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A GENERALIZATION OF $\oplus$-COFINITELY SUPPLEMENTED MODULES

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ABSTRACT. We say that a module M is a *cms-module* if, for every cofinite submodule N of M, there exist submodules K and $K^{'}$ of M such that K is a supplement of N, and K, $K^{'}$ are mutual supplements in M. In this article, the various properties of cms-modules are given as a generalization of $\oplus$-cofinitely supplemented modules. In particular, we prove that a π-projective module M is a cms-module if and only if M is $\oplus$-cofinitely supplemented. Finally, we show that every free R-module is a cms-module if and only if R is semiperfect.
Keywords: Supplements, cofinite submodule, ($\oplus$-)cofinitely supplemented module.
MSC(2010): Primary: 16D10; Secondary: 16N80.

1. Introduction

Throughout this paper, it is assumed that R is an associative ring with identity and all modules are unital right R-modules. Let R be such a ring and let M be an R-module. The notation $K \subseteq M$ ($K \subset M$) means that K is a (proper) submodule of M. A submodule N of M is called *cofinite* in M if the factor module $\frac{M}{N}$ is finitely generated. A module M is called *extending* if every submodule is essential in a direct summand of M [3]. Here a submodule $K \leq M$ is said to be *essential* in M, denoted as $K \trianglelefteq M$, if $K \cap N \neq 0$ for every non-zero submodule $N \leq M$. Dually a proper submodule S of M is called *small (in M)*, denoted as $S << M$, if $M \neq S + L$ for every proper submodule L of M [12]. The Jacobson radical of M will be denoted by $Rad(M)$. It is known that $Rad(M)$ is the sum of all small submodules of M.

A non-zero module M is said to be *hollow* if every proper submodule of M is small in M, and it is said to be *local* if it is hollow and is finitely generated. A module M is local if and only if it is finitely generated and $Rad(M)$ is maximal

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(see [3, 2.12 §2.15]). A ring R is said to be *local* if J is maximal, where J is the Jacobson radical of R.

An R-module M is called *supplemented* if every submodule of M has a supplement in M. Here a submodule $K \subseteq M$ is said to be a *supplement* of N in M if K is minimal with respect to $N + K = M$, or equivalently, $N + K = M$ and $N \cap K \ll K$ [12]. A supplement submodule X of M is then defined when X is a supplement of some submodule of M. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented. For a module M, two submodules N and K of M are called *mutual supplements* if, $M = N + K$, $N \cap K \ll K$ and $N \cap K \ll N$ [3]. Alizade et al. [1] have defined cofinitely supplemented modules as a proper generalization of supplemented modules. They call a module M *cofinitely supplemented* if every cofinite submodule N of M has a *supplement* in M, and give characterizations of these modules over any ring and commutative domain (see [1]).

A module M is called *lifting* (or D_1*-module*) if, for every submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} << \frac{M}{K}$. Mohamed and Müller has generalized the concept of lifting modules to $\oplus$-supplemented modules. M is called $\oplus$*-supplemented* if every submodule N of M has a supplement that is a direct summand of M [7]. Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [7, Lemma A.4 (2)]). It is shown in [7, Proposition A.7 and Proposition A.8] that if R is a Dedekind domain, every supplemented R-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented.

In [4], Çalışıcı and Pancar call a module M $\oplus$*-cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. They gave in the same paper some properties of these module. In particular, it is shown in [4, Theorem 2.9] that every free R-module is $\oplus$-cofinitely supplemented if and only if R is semiperfect. Now we generalize these modules, and so we define cms-modules.

In this paper, we provide the some properties of cms-modules. Some examples are given to separate cms-modules and $\oplus$-cofinitely supplemented modules. We prove that a π-projective module M is a cms-module if and only if M is $\oplus$-cofinitely supplemented. In Proposition 2.5, we show that if M is cofinitely supplemented and f-supplemented, then it is a cms-module. We obtain a new characterization of semiperfect rings by using this result. We give some conditions for factor modules (in particular, cofinite direct summands) of a cms-module to be a cms-module. We prove that a refinable module M is $\oplus$-cofinitely supplemented if and only if M is a cms-module if and only if it is cofinitely supplemented.

2. CMS-MODULES

In this section, we define the concept of cms-modules and give various properties of them.

Definition 2.1. Let M be a module. Then M is called a *cms-module* if, for every cofinite submodule N of M, there exist submodules K and $K^{'}$ of M such that K is a supplement of N, and K, $K^{'}$ are mutual supplements in M.

