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RINGS FOR WHICH EVERY SIMPLE MODULE IS ALMOST INJECTIVE

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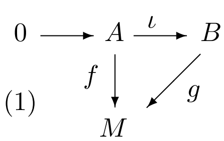
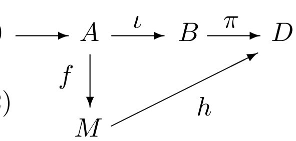
Abstract. We introduce the class of "right almost V-rings" which is properly between the classes of right V-rings and right good rings. A ring R is called a right almost V-ring if every simple R-module is almost injective. It is proved that R is a right almost V-ring if and only if for every R-module M, any complement of every simple submodule of M is a direct summand. Moreover, R is a right almost V-ring if and only if for every simple R-module S, either S is injective or the injective hull of S is projective of length 2. Right Artinian right almost V-rings and right Noetherian right almost V-rings are characterized. A 2×2 upper triangular matrix ring over R is a right almost V-ring precisely when R is semisimple.
Keywords: Almost injective modules, V-rings, almost V-rings.
MSC(2010): Primary: 16D70; Secondary: 16D80, 16P20, 16P40.

1. Introduction

Throughout rings will have unity and modules will be unitary right modules. An R-module M is called almost injective if for every embedding $\iota : A \to B$ of R-modules and every homomorphism $f : A \to M$, either there exists a homomorphism $g : B \to M$ such that diagram (1) commutes, or there exists a nonzero direct summand D of B with the canonical projection $\pi : B \to D$, and a homomorphism $h : M \to D$ such that diagram (2) commutes:

(1)


(2)


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This concept has been studied in [1,3,4,9–13]. The reader is referred to [13] for a recent survey on this subject. Besides the other relations of almost injective property to some classical rings mentioned in [13], this property supplies an answer to a question in the theory of extending modules. Recall that a module M is extending (or CS) if every complement submodule of M is a direct summand. In the study of extending modules, a question that has attracted more attention is when a direct sum of such modules is extending. The almost injective property provides an answer to this question in some special cases; see [1, Remark 10] and [5, 1.19]. In this paper, we deal with the rings R for which every simple R-module is almost injective, called right almost V-rings. Recall that R is a right V-ring if every simple R-module is injective. Right V-rings are introduced by Villamayor for associative rings and for commutative rings it was shown by Kaplansky that these rings coincide with von Neumann regular rings. But, the latter fact fails for non-commutative rings in general. In fact, this failure was responsible for the extensive research in the literature concerning V-rings. Our work in this article is somewhat related to V-rings, too. Every right V-ring and every 2×2 upper triangular matrix ring over a semisimple ring is a right almost V-ring. A ring R is called a right good ring if $R/\text{Rad}(R)$ is a right V-ring. Right almost V-rings form a class of rings which properly lies between the classes of right V-rings and right good rings (Proposition 2.3, Examples 2.6 and 2.10), so they are worth studying. Moreover, these rings are of interest for us since over such rings, every finite direct sum of Artinian extending modules is extending (Corollary 2.14 and Proposition 3.4).

In Section 2, we obtain several results on right almost V-rings. Every ring Morita equivalent to a right almost V-ring is a right almost V-ring (Proposition 2.8). Let us say that a module M is simple-extending if every complement of any simple submodule of M is a direct summand. It is an obvious fact that every module over a right V-ring is simple-extending. We prove that a ring R for which every R-module is simple-extending is precisely a right almost V-ring (Theorem 2.9). For the close connection between right almost V-rings and simple-extending modules, we end the section by investigating some properties of simple-extending modules. Among others, it is shown that every direct sum of (two) cocyclic R-modules is simple-extending if and only if every cocyclic R-module has length at most 2 (Theorem 2.15).

In Section 3 we study right almost V-rings with chain conditions. First, we prove that R is a right almost V-ring if and only if for every simple R-module S, either S is injective or $E(S)$ is projective of length 2 (Theorem 3.1). In the sequel, we determine right almost V-rings with chain conditions. It is shown that R is a right Artinian right almost V-ring, if and only if, every R-module is extending, if and only if, R is an Artinian serial ring with $\text{Rad}^2(R) = 0$ (Corollary 3.5). Moreover, R is a right Noetherian right almost V-ring if and only if there exists a two-sided ideal K of R such that R/K is a right V-ring,

K is a direct summand of R, and K is a direct sum of minimal right ideals and injective right ideals of length 2 (Theorem 3.6). Finally, we show that a 2×2 upper triangular matrix ring over a ring R is a right almost V-ring if and only if R is semisimple (Theorem 3.8).

2. right almost V-rings

Definition 2.1. We say that R is a right almost V-ring if every simple R-module is almost injective.

Clearly, every right V-ring is a right almost V-ring, however we will give examples of right almost V-rings which are not right V-rings; see Example 2.10 and Corollary 3.9.

The first result, in particular, shows that if R is a right almost V-ring such that $\mathrm{Soc}(R_R)$ is injective, then R is a right V-ring.

