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EVERY CLASS OF S-ACTS HAVING A FLATNESS PROPERTY IS CLOSED UNDER DIRECTED COLIMITS

H. QIAO*, L. WANG AND X. MA

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Abstract. Let S be a monoid. In this paper, we prove every class of S-acts having a flatness property is closed under directed colimits, it extends some known results. Furthermore this result implies that every S-act has a flatness cover if and only if it has a flatness precover.
Keywords: Flatness property, colimit, closed.
MSC(2010): Primary: 20M30; Secondary: 20M50.

1. Introduction

Throughout this paper, S always stands for a monoid. A nonempty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right; that is, there exists a mapping $A \times S \to A, (a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for every $a \in A$ and all $s, t \in S$. All right S-acts and their homomorphisms form a category which is denoted by Act-S. Similarly, S-Act is the category of all left S-acts and their homomorphisms. Now we give the definition of colimits of S-acts.

Let I be a set with a preorder (that is, a reflexive and transitive relation). A direct system is a collection of S-acts $(X_i)_{i\in I}$ together with S-maps $\phi_{i,j} : X_i \to X_j$ for all $i \leq j \in I$ such that

1. $\phi_{i,i} = 1_{X_i}$, for all $i \in I$; and
2. $\phi_{j,k}\phi_{i,j} = \phi_{i,k}$, whenever $i \leq j \leq k$.

The colimit of the system $(X_i)_{i\in I}$ is an S-act X together with S-maps $\alpha_i : X_i \to X$ such that

1. $\alpha_i = \alpha_j\phi_{i,j}$, whenever $i \leq j$,
2. If Y is an S-act and $\beta_i : X_i \to Y$ are S-maps such that $\beta_i = \beta_j\phi_{i,j}$ whenever $i \leq j$, then there exists a unique S-map $\psi : X \to Y$ such that the

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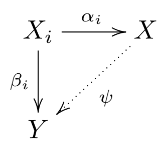
diagram

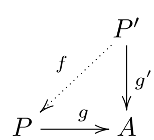


commutes for all $i \in I$.

If the indexing set I satisfies the property that for all $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$ then we say that I is *directed*. In this case we call the colimit a *directed colimit*.

By [[1]], the colimit of S-acts is easy to demonstrate. In fact let $\lambda_i : X_i \to \dot{\bigcup\limits_{i\in I}} X_i$ be the natural inclusion and let ρ be the right congruence on $\dot{\bigcup\limits_{i\in I}} X_i$ generated by $R = \{(\lambda_i(x_i), \lambda_j(\phi_{i,j}(x_i)))|x_i \in X_i, i \leq j \in I\}$. Then $X = (\dot{\bigcup\limits_{i\in I}} X_i)/\rho$ and $\alpha_i : X_i \to X$ given by $\alpha_i(x_i) = \lambda_i(x_i)\rho$ are such that (X, α_i) is a colimit of $(X_i, \phi_{i,j})$.

Let S be a monoid, A an S-act, and $\mathcal{X}$ a class of S-acts. In 2012, Bailey and Renshaw initiated the study of Enochs' notion of cover to the category of acts over monoids. They introduced the concept of an $\mathcal{X}$-cover and $\mathcal{X}$-precover for a class $\mathcal{X}$ of S-acts. This is the analogue of Enochs' definition for covers of modules over rings. An S-map $g : P \to A$ for some $P \in \mathcal{X}$ is called an *$\mathcal{X}$-precover* of an S-act A, if for every S-map $g' : P' \to A$, for $P' \in \mathcal{X}$, there exists an S-map $f : P' \to P$ with $g' = gf$. That is the following diagram



commutes. If in addition the precover satisfies the condition that each S-map $f : P \to P$ with $gf = g$ is an isomorphism, then we shall call it an *$\mathcal{X}$-cover.*

Pullbacks in the category of left S-acts are defined as in any category. Note that pullbacks do not necessarily exist in this category. If a pullback of the homomorphisms f: ${}_SM \to_S Q$ and g: ${}_SN \to_S Q$ does exist in the category of left S-acts, then it is determined up to isomorphism, and we may assume that it is equal to

$$P = \{(m, n) \in_S M \times_S N | f(m) = g(n)\}$$

together with the restrictions p_1 and p_2 of the projections of ${}_SM \times_S N$ onto ${}_SM$ and ${}_SN$, respectively. The pullback diagram

$$\begin{array}{ccc} {}_SP & \xrightarrow{p_1} & {}_SM \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ {}_SN & \xrightarrow[g]{} & {}_SQ \end{array}$$

in the category of left S-acts will be henceforth denoted by $P(M, N, f, g, Q)$.

