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# ON SILVERMAN'S CONJECTURE FOR A FAMILY OF ELLIPTIC CURVES

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Abstract. Let $E$ be an elliptic curve over $\mathbb{Q}$ with the given Weierstrass equation $y^2 = x^3 + ax + b$. If $D$ is a squarefree integer, then let $E^{(D)}$ denote the $D$-quadratic twist of $E$ that is given by $E^{(D)} : y^2 = x^3 + aD^2x + bD^3$. Let $E^{(D)}(\mathbb{Q})$ be the group of $\mathbb{Q}$-rational points of $E^{(D)}$. It is conjectured by J. Silverman that there are infinitely many primes $p$ for which $E^{(p)}(\mathbb{Q})$ has positive rank, and there are infinitely many primes $q$ for which $E^{(q)}(\mathbb{Q})$ has rank 0. In this paper, assuming the parity conjecture, we show that for infinitely many primes $p$, the elliptic curve $E_n^{(p)} : y^2 = x^3 - np^2x$ has odd rank and for infinitely many primes $p$, $E_n^{(p)}(\mathbb{Q})$ has even rank, where $n$ is a positive integer that can be written as biquadrates sums in two different ways, i.e., $n = u^4 + v^4 = r^4 + s^4$, where $u, v, r, s$ are positive integers such that $\gcd(u, v) = \gcd(r, s) = 1$. More precisely, we prove that: if $n$ can be written in two different ways as biquartic sums and $p$ is prime, then under the assumption of the parity conjecture $E_n^{(p)}(\mathbb{Q})$ has odd rank (and so a positive rank) as long as $n$ is odd and $p \equiv 5, 7 \pmod 8$ or $n$ is even and $p \equiv 1 \pmod 4$. In the end, we also compute the ranks of some specific values of $n$ and $p$ explicitly.
**Keywords:** Silverman's conjecture, elliptic curve, quadratic twist, rank, parity conjecture.
**MSC(2010):** Primary: 11G05; Secondary: 14H52.

## 1. Introduction

An elliptic curve $E$ over the rational field $\mathbb{Q}$ is the projective curve associated to an affine equation of the form

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}, \tag{1.1}$$

where the cubic polynomial $x^3 + ax + b$ has distinct roots. This is called the Weierastrass normal form. By the Mordell-Weil theorem it is known that the

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set of rational points $E(\mathbb{Q})$ is a finitely generated abelian group, as such it takes the following decomposition:

$$E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}}, \tag{1.2}$$

where $r$ is a nonnegative integer called the *rank* of $E$ and $E(\mathbb{Q})_{\text{tors}}$ is the finite abelian group consisting of all the elements of finite order in $E(\mathbb{Q})$ (see [20, Theorem 6.7, page 239] for more information).
Let us recall the definition of the quadratic twist of an elliptic curve. If $D$ is a squarefree integer, then let $E^{(D)}$ denote the $D$-quadratic twist of $E$ that is given by $E^{(D)}: y^2 = x^3 + aD^2x + bD^3$. The group of $\mathbb{Q}$-rational points of $E^{(D)}$ is shown by $E^{(D)}(\mathbb{Q})$.

In [11], the authors considered the family of elliptic curves defined by

$$E_n : y^2 = x^3 - nx, \tag{1.3}$$

where $n$ is a positive integer that can be written as biquadrates sums in two different ways, i.e., $n = u^4 + v^4 = r^4 + s^4$ where $\gcd(u, v) = \gcd(r, s) = 1$. The 4-tuple $(u, v, r, s)$ with positive integer coordinates satisfying the above conditions is called a primitive solution. In [11, Theorem 1.1] the authors proved that for such $n$ the elliptic curve $y^2 = x^3 - nx$, has rank $\geq 3$. If moreover $n$ is odd and the parity conjecture is true, then it has even rank $\geq 4$.