Proposition 2.2. *Let R be a right almost V-ring, and S be a simple R-module. If S cannot be embedded in R, then S is injective.*

Proof. Assume that S is not injective. Then there exist an essential right ideal I of R and a homomorphism $f : I \to S$ such that f cannot be extended to a homomorphism of R to S. Since R is a right almost V-ring, there exist a nonzero direct summand D of R, say $R = D \oplus D'$, and a homomorphism $h : S \to D$ such that $\pi\iota = hf$, where $\iota : I \to R$ is the natural embedding and $\pi : R \to D$ is the canonical projection. If $h = 0$, then $I \leq \ker \pi = D'$ and so D' is essential in R. This implies that $D = 0$ which is a contradiction. Thus h is an embedding of S into R. □

Proposition 2.3. *Let R be a right almost V-ring.*

(i) $R/\mathrm{Rad}(R)$ *is a right V-ring.*

(ii) $R/\mathrm{Soc}(R_R)$ *is a right V-ring.*

(iii) $\mathrm{Rad}(M) \leq \mathrm{Soc}(M)$ *and* $\mathrm{Rad}(M) \ll M$, *for every R-module M.*

Proof. (i). By [6, 2.13], it suffices to show that if I is a right ideal of R containing $\mathrm{Rad}(R)$, then for every $x \in R \setminus I$, there exists a maximal right ideal K of R such that $I \leq K$ and $x \notin K$. Let L be a right ideal of R maximal with respect to the properties that $I \leq L$ and $x \notin L$. Set $S = (L + xR)/L$. Clearly, S is simple, and since R is a right almost V-ring, S is almost injective. If the natural epimorphism $f : L + xR \to S$ extends to $g : R \to S$, then $\ker g$ is a maximal right ideal of R such that $I \leq \ker g$ and $x \notin \ker g$. If there are a nonzero direct summand D of R, say $R = D \oplus D'$, and a homomorphism $h : S \to D$ such that $\pi\iota = hf$, where $\iota : L + xR \to R$ is the natural embedding and $\pi : R \to D$ is the canonical projection, then $\mathrm{Rad}(R) \leq L \leq D'$. Thus $\mathrm{Rad}(R) = \mathrm{Rad}(D) \oplus \mathrm{Rad}(D')$ implies that $\mathrm{Rad}(D) = 0$. Moreover, $\pi(x) \neq 0$, for otherwise $x \in D'$, and so $x \notin L \oplus D$, which contradicts the maximal property of L. Hence there exists a maximal submodule A of D such that $\pi(x) \notin A$. Thus $K = A \oplus D'$ is a maximal right ideal of R such that $I \leq K$ and $x \notin K$.

(iii). First we show that $\mathrm{Rad}(R) \leq \mathrm{Soc}(R_R)$. Let $x \in \mathrm{Rad}(R) \setminus \mathrm{Soc}(R_R)$. Clearly, $E(R_R) = E(xR) \oplus E'$ for some submodule E' of $E(R)$, and so $\mathrm{Rad}(E(R_R)) = \mathrm{Rad}(E(xR)) \oplus \mathrm{Rad}(E')$. Thus $x \in \mathrm{Rad}(E(xR))$. Now, assume that L is a maximal submodule of xR containing $\mathrm{Soc}(xR)$, and set $S = xR/L$. Since R is a right almost V-ring, S is almost injective. Let $f : xR \to S$ be the natural epimorphism, and $\iota : xR \to E(xR)$ be the natural embedding. If there exists $g : E(xR) \to S$ such that $g\iota = f$, then $\ker g$ is a maximal submodule of $E(xR)$. Hence $x \in \ker g$, and so $x \in \ker f$ which is impossible. Thus there exist a nonzero direct summand E of $E(xR)$ and $h : S \to E$ such that $\pi\iota = hf$, where $\pi : E(xR) \to E$ is the natural projection. Without loss of generality we can assume that E is the injective hull of $h(S)$. Let $a = \pi(x)$. Since $aR = \pi(xR) = h(S)$ is a simple R-module, we conclude that $aR \leq \mathrm{Soc}(E(xR)) = \mathrm{Soc}(xR) \leq \ker f$. Hence $aR = \pi(aR) = hf(aR) = 0$ which is impossible. Therefore $\mathrm{Rad}(R) \leq \mathrm{Soc}(R_R)$, as desired. Now let M be an R-module. By (i), $R/\mathrm{Rad}(R)$ is a right V-ring, and so $\mathrm{Rad}(M/M\mathrm{Rad}(R_R)) = 0$. Hence $\mathrm{Rad}(M) \leq M\mathrm{Rad}(R)$. But $M\mathrm{Rad}(R) \leq \mathrm{Rad}(M)$, and so $\mathrm{Rad}(M) = M\mathrm{Rad}(R)$. Thus by what we have shown above, $\mathrm{Rad}(M) = M\mathrm{Rad}(R) \leq M\mathrm{Soc}(R_R) \leq \mathrm{Soc}(M)$. Moreover, by [5, 2.8(9)], $\mathrm{Soc}(\mathrm{Rad}(M)) \ll M$, hence $\mathrm{Rad}(M) \ll M$.

