hJP^{k,p_i^{e_i}}\right]$.*

Proof. Let us consider the case $hJ^{k,m} =$lcm $\left[hJ^{k,p_i^{e_i}}\right]$. The statement, "$hJ^{k,p_i^{e_i}}$ is the length of the period of $\left\{J_n^{k,p_i^{e_i}}\right\}$," implies that the sequence $\left\{J_n^{k,p_i^{e_i}}\right\}$ repeats only after blocks of length $u \cdot hJ^{k,p_i^{e_i}}$, $(u \in \mathbb{N})$; and the statement, "$hJ^{k,m}$ is the length of the period $\left\{J_n^{k,m}\right\}$," implies that $\left\{J_n^{k,p_i^{e_i}}\right\}$ repeats after $hJ^{k,m}$ terms for all values i. Thus, $hJ^{k,m}$ is of the form $u \cdot hJ^{k,p_i^{e_i}}$ for all values of i, and since any such number gives a period of $\left\{J_n^{k,m}\right\}$. Then we get that $hJ^{k,m} =$lcm $\left[hJ^{k,p_i^{e_i}}\right]$.

The proof of the case $hJP^{k,m}$ =lcm$\left[hJP^{k,p_i^{e_i}}\right]$ is similar to the above and is omitted. □

3. The generalized order-k Jacobsthal sequences and the generalized order-k Jacobsthal-Padovan sequences in finite groups

Definition 3.1. For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_k\}$ we define the generalized order-k Jacobsthal orbit $J_A^k(G)$ with respect to the generating set A to be the sequence $\{x_i\}$ of the elements of G such that

$$x_i = a_{i+1} \text{ for } 0 \leq i \leq k-1,\ x_{i+k} = \begin{cases} (x_i)^2 (x_{i+1}), & k = 2, \\ (x_i) \cdots (x_{i+k-2})^2 (x_{i+k-1}), & k \geq 3 \end{cases}$$

for $i \geq 0$.

Definition 3.2. For a finitely generated group $G = \langle A \rangle$, where

$$A = \{a_1,\ a_2,\ \cdots,\ a_k\}\,(k \geq 3)$$

we define the generalized order-k Jacobsthal-Padovan orbit $JP_A^k(G)$ with respect to the generating set A to be the sequence $\{x_i\}$ of the elements of G such that

$$x_0 = a_1,\ x_1 = a_2,\ \cdots,\ x_{k-1} = a_k,\ x_k = a_k,$$

$x_{i+k+1} = (x_i)(x_{i+1}) \cdots (x_{i+k-2})^2 (x_{i+k-1})$ for $i \geq 0$.

Theorem 3.3. *A generalized order-k Jacobsthal orbit and a generalized order-k Jacobsthal-Padovan orbit of a finite group are periodic.*

Proof. Let us consider the generalized order-k Jacobsthal orbit and let n be the order of G. Since there are n^k distinct k-tuples of elements of G, at least one of the k-tuples appears twice in a generalized order-k Jacobsthal orbit of G. Thus, the subsequence following this k-tuples. Because of the repeating, the generalized order-k Jacobsthal orbit is periodic.
The proof for a generalized order-k Jacobsthal-Padovan orbit of a finite group is similar to the above and is omitted. □

We denote the lengths of the periods of the generalized order-k Jacobsthal orbit $J_A^k(G)$ and the generalized order-k Jacobsthal-Padovan orbit $JP_A^k(G)$ by $LJ_A^k(G)$ and $LJP_A^k(G)$, respectively, respectively and we call them the generalized order-k Jacobsthal length and the generalized order-k Jacobsthal-Padovan length of G, recpectively.
From the definitions it is clear that the generalized order-k Jacobsthal length and the generalized order-k Jacobsthal-Padovan length of a group depend on the chosen generating set and the order in which the assignments of $x_0, x_1, \ldots, x_k$ are made.
We will now address the generalized order-k Jacobsthal lengths and the generalized order-k Jacobsthal-Padovan lengths of specific classes of finite groups.

We use the natural generating set for D_{2n}, as in [2], defined as satisfying $D_{2n} = \left\langle x, y : x^2 = y^n = (xy)^2 = e \right\rangle$. This is extended to direct product by using the following well known method of construction:
If $G_1 = \langle A : R_1 \rangle$ and $G_2 = \langle B : R_2 \rangle$, then $G_1 \times G_2 = \langle A, B : R_1,\ R_2,\ [A, B] \rangle$ where $[A, B] = \{[a, b] : a \in A,\ b \in B\}$, see [7].
The direct product $D_{2n} \times \mathbb{Z}_{2m}$, $(n,\ m \geq 3)$ is defined by the presentation

$$D_{2n} \times \mathbb{Z}_{2m} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2m} = [x, z] = [y, z] = e \right\rangle.$$