The Diophantine equation

$$n = u^4 + v^4 = r^4 + s^4, \tag{1.4}$$

was first studied by Euler [7] in 1772 and since then has been considered by numerous mathematicians. Among quartic Diophantine equations, (1.4) has a distinct feature for its simple structure, the almost perfect symmetry between the variables and close relationship with the theory of elliptic functions. The simplest parametric solution of (1.4) that was constructed by Euler [10, Page 201, (13.7.11)] is:

$$\begin{cases} u = a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6, \\ v = a^6b - 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7, \\ r = a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6, \\ s = a^6b + 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7. \end{cases} \tag{1.5}$$

The first known examples of solutions, and among these the solution in least positive integers, i.e., $(u, v, r, s) = (134, 133, 158, 59)$, were cmputed already by Euler [7–9]. But this parametric solution does not exhaust all the possible solutions of the equation (1.4) (see [12, page 21]). Some others were found by later researchers (see [6, pp. 644-647]), but it was not until the advent of computers that systematic searches could be conducted. The most extensive lists published to date are due to Lander and Parkin [13] and Lander, Parkin and Selfridge [14]. Zajta [22], discusses the method for finding such solutions and presents a list of 218 numerical solutions in the range $\max(u, v, r, s) < 10^6$. Choudhry [3], presents a method of deriving new solutions of equation (1.4) starting from a given solution. According to his method, by taking $(\pm 133, \pm 134, \pm 158, \pm 59)$ one can obtain:

| $u$ | $v$ | $r$ | $s$ |
|---|---|---|---|
| 1054067 | 545991 | 522059 | 1057167 |
| 10381 | 10203 | 12231 | 2903 |
| 1453319 | 829418 | 1486969 | 461882 |
| 1137493 | 654854 | 60779 | 1167518 |
| 114613 | 111637 | 134413 | 34813 |
| 6565526 | 3687711 | 6710751 | 1967986 |
| 12178821457 | 7038985479 | 783453421 | 12505169907 |

## 2. Silverman's conjecture and some known results related to it

In this section, we first state Silverman's conjecture and then we briefly discuss some known results related to it.

**Conjecture 2.1** (Silveman's Conjecture)**.** ([16, page250]). *If $E$ is an elliptic curve over the rational field $\mathbb{Q}$, then there are infinitely many primes $p$ for which $E^{(p)}(\mathbb{Q})$, has positive rank, and there are infinitely many primes $q$ for which $E^{(q)}(\mathbb{Q})$ has rank 0 .*

By using 2-descents (see [18, Theorem 3.1, page 229]), one can prove part of this conjecture for the congruent number elliptic curve

$$E' : y^2 = x^3 - x.$$

For instance it is known for prime $p$ that if $p \equiv 3 \pmod 8$, then $E'^{(p)}$ has rank 0 and if $p \equiv 5 \pmod 8$, then $E'^{(2p)}(\mathbb{Q})$ has rank 0. Moreover, it is known that $E'^{(pqr)}(\mathbb{Q})$ has rank 0 if $p$, $q$ and $r$ are primes satisfying

$$p \equiv 1 \pmod 8, \; q \equiv 3 \pmod 8, \; r \equiv 3 \pmod 8, \text{ and } \left(\frac{p}{q}\right) = -\left(\frac{p}{r}\right),$$

where $\left(\frac{p}{q}\right)$ is the Legendre symbol.

On the other hand, for the second part of the conjecture, i.e., the part with positive rank, Monsky [15, Corollary 5.15, page 66] proved that if $p_1$, $p_3$, $p_5$ and $p_7$ denote primes $\equiv 1, 3, 5, 7 \pmod 8$, then the $D$-quadratic twist of $E'$ has positive rank, where $D$ runs through the following numbers:

$$p_5, \; p_7, \; 2p_7 \text{ and } 2p_3,$$

$$p_3p_7, \; p_3p_5, \; 2p_3p_5 \text{ and } 2p_5p_7,$$

$$p_1p_5 \text{ provided } \left(\frac{p_1}{p_5}\right) = -1, \quad p_1p_7 \text{ and } 2p_1p_7 \text{ provided } \left(\frac{p_1}{p_7}\right) = -1,$$

and

$$2p_1p_3 \text{ provided } \left(\frac{p_1}{p_3}\right) = -1.$$

Ono [16, Corollary 3.1, page 349], showed that for some special curves $E$ there is a set $S$ of primes $p$ with density $\frac{1}{3}$ for which if $D = \prod p_j$ is a squarefree integer where $p_j \in S$, then $E^{(D)}$ has rank 0. In particular $E^{(p)}$ has rank 0 for every $p \in S$.