(ii). By (iii), $R/\mathrm{Soc}(R_R)$ is a factor ring of $R/\mathrm{Rad}(R)$, and so it is a right almost V-ring by (i). □

Corollary 2.4. *If M is a singular module over a right almost V-ring R, then* $\mathrm{Rad}(M) = 0$.

Proof. Let $\overline{R} = R/\mathrm{Soc}(R_R)$. Since the annihilator of every element of M is an essential right ideal of R, we conclude that $M\mathrm{Soc}(R_R) = 0$. Hence M is an $\overline{R}$-module, and so by Proposition 2.3 (ii), $\mathrm{Rad}(M_{\overline{R}}) = 0$. Thus $\mathrm{Rad}(M_R) = 0$. □

Following [15], a submodule A of M is called an absolute summand of M if $A \oplus C = M$, for every complement C of A in M. Every injective submodule of M is an absolute summand of M.

Corollary 2.5. *The following statements are equivalent for a ring R.*

(1) *R is a right almost V-ring and every minimal right ideal of R is an absolute summand.*

(2) *R is a right almost V-ring and every minimal right ideal of R is injective.*

(3) *R is a right V-ring.*

Proof. (1) $\Rightarrow$ (3). By Proposition 2.3, it is enough to prove that $\mathrm{Rad}(R) = 0$. Then since $\mathrm{Rad}(R) \leq \mathrm{Soc}(R_R)$, it suffices to show that $\mathrm{Rad}(R)$ contains no minimal right ideal of R. Assume that $I \leq \mathrm{Rad}(R)$ is a minimal right ideal of R, and C is a complement of I in R. By hypothesis, $I \oplus C = R$, and so $C = R$ which is impossible. This shows that $\mathrm{Rad}(R)$ contains no minimal right ideal of R, as desired.

(3) ⇒ (2) ⇒ (1). These implications are obvious. □

Recall that a ring R is called a right good ring if $f(\mathrm{Rad}(R)) = \mathrm{Rad}(f(R))$, for every R-homomorphism $f : R \to X$. The ring R is a right good ring if and only if $R/\mathrm{Rad}(R)$ is a right V-ring; see [17, 23.7]. By Proposition 2.3(i), every right almost V-ring is a right good ring. The next example shows that the class of right good rings properly contains the class of right almost V-rings.

Example 2.6. Let $R = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. Then $\mathrm{Rad}(R) = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$. Since $R/\mathrm{Rad}(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, it is a V-ring. Now, let $I = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, $S = I/J$, $f : I \to S$ be the natural epimorphism, and $\iota : I \to R$ be the natural embedding. Clearly, S is a simple R-module. If there exists a homomorphism $g : R \to S$ such that $g\iota = f$, then $\ker g$ is a maximal right ideal of R containing $J = \ker f$. Thus $\ker g = I$, which is impossible. On the other hand, if there exists a nonzero direct summand D of R, say $R = D \oplus D'$, and a homomorphism $h : S \to R$ such that $hf = \pi\iota$, where $\pi : R \to D$ is the canonical projection, then $J = \ker f \leq \ker \pi = D'$. Therefore $D' = R$, which is impossible. This shows that R is not a right almost V-ring.

Recall that R is a right generalized V-ring if every simple R-module is either projective or injective, or equivalently, every singular simple R-module is injective. In the following we show that a right nonsingular right almost V-ring is a right generalized V-ring.

Proposition 2.7. *Let R be a right almost V-ring.*
(i) *If $Z(R_R) \cap \mathrm{Rad}(R) = 0$, then R is a right generalized V-ring.*
(ii) *$R/Z_2(R_R)$ is a right generalized V-ring.*

Proof. (i). Let I be a proper essential right ideal of R. Then R/I is singular, and so by Corollary 2.4, $\mathrm{Rad}(R/I) = 0$. Thus I is an intersection of maximal right ideals of R. So by [15, Theorem 3.3(1)], R is a right generalized V-ring.

(ii). This follows from (i) and the fact that $R/Z_2(R_R)$ is a right nonsingular ring. □

Proposition 2.8. *Every ring Morita equivalent to a right almost V-ring is a right almost V-ring.*

Proof. This follows from the facts that being an exact sequence, a direct summand and a simple module are preserved under Morita equivalences. □

Recall that a module M is extending (or CS) if every complement (equivalently, every closed submodule) of M is a direct summand. Let us say that M is a simple-extending module if every complement of any simple submodule of M is a direct summand of M. Every direct summand of a simple-extending module inherits the property. The next result gives an equivalent condition for a right almost V-ring in terms of the simple-extending property.