The usual notation $G_1 \times_\varphi G_2$ is used for the semidirect product of the group G_1 by G_2, where $\varphi : G_2 \to \operatorname{Aut}(G_1)$ is a homomorphism such that $b\varphi = \varphi_b$ where $\varphi_b : G_1 \to G_1$ is an element $\operatorname{Aut}(G_1)$.
The semidirect product $D_{2n} \times_\varphi \mathbb{Z}_{2m}$, $(n,\ m \geq 3)$ is defined by the presentation

$$D_{2n} \times_\varphi \mathbb{Z}_{2m} = \left\langle x, y, z : x^2 = y^n = (xy)^2 = z^{2m} = e,\ z^{-1}xzx = e,\ z^{-1}yzy = e \right\rangle,$$

where if $\mathbb{Z}_{2m} = \langle z \rangle$, then $\varphi : \mathbb{Z}_{2m} \to \operatorname{Aut}(D_{2n})$ is a homomorphism such that $z\varphi = \varphi_z$; $\varphi_z : D_{2n} \to D_{2n}$ is defined by $x\varphi_z = x$ and $y\varphi_z = y^{-1}$.
For more information see [6].

Theorem 3.4. *$LJ^3_{(x,y,z)}(D_{2n} \times \mathbb{Z}_{2m}) = hJ^{3,2m}$.*
ii)

Proof. The orbit $J^3_{(x,y,z)}(D_{2n} \times \mathbb{Z}_{2m})$ is

$$x,\ y,\ z,\ xy^2z,\ xyz^3,\ xyz^6,\ y^{-1}z^{13},\ xz^{27},\ yz^{59},\ z^{126}, \ldots.$$

Using the above information, the orbit $J^3_{(x,y,z)}(D_{2n} \times \mathbb{Z}_{2m})$ becomes:

$$\begin{array}{l} x_0 = x,\ x_1 = y,\ x_2 = z,\ \ldots, \\ x_7 = xz^{J^3_6},\ x_8 = yz^{J^3_7},\ x_9 = z^{J^3_8},\ \ldots, \\ x_{14} = xz^{J^3_{13}},\ x_{15} = yz^{J^3_{14}},\ x_{16} = z^{J^3_{15}},\ \ldots, \\ x_{7.i} = xz^{J^3_{7.i-1}},\ x_{7.i+1} = yz^{J^3_{7.i}},\ x_{7.i+2} = z^{J^3_{7.i+1}},\ \ldots. \end{array}$$

The smallest non-trivial integer satisfiying the above conditions occurs when the period is $\operatorname{lcm}\left[7,\ hJ^{3,2m}\right] = hJ^{3,2m}$.

□

Theorem 3.5. *$LJP^3_{(x,y,z)}(D_{2n} \times \mathbb{Z}_{2m})$=lcm$\left[12,\ hJP^{3,2m}\right]$.*

Proof. The orbit $JP^3_{(x,y,z)}(D_{2n} \times \mathbb{Z}_{2m})$ is

$$\begin{array}{l} x,\ y,\ z, z,\ xy^2z,\ yz^3,\ xy^2z^4,\ yz^6,\ y^{-2}z^{11}, \\ y^2z^{17},\ xy^2z^{27}, y^{-1}z^{45}, xz^{72}, yz^{116}, z^{189}, z^{305}, \ldots. \end{array}$$

Using the above information, the orbit $JP^3_{(x,y,z)}(D_{2n} \times \mathbb{Z}_{2m})$ becomes:

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = z,\ x_3 = z,\ \ldots,\\
&x_{12} = xz^{JP^3(11)},\ x_{13} = yz^{JP^3(12)},\ x_{14} = z^{JP^3(13)},\ x_{15} = z^{JP^3(14)},\ \ldots,\\
&x_{24} = xz^{JP^3(23)},\ x_{25} = yz^{JP^3(24)},\ x_{26} = z^{JP^3(25)},\ x_{27} = z^{JP^3(26)},\ \ldots,\\
&x_{12.i} = xz^{JP^3(12.i-1)},\ x_{12.i+1} = yz^{JP^3(12.i)},\\
&x_{12.i+2} = z^{JP^3(12.i+1)},\ x_{12.i+3} = z^{JP^3(12.i+2)},\ \cdots.
\end{aligned}$$

The smallest non-trivial integer satisfying the above conditions occurs when the period is $\operatorname{lcm}\left[12,\ hJP^{3,2m}\right]$. □

Theorem 3.6. $LJ^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m}) = \begin{cases} \operatorname{lcm}\left[7.\frac{n}{4}, hJ^{3,2m}\right] & \text{if } n \equiv 0\,(mod\ 4)\,, \\ \operatorname{lcm}\left[7.\frac{n}{2}, hJ^{3,2m}\right] & \text{if } n \equiv 2\,(mod\ 4)\,, \\ \operatorname{lcm}\left[7.n, hJ^{3,2m}\right] & \text{if } \quad \text{Otherwise.} \end{cases}$

.