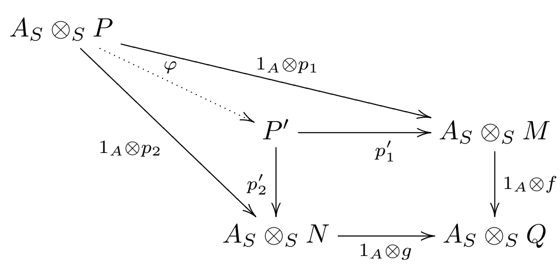
Tensoring the pullback diagram $P(M, N, f, g, Q)$ by any right S-act A_S produces the commutative diagram

$$\begin{array}{ccc} A_S \otimes_S P & \xrightarrow{1_A \otimes p_1} & A_S \otimes_S M \\ {\scriptstyle 1_A \otimes p_2}\downarrow & & \downarrow{\scriptstyle 1_A \otimes f} \\ A_S \otimes_S N & \xrightarrow[1_A \otimes g]{} & A_S \otimes_S Q \end{array}$$

in the category of sets. For the pullback of mappings $1_A \otimes f$ and $1_A \otimes g$ in the category of sets we may take

$$P' = \{(a \otimes m, a' \otimes n) \in (A_S \otimes_S M) \times (A_S \otimes_S N) | a \otimes f(m) = a' \otimes g(n)\}.$$

It follows from the definition of pullbacks that there exists a unique mapping $\varphi : A_S \otimes_S P \to P'$ such that the diagram



is commutative. This mapping is given by

$$\varphi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

for every $a \in A_S$ and $(m, n) \in {}_SP$, and will be called the *mapping φ corresponding to the pullback diagram* $P(M, N, f, g, Q)$(for A_S).

All kinds of flatness properties of S-acts are investigated in many articles, such as [[1]- [10]]. For a complete discussion of flatness of S-acts, the reader is referred to [[3, 7]]. And all the flatness properties and their relations are as follows.

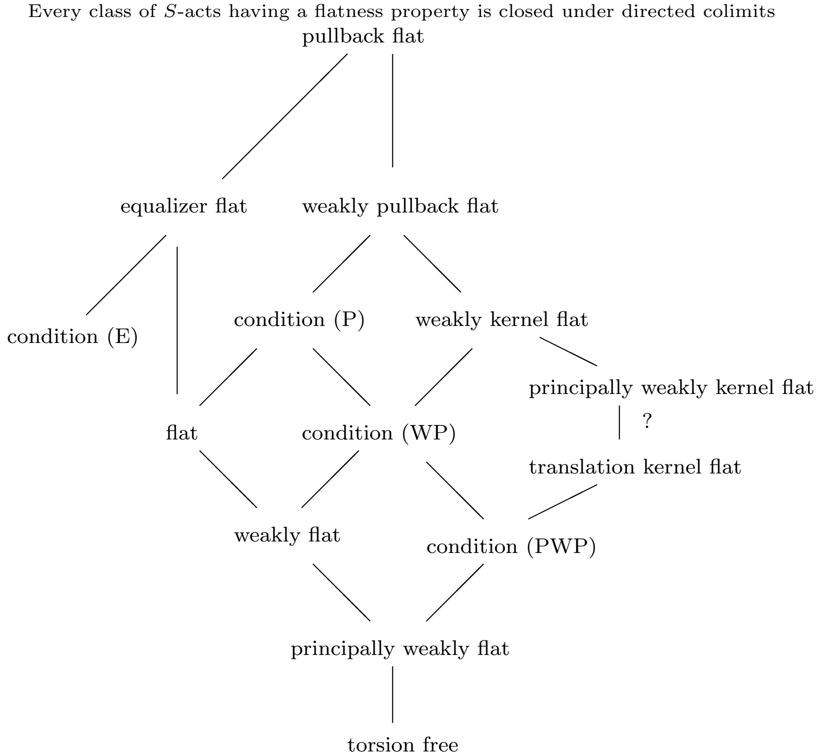


Figure 1

In 2012, Bailey and Renshaw prove the following result.