Theorem 2.9. *A ring R is a right almost V-ring if and only if every R-module M is simple-extending.*

Proof. ($\Rightarrow$). Let S be a simple submodule of an R-module M, and C be a complement of S in M. By hypothesis, S is almost injective. Assume that $f : S \oplus C \to S$ is the canonical projection, and $\iota : S \oplus C \to M$ is the natural embedding. If there exists $g : M \longrightarrow S$ such that $g\iota = f$, then $\ker g$ is a maximal submodule of M which contains C but not S. Since C is a complement of S we conclude that $\ker f = C$. Thus $S \oplus C = M$, and so C is a direct summand of M. Now, assume that there exist a nonzero direct summand D of M, say $M = D \oplus D'$, and a homomorphism $h : S \to D$ such that $\pi\iota = hf$, where $\pi : M \to D$ is the canonical projection. So $C \leq \ker \pi = D'$. Since $S \oplus C \leq_e M$ we conclude that $S \cap D' = 0$. Thus $C = D'$ is a direct summand of M. This shows that M is simple-extending.

($\Leftarrow$). Step 1: First we show that $R/\mathrm{Soc}(R_R)$ is a right V-ring. By [16, Theorem 23], it suffices to show that if S is a simple R-module and M is an extending R-module, then $S \oplus M$ is extending. Let C be a closed submodule of $S \oplus M$. If $S \leq C$, then $C = S \oplus (C \cap M)$. It is easy to see that $C \cap M$ is a closed submodule of M, and so it is a direct summand of M since M is extending. Thus C is a direct summand of $S \oplus M$. If $S \nleq C$, then $S \oplus C = S \oplus \pi(C)$, where $\pi : S \oplus M \to M$ is the canonical projection. Since M is extending, there exists a direct summand D of M such that $\pi(C) \leq_e D$. Thus $S \oplus C = S \oplus \pi(C) \leq_e S \oplus D$. Hence by [14, Proposition 6.22], C is a complement of S in $S \oplus D$. But $S \oplus D$ is simple-extending by hypothesis, and so C is a direct summand of $S \oplus D$. Hence C is a direct summand of $S \oplus M$. This implies that $S \oplus M$ is extending, as desired.

Step 2: If S is a singular simple R-module, we show that $Z(E(S)) = S$. Since S is singular we conclude that $\mathrm{Soc}(R_R) \leq \mathrm{ann}_R(S)$. As shown in step 1, $R/\mathrm{Soc}(R_R)$ is a right V-ring, and so S is an injective $R/\mathrm{Soc}(R_R)$-module. By [8, Exercise 5J], the injective hull of S as $R/\mathrm{Soc}(R_R)$-module is $\mathrm{ann}_{\mathrm{E}(S)}(\mathrm{Soc}(R_R))$. Thus $\mathrm{ann}_{\mathrm{E}(S)}(\mathrm{Soc}(R)) = S$, and since $Z(E(S))$ is singular we conclude that $Z(E(S)) \leq \mathrm{ann}_{\mathrm{E}(S)}(\mathrm{Soc}(R_R)) = S$. On the other hand, $S \leq Z(E(S))$ since S is singular, and hence $Z(E(S)) = S$.

Step 3: Finally, we show that a simple R-module S is almost injective. Let A be an essential submodule of an R-module B, and $f : A \to S$ be a nonzero R-homomorphism. Set $K = \ker f$. Clearly, K is a maximal submodule of A. If $K \not\leq_e A$, then there exists a simple submodule K' of A such that $A = K \oplus K'$. Let C be a complement of K' in B which contains K. By hypothesis, B is simple-extending, hence C is a direct summand of B, say $B = C \oplus C'$. Assume that $\pi : B \to C'$ is the canonical projection, and $\theta = (f|_{K'})^{-1}$. Then $h = \pi\theta : S \to C'$ is an R-homomorphism such that $hf = \pi\iota$, where $\iota : A \to B$ is the natural embedding. On the other hand, if $K \leq_e A$ then $K \leq_e B$. Clearly, there exists a homomorphism $g : B \to E(S)$ such that $g\iota = f$. Since $K \leq \ker g$ we conclude that $\ker g \leq_e B$, and so $\operatorname{Im} g \leq Z(E(S))$. Thus by what we have shown in step 2, $\operatorname{Im} g \leq S$. So $g : B \to S$ is a homomorphism such that $g\iota = f$. □

By Theorem 2.9, we give an example of a right almost V-ring which is not a right V-ring. More examples are provided by Corollary 3.9.

Example 2.10. Let $R = \mathbb{Z}_{p^2}$, where p is a prime number. Clearly, R is an Artinian serial ring with $\operatorname{Rad}^2(R) = 0$. So by [6, 13.5($(a) \Leftrightarrow (g)$)] and Theorem 2.9, R is a right almost V-ring. However, R is not a right V-ring since $\operatorname{Rad}(R) \neq 0$.

We end this section by exploring some properties of simple-extending modules which are useful in the study of right almost V-rings. The following example shows that the notions of an extending module and a simple-extending module are really different.

Example 2.11. Let R be an infinite direct product of fields. Since R is a V-ring, Theorem 2.9 implies that every R-module is simple-extending. However, not every R-module can be extending, for otherwise, R would be Artinian by [6, 13.5(g)].