From the above definition it is clear that every supplemented module is a cms-module. But every cms-module is not always supplemented. For example, let R (e.g. $\mathbb{Z}$) be a non-local Dedekind domain which is not a field and Q be a quotient field of R. Consider the right R-module $M = Q^{(I)}$, where I is any index set. Since M has not any maximal submodule, M is a unique cofinite submodule of M. So M is a cms-module. Suppose that M is supplemented. Then Q is supplemented as a factor module of M. By [13], this implies that R is local, a contradiction. Therefore M is not supplemented. It is easy to see that every finitely generated cms-module is supplemented.

Resulting from all direct summands are mutual supplements to each other, every $\oplus$-cofinitely supplemented module is a cms-module. Under given definitions, we clearly have the following implication on modules:

$$\begin{array}{c} \oplus - cofinitely\ supplemented\ modules \\ \downarrow \\ cms - modules \\ \downarrow \\ cofinitely\ supplemented\ modules \end{array}$$

But we shall give example of a cms-module which is not $\oplus$-cofinitely supplemented.

Example 2.2. (See [6]) Let F be any field and $R = F[[X, Y]]$ the ring of formal power series over F indeterminates X, Y. Then R is a local commutative Noetherian domain. Now suppose that M is the Noetherian right R-module J. Therefore $M = XR+YR$. By [12, 42.6], since R is a local ring, every submodule of M is supplemented and so it is a cms-module. It follows from [6, Corollary 2.4] that M is not $\oplus$-supplemented. Since M is finitely generated, M is not $\oplus$-cofinitely supplemented.

In [9, 1.4], a module M is called *uniserial* if its lattice of submodules is a chain. M is said to be *serial* if M is a direct sum of uniserial modules. A ring R is *right (left) serial* if the module R_R (${}_RR$) is serial. In [3, 29.10] a ring R is artinian serial with $J^2 = 0$ if and only if every R-module is lifting if and only if every R-module is extending.

Example 2.3. (See [5]) Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = \frac{R}{W}$ is commutative, $dim(_Q W) = 1$, and $dim(W_Q) = 3$. Then R is left serial but not right serial. Let $W = w_1 R \oplus w_2 R \oplus w_3 R$. By [5, Proposition 4.9], there exist five isomorphism types of indecomposable R-modules defined in [5, Lemmas 4.1§4.2], where $X_5 = \frac{R_R \oplus R_R}{(w_1,0)R+(0,w_1)R+(w_2,w_3)R}$ is an indecomposable R-module of length 5 which is not local. Hence, X_5 is not ⊕-supplemented by [6, Lemma 3.1]. Since X_5 is 2-generated, it is not ⊕-cofinitely supplemented. Applying [12, 42.6], since R is local, we obtain that X_5 is supplemented. Therefore X_5 is a cms-module.

A module M is called *π-projective* if, for every two submodules U, V of M and identity homomorphism $I_M : M \longrightarrow M$ with $M = U + V$, there exists $f \in End(M)$ with $Im(f) \subseteq U$ and $Im(I_M - f) \subseteq V$ [12, 41.13].

Proposition 2.4. *Let M be a π-projective module. If M is a cms-module, then M is a ⊕-cofinitely supplemented module.*

Proof. Let N be any cofinite submodule of M. By the hypothesis, there exist submodules K and $K^{'}$ of M such that K is a supplement of N, and K, $K^{'}$ are mutual supplements in M. Since M is a π-projective module, in accordance with [3, 20.9], $K \cap K^{'} = 0$ and hence $M = K \oplus K^{'}$. Therefore M is a ⊕-cofinitely supplemented module. □

Recall from [12, 41.1] that a module M is *f-supplemented* if every finitely generated submodule of M has a supplement in M.

Proposition 2.5. *Let M be a cofinitely supplemented module.*

(1) *If M is f-supplemented, then it is cms.*
(2) *If every proper cofinite submodule of M is supplemented, then M is a cms-module.*

Proof. (1) For any cofinite submodule $U \subseteq M$, it follows from assumption that we can write $M = U + V$ and $U \cap V << V$ for some submodule $V \subseteq M$. Now

$$\frac{M}{U} \cong \frac{V}{U \cap V}$$

is finitely generated. Since $U \cap V$ is a small submodule of V, we obtain that V is finitely generated. By (1), V has a supplement in M, say $V^{'}$. Then, $M = V + V^{'}$ and $V \cap V^{'} << V^{'}$. By [12, 41.1(5)], we deduce that $V \cap V^{'} << V$. Hence, V and $V^{'}$ are mutual supplements in M.