Theorem 1.1 ([[1], Theorem 4.11]). *Let S be a monoid, let A be an S-act and let $\mathcal{X}$ be a class of S-acts closed under directed colimits. If A has an $\mathcal{X}$-precover then A has an $\mathcal{X}$-cover.*

From this result, it is clear that it is an important problem to find the classes of S-acts which are closed under directed colimits. So far, it has only been proved that every class of S-acts having some flatness properties is closed under directed colimits, such as strongly flat property, condition (P), flatness and torsion freeness. But for other flatness properties, the results are not known. In this paper, after basic results and definitions, we prove every class of S-acts having a flatness property is closed under directed colimits.

2. Main results

In order to prove our main result. We need the following two lemmas.

Lemma 2.1 ([[5], Proposition 8.1.8]). *Let S be a monoid, $a, a' \in A_S, b, b' \in_S B$. Then $a \otimes b = a' \otimes b'$ in $A_S \otimes_S B$ if and only if there exist a natural number* *k and elements* $a_1, \cdots, a_k \in A_S, b_2, \cdots, b_k \in {}_SB$, $s_1, t_1, \cdots, s_k, t_k \in S$ *such that*

$$\begin{array}{ll} a = a_1 s_1, & \\ a_1 t_1 = a_2 s_2, & s_1 b = t_1 b_2, \\ \vdots & \vdots \\ a_k t_k = a', & s_k b_k = t_k b'. \end{array}$$

Lemma 2.2 ([[9], Lemma 3.5 and Corollary 3.6])**.** *Let* $(X_i, \phi_{i,j})$ *be a direct system of* S*-acts and* S*-morphisms with a directed index set and with directed colimit* (X, α_i)*. Then* $\alpha_i(x_i) = \alpha_j(x_j)$ *if and only if* $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$ *for some* $k \geq i, j$*. Consequently* α_i *is a monomorphism if and only if* $\phi_{i,k}$ *is a monomorphism for all* $k \geq i$*.*

We begin by proving the following result.

Lemma 2.3. *Let* S *be a monoid, let* $(A_i, \phi_{i,j})$ *be a direct system of* S*-acts with directed index set* I *and let* (A, α_i) *be the directed colimit. Suppose for each* A_i *the mapping* φ *corresponding to the pullback diagram* $P(M, N, f, g, Q)$ *is surjective, then for* A *the mapping* φ *corresponding to the pullback diagram* $P(M, N, f, g, Q)$ *is surjective.*

Proof. Suppose that $(x \otimes m, y \otimes n)$ belongs to the pullback of $P(A \otimes M, A \otimes N, 1_A \otimes f, 1_A \otimes g, A \otimes Q)$, where $x, y \in A, m \in M, n \in N$. Then $x \otimes f(m) = y \otimes g(n)$ in $A \otimes Q$. If we can find some $a \in A$, $(m', n') \in P(M, N, f, g, Q)$ such that $\varphi(a \otimes (m', n')) = (x \otimes m, y \otimes n)$, then the result follows.

Since $x \otimes f(m) = y \otimes g(n)$, by Lemma 2.1 there exist a natural number k and elements $a_1, \cdots, a_k \in A_S, q_2, \cdots, q_k \in {}_SQ$, $s_1, t_1, \cdots, s_k, t_k \in S$ such that

$$\begin{array}{ll} x = a_1 s_1, & \\ a_1 t_1 = a_2 s_2, & s_1 f(m) = t_1 q_2, \\ \vdots & \vdots \\ a_k t_k = y, & s_k q_k = t_k g(n). \end{array}$$

Denote x by a_0 and y by a_{k+1}, so there exist $a'_{i_j} \in A_{i_j}$ with $a_j = \alpha_{i_j}(a'_{i_j})$, where $i_j \in I$ and $j = 0, 1, \cdots, k, k+1$. Hence we have

$$\begin{array}{ll} \alpha_{i_0}(a'_{i_0}) = \alpha_{i_1}(a'_{i_1} s_1), & \\ \alpha_{i_1}(a'_{i_1} t_1) = \alpha_{i_2}(a'_{i_2} s_2), & s_1 f(m) = t_1 q_2, \\ \vdots & \vdots \\ \alpha_{i_k}(a'_{i_k} t_k) = \alpha_{i_{k+1}}(a'_{i_{k+1}}), & s_k q_k = t_k g(n). \end{array}$$