Proposition 2.12. *Let M be a simple-extending module. Then every complement of a finitely generated semisimple submodule is a direct summand of M.*

Proof. By induction, it suffices to show that a complement C of $S_1 \oplus S_2$ in M is a direct summand, where S_1, S_2 are two simple submodules of M. There exists a complement C_1 of S_1 in M such that $S_2 \oplus C \leq C_1$. Since M is simple-extending, C_1 is a direct summand of M. On the other hand, it is easy to see that C is a complement of S_2 in C_1. But C_1 is a direct summand of M, and so it is simple-extending. Thus C is a direct summand of C_1. Hence C is a direct summand of M, as desired. □

Corollary 2.13. *A finitely cogenerated module M is simple-extending if and only if it is extending.*

Proof. Let M be simple-extending, and C be a closed submodule of M which is a complement of a submodule N. Since M is finitely cogenerated, $\mathrm{Soc}(M)$ is finitely generated and essential, and so is $\mathrm{Soc}(N)$. Thus C is a complement of $\mathrm{Soc}(N)$, and so by Proposition 2.12, C is a direct summand of M. This shows that M is extending. □

It is well known that a direct sum of extending modules is not necessarily extending. In the literature, there are some conditions under which a direct sum of certain extending modules is extending. The next result shows that every finite direct sum of finitely cogenerated extending modules over a right almost V-ring is extending.

Corollary 2.14. *If R is a right almost V-ring, then every finite direct sum of finitely cogenerated extending R-modules is extending.*

Proof. Let M be a finite direct sum of finitely cogenerated extending R-modules. Then M is finitely cogenerated, and by Theorem 2.9, it is simple-extending. Thus by Corollary 2.13, M is extending. □

Recall that a module M is called cocyclic if there exists an essential simple submodule in M. In fact, M is cocyclic, if and only if, M is an uniform module with nonzero socle, if and only if, M is isomorphic to a nonzero submodule of the injective hull of a simple module. In the following, by adapting the method of [6, Theorem 13.1], we show that every direct sum of cocyclic R-modules is simple-extending, if and only if, every cocyclic R-module has length at most 2. This is used in obtaining more characterizations of right almost V-rings in the next section.

Theorem 2.15. *The following statements are equivalent for a ring R.*
(1) *Every direct sum of (two) cocyclic R-modules is simple-extending.*
(2) *Every direct sum of (two) cocyclic R-modules is extending.*
(3) *Every cocyclic R-module has length at most 2.*

Proof. (1) $\Rightarrow$ (3). First we show that the radical of any uniform R-module is either zero or simple. For this it suffices to show that if U is an injective uniform R-module, then $\mathrm{Rad}(U)$ is either zero or simple. Let $0 \neq x \in \mathrm{Rad}(U)$, and L be a nonzero maximal submodule of xR. The hypothesis implies that $U \oplus xR/L$ is simple-extending, and so by Corollary 2.13, $U \oplus xR/L$ is extending. Thus by [6, Lemma 7.3(i)], the natural epimorphism $f : xR \to xR/L$

extends to a homomorphism $\overline{f} : U \to xR/L$. But $xR \leq \mathrm{Rad}(U) \leq \ker \overline{f}$, and hence $xR \leq \ker f$, which is impossible. This shows that xR is simple, for every $0 \neq x \in \mathrm{Rad}(U)$. Hence $\mathrm{Rad}(U)$ is simple.

For proving (3), it suffices to show that an injective cocyclic R-module M has length at most 2 (Note that the injective hull of a cocyclic R-module is cocyclic). If $\mathrm{Rad}(M) = 0$, then M is simple. Otherwise, by what we have shown above, $\mathrm{Rad}(M)$ is simple. So if we show that M has only one maximal submodule, then M has length at most 2. Let L_1 and L_2 be two distinct maximal submodule of M. Assume that $f : L_i \to L_j$ is a monomorphism, for $i, j \in \{1, 2\}$. The injectivity of M implies that f extends to an endomorphism g of M such that $\ker(g) \cap L_i = 0$. Since M is uniform we conclude that $\ker(g) = 0$, and so g is left-invertible. But $\mathrm{End}(M)$ is local, hence g is an isomorphism. Therefore $f(L_i)$ is a maximal submodule M, and so f is an isomorphism. Then if $i = j$, the facts that $\ker(f) \cap \ker(1 - f) = 0$ and M is uniform imply that the endomorphism rings of L_1 and L_2 are local. On the other hand, by hypothesis, $L_i \oplus L_j$ is simple-extending. Thus by Corollary 2.13, $L_i \oplus L_j$ is extending, for $i, j \in \{1, 2\}$. Thus by [6, Lemma 7.3(ii)], L_1 is L_1-injective and L_2-injective. But $L_1 + L_2 = M$, hence L_1 is M-injective, and so it is a direct summand of M, which is impossible. Thus M has only one maximal submodule, as desired.

(3) $\Rightarrow$ (2). Let N be a direct sum of essential extensions of simple R-modules. By hypothesis, an essential extension of a simple R-module S is either S or $E(S)$. Hence $N = N_1 \oplus N_2$, where N_1 is semisimple and N_2 is a direct sum of injective modules of length 2. Thus by [6, Lemma 8.14], N is extending.

(2) $\Rightarrow$ (1). Since every extending module is simple-extending, the implication is clear. □

Corollary 2.16. *Let R be a ring for which every direct sum of two cocyclic R-module is simple-extending. Then R is a right max ring with* $\mathrm{Rad}(R)^2 = 0$.