## 3. Main results

Our main results are as follows:

**Theorem 3.1.** *Let $n$ be an odd number of the form (1.4). Then under the assumption of the parity conjecture we have the following:*

(i) *For any primes $p \equiv 5,\ 7 \pmod 8$, the rank of $E_n^{(p)}(\mathbb{Q})$ is odd and therefore positive. Consequently, the first part of Silverman's conjecture is true.*
(ii) *For any primes $p \equiv 1,\ 3 \pmod 8$, the rank of $E_n^{(p)}(\mathbb{Q})$ is even.*

**Theorem 3.2.** *Let $n$ be an even number of the form (1.4). Then under the assumption of the parity conjecture we have the following:*

(i) *For any primes $p \equiv 1 \pmod 4$, the rank of $E_n^{(p)}(\mathbb{Q})$ is odd and therefore positive. Consequently, the first part of Silverman's conjecture is true.*
(ii) *For any primes $p \equiv 3 \pmod 4$, the rank of $E_n^{(p)}(\mathbb{Q})$ is even.*

Before giving the proof of these theorems, we state a couple of necessary facts from the literature.

First of all, the *Parity Conjecture* which states that an elliptic curve $E$ over $\mathbb{Q}$ with the rank $r$ satisfies

$$\omega(E) = (-1)^r,$$

where $\omega(E)$ is the sign of the functional equation of the Hasse-Weil $L$-function $L(E, s)$. It is known [1] that for $n \not\equiv 0 \pmod 4$, a fourth power free integer, the sign of the functional equation, denoted $\omega(E_n)$, for the elliptic curve

$$E_n : y^2 = x^3 - nx,$$

is given by

$$\omega(E_n) = \operatorname{sgn}(-n) \cdot \epsilon(n) \cdot \prod_{l^2 \| n} \left(\frac{-1}{l}\right), \tag{3.1}$$

where $l \geq 3$ denotes a prime and

$$\epsilon(n) = \begin{cases} -1, & n \equiv 1, 3, 11, 13 \pmod{16}, \\ 1, & n \equiv 2, 5, 6, 7, 9, 10, 14, 15 \pmod{16}. \end{cases} \tag{3.2}$$

(see Ono and Ono [17, Equations (1), (2), and (3)]).

Second, the following proposition and Theorem 3.6 are also useful in the proof of our main results. Let us recall the following lemma which will be essential in the proof of Proposition 3.4.

**Lemma 3.3.** *Let $n$ be a nonzero integer, and let $p$ be an odd prime not dividing $n$. Then*

$$p \mid x^2 + ny^2,\ \gcd(x, y) = 1 \Longleftrightarrow \left(\frac{-n}{p}\right) = 1.$$

*Proof.* See ([4, Lemma 1.7, page 13]). □

**Proposition 3.4.** *Let $n = u^4 + v^4 = r^4 + s^4$ be such that $\gcd(u, v) = 1$. If $l \mid n$ for an odd prime number $l$, then $l = 8k + 1$ for some $k \in \mathbb{Z}$.*

*Proof.* Without loss of generality we can assume that $n$ is not divisible by 4. We use Lemma 3.3. Let $l$ be an odd prime factor of $n$. One can write

$$n = u^4 + v^4 = (u^2 - v^2)^2 + 2(uv)^2,$$

so,

$$l \mid (u^2 - v^2)^2 + 2(uv)^2.$$

According to Lemma 3.3, $x = u^2 - v^2$, $y = uv$ and $m = 2$. Therefore, $\left(\frac{-2}{l}\right) = 1$ which implies that $l = 8k + 1$ or $l = 8k + 3$. On the other hand, $n = (u^2 + v^2)^2 - 2(uv)^2$, so

$$l \mid (u^2 + v^2)^2 - 2(uv)^2.$$

We get $\left(\frac{2}{l}\right) = 1$ which implies that $l = 8k + 1$ or $l = 8k + 7$. Putting these two results together we get $l = 8k + 1$. □

**Remark 3.5.** By Proposition 3.4, every prime number $l$ dividing $n$ is in the form $8k + 1$ which in turn is in the form $4k + 1$ as well. Therefore by the quadratic reciprocity law (see [21, page153] ), for every prime factor $l$ of $n$ such that $l^2 \| n$ we have $\left(\frac{-1}{l}\right) = 1$. This shows that these prime factors can be ignored in evaluation of $\omega(E_n)$.

**Theorem 3.6.** *(Dirichlet's Theorem) If $n$ is a positive integer and $a$ and $b$ have no common divisor except 1, then there are infinitely many primes of the form $an + b$.*

*Proof.* See ([10, Theorem 15, page 13]). □

By the above facts in our disposal, we are now ready to investigate Silverman's conjecture for the elliptic curve (1.3). To this end, we first take the $p$-quadratic twists

$$E_n^{(p)} : y^2 = x^3 - p^2nx, \tag{3.3}$$

where $n$ is in the form (1.4) and the primes $p$ satisfying $\gcd(p, n) = 1$, and then compute the sign of $\omega(E_n^{(p)})$ to characterize the parity of the rank for each curve.