Since I is directed, by Lemma 2.2 we can always find some $l \geq i_0, i_1, \cdots, i_{k+1}$ such that

$$\begin{aligned} &\phi_{i_0,l}(a'_{i_0}) = \phi_{i_1,l}(a'_{i_1})s_1, && \\ &\phi_{i_1,l}(a'_{i_1})t_1 = \phi_{i_2,l}(a'_{i_2})s_2, && s_1 f(m) = t_1 q_2, \\ &\qquad \vdots && \qquad \vdots \\ &\phi_{i_k,l}(a'_{i_k})t_k = \phi_{i_{k+1},l}(a'_{i_{k+1}}), && s_k q_k = t_k g(n). \end{aligned}$$

Hence we have $\phi_{i_0,l}(a'_{i_0}) \otimes f(m) = \phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes g(n)$ in $A_l \otimes Q$. That is $(\phi_{i_0,l}(a'_{i_0}) \otimes m, \phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n)$ belongs to the pullback of $P(A_l \otimes M, A_l \otimes N, 1_{A_l} \otimes f, 1_{A_l} \otimes g, A_l \otimes Q)$. By assumption, for A_l the mapping φ corresponding to the pullback diagram $P(M,N,f,g,Q)$ is surjective, so there exist $a'' \in A_l, m' \in M$ and $n' \in N$ such that $\varphi(a'' \otimes (m',n')) = (\phi_{i_0,l}(a'_{i_0}) \otimes m, \phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n)$, where $f(m') = g(n')$. That is $\phi_{i_0,l}(a'_{i_0}) \otimes m = a'' \otimes m'$ and $\phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n = a'' \otimes n'$.

Since $\phi_{i_0,l}(a'_{i_0}) \otimes m = a'' \otimes m'$, by Lemma 2.1, there exist a natural number p and elements $c_1, \cdots, c_p \in A_l, m_2, \cdots, m_p \in {}_S M$, $u_1, v_1, \cdots, u_p, v_p \in S$ such that

$$\begin{aligned} &\phi_{i_0,l}(a'_{i_0}) = c_1 s_1, && \\ &c_1 t_1 = c_2 s_2, && u_1 m = v_1 m_2, \\ &\qquad \vdots && \qquad \vdots \\ &c_p t_p = a'', && u_p m_p = v_p m'. \end{aligned}$$

Acting α_l on left column equations and since $x = \alpha_{i_0}(a'_{i_0}) = \alpha_l \phi_{i_0,l}(a'_{i_0})$ we have

$$\begin{aligned} &x = \alpha_l(c_1)s_1, && \\ &\alpha_l(c_1)t_1 = \alpha_l(c_2)s_2, && u_1 m = v_1 m_2, \\ &\qquad \vdots && \qquad \vdots \\ &\alpha_l(c_p)t_p = \alpha_l(a''), && u_p m_p = v_p m'. \end{aligned}$$

Hence $x \otimes m = \alpha_l(a'') \otimes m'$ in $A \otimes M$. Since $\phi_{i_{k+1},l}(a'_{i_{k+1}}) \otimes n = a'' \otimes n'$, by a similar way we can prove $y \otimes n = \alpha_l(a'') \otimes n'$ in $A \otimes N$. Now we finally have $\varphi(\alpha_l(a'') \otimes (m',n')) = (\alpha_l(a'') \otimes m', \alpha_l(a'') \otimes n') = (x \otimes m, y \otimes n)$. □

Lemma 2.4. *Let S be a monoid, let $(A_i, \phi_{i,j})$ be a direct system of S-acts with directed index set and let (A, α_i) be the directed colimit. Suppose for each A_i the mapping φ corresponding to the pullback diagram $P(M,N,f,g,Q)$ is injective, then for A the mapping φ corresponding to the pullback diagram $P(M,N,f,g,Q)$ is injective.*

Proof. For any $a, a' \in A, m, m' \in M$ and $n, n' \in N$, suppose

$$a \otimes m = a' \otimes m' \ \text{ in } A_S \otimes_S M, \quad f(m) = g(n)$$
$$a \otimes n = a' \otimes n' \ \text{ in } A_S \otimes_S N, \quad f(m') = g(n').$$

That is $\varphi(a \otimes (m,n)) = \varphi(a' \otimes (m',n'))$ belongs to the pullback of $P(A \otimes M, A \otimes N, 1_A \otimes f, 1_A \otimes g, A \otimes Q)$, where $(m,n), (m',n') \in \ A_S \otimes_S P$.