Proof. By Theorem 2.15, every cocyclic R-module has a maximal submodule. Thus by [7, 3.32D], R is a right max ring. Moreover, as shown in the proof of Theorem 2.15((1) $\Rightarrow$ (3)), $\mathrm{Rad}(E(S))$ is zero or simple, for every simple R-module S. Hence $\mathrm{Rad}(R)$ embeds in a direct product of simple modules; see [2, Corollary 18.16]. So $\mathrm{Rad}(R)^2 = 0$. □

Remark 2.17. Theorem 2.9 and Corollary 2.16 imply that a right almost V-ring is a right max ring with $\mathrm{Rad}(R)^2 = 0$.

3. Right almost V-rings with chain conditions

In this section, right Noetherian (resp., Artinian) almost V-rings are determined. First, we give a characterization of right almost V-rings according to the certain property of their simple modules.

Theorem 3.1. *The following statements are equivalent.*
(1) *R is a right almost V-ring.*
(2) *For every simple R-module S, either S is injective or $E(S)$ is projective of length* 2.

Proof. (1) $\Rightarrow$ (2). Let S be a simple non-injective R-module. There exist an essential right ideal I of R and a homomorphism $f : I \to S$ such that f cannot be extended to a homomorphism of R to S. Since R is a right almost V-ring we conclude that there exist a nonzero direct summand D, say $R = D \oplus D'$, and a homomorphism $h : S \to D$ such that $\pi\iota = hf$, where $\iota : I \to R$ is the natural embedding and $\pi : R \to D$ is the canonical projection. If $\pi\iota = 0$, then $I \leq D'$, and so D' is essential in R, which is impossible. Hence $0 \neq \pi\iota(I) = h(S)$, and so $h(S) \leq D$ is a minimal right ideal of R. Let C be a complement of $h(S)$ in D. By Theorem 2.9, C is a direct summand of D, say $D = C \oplus C'$. Let $\pi' : D \to C'$ be the canonical projection. If $X \leq C'$ such that $\pi' h(S) \cap X = 0$, then $h(S) \cap (C \oplus X) = 0$. Thus $X = 0$, and so $\pi' h(S) \leq_e C'$. This shows that C' is cocyclic and $\pi' h : S \to C'$ is a monomorphism. If $\pi' h(S) = C'$, then $\pi' h$ is an isomorphism. So $(\pi' h)^{-1}\pi'\pi : R \to S$ is a homomorphism such that $(\pi' h)^{-1}\pi'\pi\iota = f$, which is a contradiction to the first assumption that f cannot be extended to a homomorphism of R to S. Thus $\pi' h(S) \neq C'$, and so by Theorems 2.9 and 2.15, C' and $E(C')$ are cocyclic R-modules of length 2. Hence $E(C') = C'$ is a direct summand of R. Thus $E(C')$ is projective of length 2. Since $S \cong \pi' h(S) \leq_e C'$ we conclude that $E(S)$ is projective of length 2.

(2) $\Rightarrow$ (1). Let S be a non-injective simple R-module. There exist an essential submodule A of a module B, and a homomorphism $f : A \to S$ such that f cannot be extended to a homomorphism of B to S. Since $E(S)$ is injective, there exists a homomorphism $g : B \to E(S)$ such that $g\iota = \iota' f$, where $\iota : A \to B$ and $\iota' : S \to E(S)$ are natural embeddings. The assumption on f implies that $f \neq 0$, and so $g(B)$ is a nonzero submodule of $E(S)$. Thus $S \leq g(B)$. Again, the assumption on f implies that S is a proper submodule of $g(B) \leq E(S)$. Since $E(S)$ has length 2 by hypothesis, we conclude that $g(B) = E(S)$, and so $B/\ker(g) \cong E(S)$. But, by hypothesis, $E(S)$ is projective, so $\ker(g)$ is a direct summand of B, say $B = \ker(g) \oplus L$. Clearly, $g|_L : L \to g(L)$ is an isomorphism, and $S \leq g(L)$. Then $(g|_L)^{-1}\iota' : S \to L$ is a homomorphism such that $(g|_L)^{-1}\iota' f = \pi\iota$. where $\pi : B \to L$ is the natural projection. This shows that R is a right almost V-ring. □

Remark 3.2. A ring R is called right co-Noetherian if the injective hulls of simple R-modules are Artinian. Theorem 3.1 implies that every right almost V-ring is right co-Noetherian.

By Proposition 2.3, if R is a right almost V-ring, then $R/\mathrm{Rad}(R)$ is a right V-ring and $\mathrm{Rad}(R) \leq \mathrm{Soc}(R_R)$. The converse implication is not always true. For example, the ring R of Example 2.6 is not a right almost V-ring but $R/\mathrm{Rad}(R)$ is a right V-ring and $\mathrm{Rad}(R) = \mathrm{Soc}(R_R)$. The next result shows that the converse implication holds if $E(A)$ is projective of length 2, for every simple right ideal A contained in $\mathrm{Rad}(R)$.