(2) Let U be any cofinite submodule of M. Since M is cofinitely supplemented module, there exists a submodule $V \subseteq M$ that $M = U + V$ and $U \cap V \ll V$. By the hypothesis, $U = (U \cap V) + T$ and $(U \cap V) \cap T = V \cap T \ll T$ for some submodule $T \subseteq U$. Now $M = U + V = (U \cap V) + T + V = V + T$.

Note that $V \cap T \ll M$. Since V is a supplement of U in M, we have $V \cap T \ll V$ by [12, 41.1(5)]. Therefore M is a cms-module. □

We don't know whether or not any factor module of a cms-module is a cms-module. But we prove that a factor module of a cms-module by a fully invariant submodule is a cms-module in the following theorem.

Recall from [12, 6.4] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. A module M is called *duo*, if every submodule of M is fully invariant [8].

Theorem 2.6. *Let M be a cms-module and N be a fully invariant submodule of M. Then $\frac{M}{N}$ is a cms-module.*

Proof. Let $\frac{U}{N}$ be any cofinite submodule of $\frac{M}{N}$.

$$\frac{\frac{M}{N}}{\frac{U}{N}} \cong \frac{M}{U}$$

is finitely generated. So U is cofinite in M. Since M is a cms-module, then there exist submodules V and $V^{'}$ of M such that V is a supplement of U, and V, $V^{'}$ are mutual supplements in M. It is clear that $\frac{V+N}{N}$ is a supplement of $\frac{U}{N}$ in $\frac{M}{N}$. Since $V \cap V^{'} \ll V^{'}$, $V \cap V^{'} \ll V$ and N is a fully invariant submodule of M, then $\frac{V+N}{N} \cap \frac{V^{'}+N}{N} \subseteq \frac{(V\cap V^{'})+N}{N} \ll \frac{V+N}{N}$ and $\frac{V+N}{N} \cap \frac{V^{'}+N}{N} \subseteq \frac{(V\cap V^{'})+N}{N} \ll \frac{V^{'}+N}{N}$. Thus M is a cms-module. □

Since $Rad(M)$ is a fully invariant submodule of a module M, we obtain the following corollary as an immediate consequence of Theorem 2.6.

Corollary 2.7. *If M is a cms-module, then every cofinite submodule of $\frac{M}{Rad(M)}$ is a direct summand.*

Proposition 2.8. *Let $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ be a short exact sequence such that N is small in a module $M^{'}$, whenever $N \subset M^{'}$. If K is a cms-module, then M is a cms-module.*

Proof. Without loss of generality we will assume that $N \subseteq M$. Then, $\frac{M}{N} \cong K$ is a cms-module. Let U be any cofinite submodule of M,

$$\frac{M}{U+N} \cong \frac{\frac{M}{U}}{\frac{U+N}{U}}$$

and, so

$$\frac{\frac{M}{N}}{\frac{U+N}{N}} \cong \frac{M}{U+N}$$

is finitely generated. Then $\frac{U+N}{N}$ is a cofinite submodule of $\frac{M}{N}$. Since $\frac{M}{N}$ is cms-module, then there exist submodules $\frac{T}{N}$ and $\frac{T^{'}+N}{N}$ of $\frac{M}{N}$ such that $\frac{T}{N}$ is a supplement of $\frac{U+N}{N}$, and $\frac{T+N}{N}$, $\frac{T^{'}+N}{N}$ are mutual supplements in M. It is clear that $M = U + N + T = U + T$ and $\frac{U+N}{N} \cap \frac{T}{N} = \frac{(U \cap T)+N}{N} \ll \frac{T}{N}$. By the hypothesis $N \ll T$. Note that $M = T + T^{'}$. Then $U \cap T \ll T$ and $T \cap T^{'} \ll T$. Again by the hypothesis, $N \ll T^{'}$, from which it follows that $T \cap T^{'} \ll T^{'}$. Therefore M is a cms-module. □

Recall from [11, 1.11] that a module M is said to be *distributive* if $(X+Y) \cap Z = (X \cap Z) + (Y \cap Z)$ for any submodules X, Y, and Z of M. This means that the submodule lattice $Lat(M)$ is distributive.