Proof. The orbit $J^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m})$ is

$$x,\ y,\ z,\ xy^2z,\ z^3yx,\ z^6y^5x,\ z^{13}y^{-1},\ z^{28}x,\ z^{60}y^5,\ z^{129}y^4, \cdots.$$

Using the above information, the orbit $JP^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m})$ becomes:

$$\begin{aligned}
&x_0 = x,\ x_1 = y,\ x_2 = z, \ldots,\\
&x_7 = z^{J^3_6}x,\ x_8 = z^{J^3_7}y^5,\ x_{14} = z^{J^3_8}y^4, \ldots,\\
&x_{14} = z^{J^3_{13}}x,\ x_{15} = z^{J^3_{14}}y^9,\ x_{16} = z^{J^3_{15}}y^8, \ldots,\\
&x_{7.i} = z^{J^3_{7.i-1}}x,\ x_{7.i+1} = z^{J^3_{7.i}}y^{4.i+1},\ x_{7.i+2} = z^{J^3_{7.i+1}}y^{4.i}, \ldots.
\end{aligned}$$

So we need an i such that $4.i = n.u$ for $u \in \mathbb{N}$ and $J^3_{7.i-1} \equiv 0\,(mod\ 2m)\,,$
$J^3_{7.i} \equiv 0\,(mod\ 2m)$ and $J^3_{7.i+1} \equiv 1\,(mod\ 2m)$.
If $n \equiv 0 \bmod 4$, $i = \frac{n}{4}$. Thus, $LJ^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m}) = \operatorname{lcm}\left[7.\frac{n}{4}, hJ^{3,2m}\right]$.
If $n \equiv 2 \bmod 4$, $i = \frac{n}{2}$. Thus, $LJ^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m}) = \operatorname{lcm}\left[7.\frac{n}{2}, hJ^{3,2m}\right]$.
If $n \equiv 1 \bmod 4$ or $n \equiv 3 \bmod 4$, $i = n$. Thus,

$$LJ^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m}) = \operatorname{lcm}\left[7.n, hJ^{3,2m}\right].$$

□

Theorem 3.7. $LJP^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m}) = \begin{cases} \operatorname{lcm}\left[3n, hJP^{3,2m}\right] & \text{if } n \text{ is even}, \\ \operatorname{lcm}\left[6n, hJP^{3,2m}\right] & \text{if } n \text{ is odd}. \end{cases}$

Proof. The orbit $JP^3_{(x,y,z)}(D_{2n} \times_\varphi \mathbb{Z}_{2m})$ is

$$x,\ y,\ z,\ z,\ xy^2z,\ yz^3,\ z^4y^2x,\ z^6x,\ z^{11},\ z^{17}y^2, \cdots.$$

Using the above information, the orbit $JP^3_{(x,y,z)}(D_{2n}\times_\varphi \mathbb{Z}_{2m})$ becomes:

$$\begin{aligned}
&x_0=x,\ x_1=y,\ x_2=z,\ x_3=z,\dots,\\
&x_6=z^{JP^3(5)}y^2x, x_7=z^{JP^3(6)}y^3, x_8=z^{JP^3(7)}, x_9=z^{JP^3(8)}y^2,\dots,\\
&x_{12}=z^{JP^3(11)}y^4x, x_{13}=z^{JP^3(12)}y^5,\\
&x_{14}=z^{JP^3(13)}, x_{15}=z^{JP^3(14)}y^4,\cdots,\\
&x_{6.i}=z^{JP^3(6.i-1)}y^{2.i}x, x_{6.i+1}=z^{JP^3(6.i)}y^{2.i+1},\\
&x_{6.i+2}=z^{JP^3(6.i+1)}, x_{6.i+3}=z^{JP^3(6.i+2)}y^{2.i},\cdots.
\end{aligned}$$

So we need an i such that $2.i=n.v$ for $v\in\mathbb{N}$ and $JP^3(6.i-1)\equiv 0\,(mod\,2m)$, $JP^3(6.i)\equiv 0\,(mod\,2m)$, $JP^3(6.i+1)\equiv 1\,(mod\,2m)$ and $JP^3(6.i+2)\equiv 1\,(mod\,2m)$.
If n is even, $i=\frac{n}{2}$. Thus, $LJP^3_{(x,y,z)}(D_{2n}\times_\varphi \mathbb{Z}_{2m})=\operatorname{lcm}\left[6.\frac{n}{2},hJ^{3,2m}\right]=\operatorname{lcm}\left[3n,hJ^{3,2m}\right]$.
If n is odd, $i=n$. Thus, $LJP^3_{(x,y,z)}(D_{2n}\times_\varphi \mathbb{Z}_{2m})=\operatorname{lcm}\left[6n,hJ^{3,2m}\right]$. □

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