Corollary 3.3. *R is a right almost V-ring, if and only if, $R/\mathrm{Rad}(R)$ is a right V-ring, $\mathrm{Rad}(R)$ is semisimple, and $E(A)$ is projective of length 2, for every simple right ideal $A \leq \mathrm{Rad}(R)$.*

Proof. ($\Rightarrow$). By Proposition 2.3, $R/\mathrm{Rad}(R)$ is a right V-ring, and $\mathrm{Rad}(R)$ is semisimple. Let A be a simple right ideal of R contained in $\mathrm{Rad}(R)$. Clearly, A is not injective, and so by Theorem 3.1, $E(A)$ is projective of length 2.

($\Leftarrow$). Let S be a non-injective simple R-module. There exist an essential right ideal I of R and a homomorphism $f : I \to S$ such that f cannot be extended to a homomorphism of R to S. Clearly, $\mathrm{Rad}(R) \leq I$ since $\mathrm{Rad}(R)$ is semisimple. Assume that 'bar' denotes the image in $R/\mathrm{Rad}(R)$. Since $\overline{R}$ is a right V-ring, if $\mathrm{Rad}(R) \leq \ker(f)$, then $\overline{f} : \overline{I} \to S$ defined by $\overline{f}(\overline{a}) = f(a)$ can be extended to $\overline{h} : \overline{R} \to S$. Hence by assuming that $\pi : R \to \overline{R}$ is the natural epimorphism, $h = \overline{h}\pi$ is an extension of f, which is a contradiction. Thus $\mathrm{Rad}(R) \nleq \ker(f)$, and so there is a simple submodule A of $\mathrm{Rad}(R)$ such that $A \nleq \ker(f)$. But clearly, $\ker(f)$ is a maximal submodule of I, hence $I = A \oplus \ker(f)$. This implies that $S \cong A$, and so by hypothesis, $E(S)$ is projective of length 2. Thus by Theorem 3.1, R is a right almost V-ring. □

In the following we determine right almost V-rings with chain conditions.

Proposition 3.4. *Let M be a module over a right almost V-ring R. The following statements are equivalent.*
(1) *M is Artinian.*
(2) *M is finitely cogenerated.*
(3) *M is a serial finite length module.*
(4) *M is a finite direct sum of simple submodules and uniform injective submodules of length 2.*

Proof. (4) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (2). These implications are clear.

(2) $\Rightarrow$ (4). By [17, 21.3], $M = M_1 \oplus \cdots \oplus M_k$, where each M_i is indecomposable; moreover, $\mathrm{Soc}(M) \leq_e M$. Assume that S is a simple submodule of $\mathrm{Soc}(M_i)$, and C is a complement of S in M_i. Since R is a right almost V-ring, M_i is simple-extending. Hence C is a direct summand of M_i, and so $C = 0$ as M_i is indecomposable. This implies that $S \leq_e M_i$, and so M_i is cocyclic. Thus by Theorems 2.9 and 2.15(3), M_i has length at most 2. If M_i has length

1, then it is simple. If M_i has length 2, then $M_i = E(M_i)$ is injective since $E(M_i)$ has length at most 2 by Theorem 2.15(3). □

Corollary 3.5. *The following statements are equivalent for a ring R.*
(1) *R is a right Artinian right almost V-ring.*
(2) *Every R-module is extending.*
(3) *R is an Artinian serial ring with $\mathrm{Rad}(R)^2 = 0$.*

Proof. (1) ⇒ (2). This follows from Proposition 3.4(4) and [6, 13.5(e)].
(2) ⇒ (1). This is clear by Theorem 2.9 and [6, 13.5(g)].
(2) ⇔ (3). It was proved in [6, 13.5]. □

Proposition 3.6. *A right Noetherian ring R is right almost V-ring if and only if there exists a two-sided ideal K of R such that*
(i) *R/K is a right V-ring.*
(ii) *K is a direct summand of R.*
(iii) *K is a direct sum of minimal right ideals and injective right ideals of length 2.*

Proof. (⇒). Let C be a complement of $\mathrm{Soc}(R_R)$. Since R is right Noetherian, $\mathrm{Soc}(R_R)$ is finitely generated. Thus by Theorem 2.9 and Proposition 2.12, C is a direct summand of R, say $R = C \oplus K$. Since C is a complement of $\mathrm{Soc}(R_R)$ we conclude that $\mathrm{Soc}(K) = \mathrm{Soc}(R_R) \leq_e K$. Thus K is finitely cogenerated, and so by Proposition 3.4, K is a direct sum of minimal right ideals and injective right ideals of length 2. Now we show that K is a two-sided ideal of R. Let $r \in R$ and define the homomorphism $f : K \to rK$ by the rule $f(x) = rx$. Since rK is isomorphic to a factor module of K, it is Artinian. However C has no nonzero Artinian submodule as it is a complement of $\mathrm{Soc}(R_R)$. Thus $\pi(rK) = 0$, where $\pi : R \to C$ is the canonical projection. Hence $rK \leq K$, and so K is a two-sided ideal of R, as desired. So R/K is a right almost V-ring. Moreover, $R/K \cong C$ has zero socle. Hence by Proposition 2.3(ii), R/K is a right V-ring.