Proposition 2.9. *Let M be a distributive cms-module and N be a cofinite direct summand of M. Then N is a cms-module.*

Proof. Let L be any cofinite submodule of N. Then $\frac{N}{L}$ is finitely generated. Since N is a direct summand of M, there exists a finitely generated submodule $N^{'}$ of M such that $M = N \oplus N^{'}$. Then $N^{'} \cong \frac{M}{N}$ is finitely generated. Furthermore $M = N + N^{'} + L$ and $N \cap (N^{'} + L) = L$. Since

$$\frac{(N^{'}+L)}{L} \cong \frac{N^{'}}{N^{'} \cap L} = \frac{N^{'}}{0} \cong N^{'}$$

is finitely generated, then $\frac{M}{L} = \frac{N}{L} + \frac{N^{'}+L}{L}$ is finitely generated. Therefore L is a cofinite submodule of M. Since M is a cms-module, there exist submodules $L^{'}$ and $K^{'}$ of M such that $L^{'}$ is a supplement of L, and $L^{'}$, $K^{'}$ are mutual supplements in M. Then we have $N = L + (N \cap L^{'})$ and $L \cap (N \cap L^{'}) \ll L^{'}$. Since M is a distributive module, $L^{'} = (N \cap L^{'}) \oplus (N^{'} \cap L^{'})$. It follows that $L \cap (N \cap L^{'}) \ll N \cap L^{'}$. Since M is a distributive module, $K^{'} = (N \cap K^{'}) \oplus (N^{'} \cap K^{'})$. It follows that $N = (N \cap L^{'}) + (N \cap K^{'})$. So we have $(N \cap L^{'}) \cap (N \cap K^{'}) \ll N \cap K^{'}$ and $(N \cap L^{'}) \cap (N \cap K^{'}) \ll N \cap L^{'}$ due to the inequality $(N \cap L^{'}) \cap (N \cap K^{'}) \leq L^{'} \cap K^{'} \ll K^{'}$. Therefore N is a cms-module. □

Theorem 2.10. *Let $\{M_i\}_{i \in I}$ be a family of cms-modules and $M = \oplus_{i \in I} M_i$. If every cofinite submodule of M is fully invariant, then M is a cms-module.*

Proof. Let N be any cofinite submodule of M. Then $\frac{M}{N}$ is finitely generated. By the hypothesis, $N = \oplus_{i \in I}(N \cap M_i)$. Note that $\oplus_{i \in I}(\frac{M_i}{N \cap M_i}) = \frac{\oplus_{i \in I} M_i}{\oplus_{i \in I}(N \cap M_i)} = \frac{M}{N}$ is finitely generated. Then for every $i \in I$, $\frac{M_i}{N \cap M_i}$ is finitely generated. Since for every $i \in I$, M_i is a cms-module, there exist submodules K_i and T_i of M_i such that K_i is a supplement of $N \cap M_i$, and K_i and T_i are mutual supplements in M_i. Let $\oplus_{i \in I} K_i = K$ and $\oplus_{i \in I} T_i = T$, and $M = \oplus_{i \in I} M_i = \oplus_{i \in I}(N \cap M_i) + \oplus_{i \in I} K_i = N + K$, and $N \cap K = \oplus_{i \in I}(N \cap M_i) \cap \oplus_{i \in I} K_i \subseteq \oplus_{i \in I}[(N \cap M_i) \cap K_i] = \oplus_{i \in I}(N \cap K_i) \ll K$. It follows that $M = K + T$, $K \cap T \ll K$ and $K \cap T \ll T$. Therefore M is a cms-module. □

Corollary 2.11. *Let $\{M_i\}_{i \in I}$ be a family of cms-modules and $M = \oplus_{i \in I} M_i$. If M is a duo module, then M is a cms-module.*

Lemma 2.12. *Let R be a ring with identity. Then the R-module R_R is a cms-module if and only if every free R-module is a cms-module.*

Proof. ($\Rightarrow$) Let M be a free R-module. Suppose that R_R is a cms-module. Since R is π-projective, R_R is a $\oplus$-cofinitely supplemented module by Proposition 2.4. It follows that M is $\oplus$-cofinitely supplemented module by [4, Lemma 2.8]. So M is a cms-module.