We will show that $a \otimes (m,n) = a' \otimes (m',n')$ in $A_S \otimes_S P$. Since $a \otimes m = a' \otimes m'$ in $A_S \otimes_S M$, as in the proof of Lemma 2.3, we can always find some $l_1 \geq i_0, i_{p+1}$ and $a_{i_0} \in A_{i_0}$, $a'_{i_{p+1}} \in A_{i_{p+1}}$ with $\alpha_{i_0}(a_{i_0}) = a$, $\alpha_{i_{p+1}}(a'_{i_{p+1}}) = a'$ such that

$$\phi_{i_0,l_1}(a'_{i_0}) \otimes m = \phi_{i_{p+1},l_1}(a'_{i_{p+1}}) \otimes m' \ \text{ in } A_{l_1} \otimes_S M.$$

where $i_0, i_{p+1} \in I$. Similarly, we can find $l_2 \geq j_0, j_{q+1}$ and $a_{j_0} \in A_{j_0}$, $a'_{j_{q+1}} \in A_{j_{q+1}}$ with $\alpha_{j_0}(a_{j_0}) = a$, $\alpha_{j_{q+1}}(a'_{j_{q+1}}) = a'$ such that

$$\phi_{i_0,l_2}(a'_{i_0}) \otimes n = \phi_{i_{q+1},l_2}(a'_{i_{q+1}}) \otimes n' \ \text{ in } A_{l_2} \otimes_S N.$$

Then we can always find $l \geq l_1, l_2$ such that

$$\phi_{i_0,l}(a'_{i_0}) \otimes m = \phi_{i_{q+1},l}(a'_{i_{q+1}}) \otimes m' \ \text{ in } A_l \otimes_S M, \quad f(m) = g(n)$$
$$\phi_{i_0,l}(a'_{i_0}) \otimes n = \phi_{i_{q+1},l}(a'_{i_{q+1}}) \otimes n' \ \text{ in } A_l \otimes_S N, \quad f(m') = g(n').$$

Hence $\varphi(\phi_{i_0,l}(a'_{i_0}) \otimes (m,n)) = \varphi(\phi_{i_{q+1},l}(a'_{i_{q+1}}) \otimes (m',n'))$. By assumption, for A_l the mapping φ corresponding to the pullback diagram $P(M,N,f,g,Q)$ is injective, hence $\phi_{i_0,l}(a'_{i_0}) \otimes (m,n) = \phi_{i_{q+1},l}(a'_{i_{q+1}}) \otimes (m',n')$ in $A_l \otimes_S P$. By Lemma 2.1 there exist a natural number r and elements $d_1, \cdots, d_r \in A_l, (m_2,n_2), \cdots, (m_r,n_r) \in_S P$, $x_1, y_1, \cdots, x_r, y_r \in S$ such that

$$\begin{array}{ll}
\phi_{i_0,l}(a'_{i_0}) = d_1 x_1, & \\
d_1 y_1 = d_2 x_2, & x_1(m,n) = y_1(m_2,n_2), \\
\vdots & \vdots \\
d_r y_r = \phi_{i_{q+1},l}(a'_{i_{q+1}}), & x_r(m_r,n_r) = y_r(m',n').
\end{array}$$

Acting α_l on left column equations and since $a = \alpha_{i_0}(a'_{i_0}) = \alpha_l \phi_{i_0,l}(a'_{i_0})$ and $a' = \alpha_{i_{q+1}}(a'_{i_{q+1}}) = \alpha_l \phi_{i_{q+1},l}(a'_{i_{q+1}})$ we have

$$\begin{array}{ll}
a = \alpha_l(d_1) x_1, & \\
\alpha_l(d_1) y_1 = \alpha_l(d_2) x_2, & x_1(m,n) = y_1(m_2,n_2), \\
\vdots & \vdots \\
\alpha_l(d_r) y_r = a', & x_r(m_r,n_r) = y_r(m',n').
\end{array}$$

Now we have proved that $a \otimes (m,n) = a' \otimes (m',n')$ in $A_S \otimes_S P$. □

Definition 2.5 ([[6]])**.** An act A_S is called equalizer flat if the functor $A_S \otimes_S -$ preserves equalizers.