(⇐). Assume that $R = I \oplus K$, where K is a two-sided ideal of R satisfying the condition (iii) and R/K is a right V-ring. Let S be a non-injective simple R-module. By Theorem 3.1, it suffices to show that $E(S)$ is projective of length 2. If $SK = 0$, then S is a simple R/K-module, and so it is an injective R/K-module. But R/K is a projective R-module, and hence, a flat R-module. Thus by [14, Corollary (3.6A)], S is an injective R-module, which is a contradiction. Hence $SK \neq 0$, and so $SK = S$. If $E(S)I \neq 0$, then $E(S)I \cap S \neq 0$, and so $S \leq E(S)I$. Thus $S = SK \leq E(S)IK = 0$, which is impossible. This shows that $E(S)I = 0$. So $xK = xR$, for each $x \in E(S) \setminus S$. Thus the natural map $\pi : K \to xR$ is an epimorphism. But by hypothesis, $K = (\bigoplus_{i=1}^{m} A_i) \bigoplus (\bigoplus_{j=1}^{n} U_j)$, where each A_i is a minimal right ideal, and each U_j is an injective right ideal of length 2. Clearly, each U_j is uniform with simple socle. If $x\mathrm{Soc}(U_j) = 0$ for each j, then $\bigoplus_{j=1}^{n} \mathrm{Soc}(U_j) \leq \ker(\pi)$. Hence xR is a factor of the semisimple R-module $(\bigoplus_{i=1}^{m} A_i) \bigoplus (\bigoplus_{j=1}^{n} U_j/\mathrm{Soc}(U_j))$. So xR is semisimple, which is impossible as $x \in E(S) \setminus S$. This implies that $x\mathrm{Soc}(U_j) \neq 0$, for some j, and so $\mathrm{ann}(x) \cap U_j = 0$. Hence $\pi\,|_{U_j}$ is a monomorphism. Therefore xR contains an injective module of length 2. Therefore $E(S) = xR \cong U_j$. So $E(S)$ is projective of length 2, as desired. □

Corollary 3.7. *If R is a right Noetherian right almost V-ring, then R is a direct sum of right ideals such that their endomorphism rings are either simple or local.*

Proof. By Propositions 3.4 and 3.6, there exists a decomposition $R = K \oplus L$ such that K is an Artinian two-sided ideal of R and R/K is a right V-ring. So by [7, 3.20], R/K is a finite direct product of simple rings. So L is a finite direct sum of right ideals with simple endomorphism rings. On the other hand, by Proposition 3.4, K is a direct sum of minimal right ideals and injective uniform right ideals of length 2. So R is a direct sum of right ideals such that their endomorphism rings are either simple or local. □

Finally, we show that a 2×2 upper triangular matrix ring over a ring R is a right almost V-ring precisely when R is semisimple.

Theorem 3.8. *A 2×2 upper triangular matrix ring over a ring R is a right almost V-ring if and only if R is semisimple.*

Proof. ($\Rightarrow$). It suffices to show that every maximal right ideal L of R is a direct summand of R. Set $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$, and $P = \begin{pmatrix} 0 & L \\ 0 & R \end{pmatrix}$. Let 'bar' denote the image in T/P. Clearly, $S = \overline{\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}}$ is a simple T-submodule of $\overline{T}$, and $C = \overline{\begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}}$ is a complement of S. By Theorem 2.9, C is a direct summand of $\overline{T}$, say $\overline{T} = C \oplus C'$, where $C' = \overline{\begin{pmatrix} I & J \\ 0 & 0 \end{pmatrix}}$. Thus $R = L \oplus I$, that is, L is a direct summand of R, as desired.

($\Leftarrow$). Clearly, T is a (two-sided) Artinian ring. Moreover, $\mathrm{Rad}(T) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, and so $\mathrm{Rad}(T)^2 = 0$. Assume that $R = \bigoplus_{i=1}^{n} K_i$, where each K_i is a minimal right ideal of R. Then $T = (\bigoplus_{i=1}^{n} \begin{pmatrix} K_i & K_i \\ 0 & 0 \end{pmatrix}) \oplus (\bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 0 \\ 0 & K_i \end{pmatrix})$, and hence T is a serial ring. Thus by Corollary 3.5, T is a right almost V-ring. □

Corollary 3.9. *If R is a semisimple ring, then $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is a right almost V-ring but not a right V-ring.*

Proof. By Theorem 3.8, T is a right almost V-ring. Moreover, $\mathrm{Rad}(T) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \neq 0$, and so T is not a right V-ring. □

We conclude the paper with an open question. By Theorem 3.8, a 2×2 upper triangular matrix ring over a ring R is a right almost V-ring if and only if R is semisimple. On the other hand, Example 2.6 shows that a 2×2 generalized upper triangular matrix ring need not be a right almost V-ring, in general. So the following open question arises:

Question 3.10. When a 2×2 generalized upper triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is a right almost V-ring?

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