($\Leftarrow$) is obvious. □

For modules M and P, let $f : P \to M$ be an epimorphism. f is called cover if $ker(f)$ is small in P. A projective module P together with a cover $f : P \to M$ is called a *projective cover* of M. By [2, Theorem 2.1], rings whose (finitely generated) modules have a projective cover are (semi)perfect.

Theorem 2.13. *Let R be a ring with identity. Then the following statements are equivalent.*

(1) *R is semiperfect;*
(2) *R_R is $\oplus$-cofinitely supplemented;*
(3) *every free R-module is $\oplus$-cofinitely supplemented;*
(4) *R_R is a cms-module;*
(5) *every free R-module is a cms-module.*
(6) *every finitely generated R-module is a cms-module.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) It follows from [4, Theorem 2.9].

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) By Lemma 2.12 and Proposition 2.4.

(1) $\Rightarrow$ (6) Let R be a semiperfect ring. By [12, 42.6], every finitely generated R-module is supplemented. Thus every finitely generated R-module is a cms-module.

(6) $\Leftarrow$ (1) Suppose that every finitely generated R-module is a cms-module. In particular R_R is a cms-module. Since R_R is finitely generated, then R_R is supplemented. By [12, 42.6], R is semiperfect. □

Recall from [12, 21.4] that a submodule N of a module M is called *radical* if N has no maximal submodule, that is, $N = Rad(N)$. For a module M, $P(M)$ will indicate the sum of all radical submodules of M. If $P(M) = 0$, M is called *reduced*. Note that $P(M)$ is the largest radical submodule of M.

Lemma 2.14. *Let R be a Dedekind domain and M be an R-module. Then $P(M)$ is a cms-module.*

Proof. Let R be a Dedekind domain, and so R is noetherian. Here, $P(M)$ denotes the divisible part of M. Then $P(M)$ is injective by [10, proposition 2.10], hence $M = P(M) \oplus N$ for some submodule N of M. In this case N is called the reduced part of M. By [1, Lemma 4.4], $P(M)$ is the only cofinite submodule of $P(M)$. Thus $P(M)$ is a cms-module. □

Proposition 2.15. *Let R be a Dedekind domain, M be a duo R-module and N be the reduced part of M. Then M is a cms-module if and only if N is a cms-module.*

Proof. ($\Rightarrow$) Since $P(M)$ is a fully invariant submodule, then $\frac{M}{P(M)} \cong N$ is a cms-module by Theorem 2.6.

($\Leftarrow$) It is clear by Corollary 2.11 and Lemma 2.14. □

In [3, 11.26], an R-module M is called *refinable* if for any submodules $U, V \subseteq M$ with $M = U + V$, there exists a direct summand $U^{'}$ of M with $U^{'} \subseteq U$ and $M = U^{'} + V$. Every finitely generated regular module is refinable. Note that every direct summand of a refinable module is refinable.

Theorem 2.16. *Let M be a refinable module. Then the following statements are equivalent.*

(1) *M is ⊕-cofinitely supplemented;*
(2) *M is a cms-module;*
(3) *M is cofinitely supplemented.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Let N be any cofinite submodule of M. Since M is a cofinitely supplemented module, then there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll K$. So we have $N \cap K \ll M$. Since M is a refinable module, there exists a direct summand L of M such that $L \subseteq K$ and $M = N + L$. Then $N \cap L \ll L$. Thus M is a ⊕-cofinitely supplemented module. □

Corollary 2.17. *Let M be a finitely generated refinable module. Then the following statements are equivalent.*

(1) *M is ⊕-supplemented;*
(2) *M is ⊕-cofinitely supplemented;*
(3) *M is a cms-module;*
(4) *M is cofinitely supplemented;*
(5) *M is supplemented;*
(6) *every maximal submodule of M has a supplement.*

Corollary 2.18. *Let M be a refinable module. $M = \oplus_{i \in I} M_i$. Suppose that for every submodule N of M there is a cofinite submodule L of M such that $N = L + T$ or $L = N + T$ for some $T \ll M$. Then M is a cms-module if and only if M_i is a cms-module.*

Finally, we have the following fact.

Corollary 2.19. *Consider the following statements for a ring R.*

(1) *R is right perfect.*
(2) *Every right R-module is cms.*
(3) *R is semiperfect.*

Proof. (1) ⇒ (2) Since every module over a right perfect ring is supplemented, it is clear.

(2) ⇒ (3) It follows from Theorem 2.13. □

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