It is a well-known fact that every odd prime $p$ can be represented as $p \equiv 1 \pmod 4$, or $p \equiv 3 \pmod 4$. Furthermore one can easily check that square of every prime $p$ can be written as $p^2 \equiv 1 \pmod{16}$, or $p^2 \equiv 9 \pmod{16}$. It is also clear that for each $p$ and each $n$ we have $\operatorname{sgn}(-p^2n) = -1$. Having said that, we are now ready to prove our results.

**Proof of Theorem 3.1 .** $(i)$ In this case, $u$ and $v$ have opposite parities. Without loss of generality, let $u \equiv 1 \pmod 2$ and $v \equiv 0 \pmod 2$, then we have

$$n = u^4 + v^4 \equiv 1 \pmod{16}.$$

Next, based on the different choices for the primes $p$, we get the different results for $\omega(E_n^{(p)})$. We have the following possibilities:

(a): $p \equiv 1 \pmod 4$ and $p^2 \equiv 9 \pmod{16}$.

Since $n \equiv 1 \pmod{16}$, we have $p^2n \equiv 9 \pmod{16}$ and then from (3.2) we have $\epsilon(p^2n) = 1$ implying that $\omega(E_n^{(p)}) = -1$ from (3.1). Therefore, under the assumption of the parity conjecture the rank is odd and indeed it is positive. For primes of the form $p \equiv 1 \pmod 4$, we have $p = 8k + 1$, or $p = 8k + 5$ for some $k \in \mathbb{Z}$. These facts along with $p^2 \equiv 9 \pmod{16}$ implies that $p \equiv 5 \pmod 8$. By Dirichlet's theorem, there are infinitely many such primes and then, the first part of Silverman's conjecture is true.

(b): $p \equiv 3 \pmod 4$ and $p^2 \equiv 1 \pmod{16}$.

Similarly for this case from (3.2), we get $\epsilon(p^2n) = -1$. Moreover, $\left(\frac{-1}{p}\right) = -1$. So by (3.1), $\omega(E_n^{(p)}) = -1$ meaning that the rank is odd (under the assumption of the parity conjecture) and indeed it is positive again. If $p \equiv 3 \pmod 4$, then $p = 8k + 3$ or $p = 8k + 7$, for some $k \in \mathbb{Z}$. From these two and $p^2 \equiv 1 \pmod{16}$, we get $p = 8k + 7$ and by Dirichlet's theorem, there are infinitely many such primes. Therefore, in this case the first part of Silverman's conjecture is true as well.

Like the previous case, we have

$$n \equiv 1 \pmod{16},$$

and so, we can consider two different cases as follows:
(a): $p \equiv 1 \pmod 4$ and $p^2 \equiv 1 \pmod{16}$.
Since $n \equiv 1 \pmod{16}$, so $p^2 n \equiv 1 \pmod{16}$. Thus, (3.2) implies that $\epsilon(p^2 n) = -1$. It is clear that $\left(\frac{-1}{p}\right) = 1$. In this case from (3.1) we have, $\omega(E_n^{(p)}) = 1$. Therefore, the parity conjecture states the rank of the elliptic curve (3.3) is even.
(b): $p \equiv 3 \pmod 4$ and $p^2 \equiv 9 \pmod{16}$.
In this case one can easily check that $\epsilon(p^2 n)$ in (3.2) equals to 1. Moreover, $\left(\frac{-1}{p}\right) = -1$, and so from (3.1), $\omega(E_n^{(p)}) = 1$. Therefore the rank is even as well.

□

**Proof of Theorem 3.2.** (i) In this case, $u$ and $v$ are both odd and then $n = u^4 + v^4 \equiv 2 \pmod{16}$. Now, as the previous theorem, we consider different cases for the primes $p$ as follows:
(a): $p \equiv 1 \pmod 4$ and $p^2 \equiv 9 \pmod{16}$.
It is clear that $p^2 n \equiv 2 \pmod{16}$, and so by (3.2), $\epsilon(p^2 n) = 1$. Therefore, (3.1) shows that $\omega(E_n^{(p)}) = -1$ and so, under the assumption of the parity conjecture the rank is positive. We have seen that these primes are in the form of $8k + 5$ and there are infinitely many such primes. It turns out the first part of Silverman's conjecture is true.
(b): $p \equiv 1 \pmod 4$ and $p^2 \equiv 1 \pmod{16}$.

Obviously, $p^2 n \equiv 2 \pmod{16}$ and so from (3.1), $\omega(E_n^{(p)}) = -1$. Similar to the previous cases, based on the parity conjecture one can claim that the rank is positive. We know that these primes are in the form of $8k+1$ and by Dirichlet's theorem, there are infinitely many primes of this form. Consequently, the first part of Silverman's conjecture is true.

(ii) As we mentioned in the proof of (i), one can easily check that $u$ and $v$ are both odd and then $n = u^4 + v^4 \equiv 2 \pmod{16}$. Now, we consider two cases:
(a): $p \equiv 3 \pmod 4$ and $p^2 \equiv 1 \pmod{16}$.
In this case, $\left(\frac{-1}{p}\right) = -1$ and (3.2) shows that $\epsilon(p^2 n) = 1$ which together with (3.1), implies $\omega(E_n^{(p)}) = 1$. So, under the assumption of the parity conjecture the rank of $E^{(p)}(\mathbb{Q})$ is an even number.
(b): $p \equiv 3 \pmod 4$ and $p^2 \equiv 9 \pmod{16}$.
Finally in the latest case, we have $\left(\frac{-1}{p}\right) = -1$, $\epsilon(p^2 n) = 1$. Therefore, $\omega(E_n^p) = 1$, then the rank of $E_n^{(p)}(\mathbb{Q})$ in this case must be even ( by the parity conjecture). □

The following examples show computations for some specific values for $n$ and $p$. These computations were done by SAGE [19] and Cremona's MWRANK [5] softwares.

**Remark 3.7.** Determining the rank of an elliptic curve is a challenging problem and in a lot of cases MWRANK program can not compute the rank, in these cases it gives the upper and lower bounds for the rank.

**Example 3.8.** By taking $n = 635318657$ and $p < 1000000$ in Theorem 3.1 (i), the maximal rank that we found, was 5 and there are exactly two such elliptic curves, namely:

$$y^2 = x^3 - 886117685355977x,$$

$$y^2 = x^3 - 1608116501388523697x.$$

The former curve corresponds to $p = 1181$, where $p \equiv 1 \pmod 4$ and $p^2 \equiv 9 \pmod{16}$. The latter one corresponds to $p = 50311$, where $p \equiv 3 \pmod{16}$ and $p^2 \equiv 1 \pmod{16}$.

**Example 3.9.** In Tables 1 and 2, we summarized the results for the number $n = 635318657$ in Theorem 3.1 (ii). In the former case all primes $p \equiv 1 \pmod 8$ in the range 1000 have been considered for which there are no curves with rank 0. Table 2 shows the results for primes $p \equiv 3 \pmod 8$ in the range 1000. In this case we found only 7 curves having rank 0.

TABLE 1. Primes $\equiv 1 \pmod 8$ and $< 1000$

| $p$ | $rank$ |
|---|---|
| $89,\ 929,\ 977$ | $0 \le rank \le 2$ |
| $137,\ 881$ | $rank = 2$ |
| $97,\ 113,\ 233,\ 257,\ 337,\ 353,\ 409,$<br>$433,\ 521,\ 601,\ 641,\ 673,\ 761,\ 937,\ 953$ | $2 \le rank \le 4$ |
| $17,\ 41,\ 73,\ 193,\ 241,\ 281,\ 313,\ 401,\ 449,\ 457,\ 569,$<br>$577,\ 593,\ 617,\ 769,\ 809,\ 857,$ | $0 \le rank \le 4$ |

**Example 3.10.** Let $n = 156700232476402$. We considered primes $p \equiv 3 \pmod 4$ in the range the 1000 and among them, for

$$p = 59,\ 131,\ 163,\ 211,\ 307,\ 331,\ 347,\ 379,\ 571,\ 587,\ 647,\ 691,\ 911,$$

the rank of $E_n^{(p)}$ is 0.

Table 2. Primes $\equiv 3 \pmod 8$ and $< 1000$

| $p$ | $rank$ |
|---|---|
| 19, 59, 179, 491, 523, 587, 971 | $rank = 0$ |
| 11, 83, 211, 251, 283, 331, 379, 499, 547, 619, 659, 683, 811 | $rank = 2$ |
| 3, 43, 67, 107, 131, 139, 163, 227, 307, 347, 419, 443, 43, 467, 563, 571, 643 691, 739, 787, 827, 859, 883, 907, 947 | $0 \leq rank \leq 2$ |

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