Now we will prove that every directed colimit of a direct system of equalizer flat S-acts is equalizer flat.

Lemma 2.6 ([[8], Lemma 1.1 and Corollary 1.2])**.** *Let*

$$E \xrightarrow{\ l\ } X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$$

be a commutative diagram in S-Act. Then this diagram is an equalizer if and only if E is isomorphic to the S-act $E = \{x \in X | f_1(x) = f_2(x)\}$, where $l(x) = x$.

Proposition 2.7. *Let S be a monoid. Every directed colimit of a direct system of equalizer flat S-acts is equalizer flat.*

Proof. Let $(A_i, \phi_{i,j})$ be a direct system of S-acts and S-morphisms with a directed index set and with directed colimit (A, α_i). If every A_i is equalizer flat, we will prove that A is equalizer flat. Let the pair (E, l) be an equalizer in the following diagram

$$E \xrightarrow{\ l\ } X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$$

Since every equalizer flat S-act is flat, then by Lemma 2.4 and [[7], Proposition 4] A_S is flat and we have the following diagram

$$A \otimes E \xrightarrow{1_A \otimes l} A \otimes X \underset{1_A \otimes f_2}{\overset{1_A \otimes f_1}{\rightrightarrows}} A \otimes Y$$

and $1_A \otimes l$ is a monomorphism. By definition, the equalizer of $1_A \otimes f_1$ and $1_A \otimes f_2$ is $E' = \{a \otimes x \in A \otimes X | (1_A \otimes f_1)(a \otimes x) = (1_A \otimes f_2)(a \otimes x), a \in A, x \in X\}$. By the definition of an equalizer, it is clear that $A \otimes E \subseteq E'$, we only need to prove that $E' \subseteq A \otimes E$. Suppose $a \in A, x \in X$ such that $a \otimes x \in E'$. Since $a \otimes f_1(x) = a \otimes f_2(x)$ in $A \otimes Y$. By Lemma 2.1 there exist a natural number n and elements $a_1, \cdots, a_n \in A_S, y_2, \cdots, y_n \in {}_SY$, $s_1, t_1, \cdots, s_n, t_n \in S$ such that

$$\begin{aligned}
&a = a_1 s_1, && \\
&a_1 t_1 = a_2 s_2, && s_1 f_1(x) = t_1 y_2, \\
&\quad\vdots && \quad\vdots \\
&a_n t_n = a, && s_n y_n = t_n f_2(x).
\end{aligned}$$

Denote a by a_0 and a' by a_{n+1}, then there exist $a'_{i_j} \in A_{i_j}$ with $a_j = \alpha_{i_j}(a'_{i_j}), j = 0, 1, \cdots, n$. Hence we have

$$\begin{aligned} &\alpha_{i_0}(a'_{i_0}) = \alpha_{i_1}(a'_{i_1} s_1), && \\ &\alpha_{i_1}(a'_{i_1} t_1) = \alpha_{i_2}(a'_{i_2} s_2), && s_1 f_1(x) = t_1 y_2, \\ &\qquad \vdots && \qquad \vdots \\ &\alpha_{i_n}(a'_{i_n} t_n) = \alpha_{i_0}(a'_{i_0}), && s_n y_n = t_n f_2(x). \end{aligned}$$

Since I is directed, by Lemma 2.2 we can always find some $l \geq i_1, i_2, \cdots, i_n$ such that

$$\begin{aligned} &\phi_{i_0,l}(a'_{i_0}) = \phi_{i_1,l}(a'_{i_1}) s_1, && \\ &\phi_{i_1,l}(a'_{i_1}) t_1 = \phi_{i_2,l}(a'_{i_2}) s_2, && s_1 f_1(x) = t_1 y_2, \\ &\qquad \vdots && \qquad \vdots \\ &\phi_{i_n,l}(a'_{i_n}) t_n = \phi_{i_0,l}(a'_{i_0}), && s_n y_n = t_n f_2(x). \end{aligned}$$

This means that $\phi_{i_0,l}(a'_{i_0}) \otimes f_1(x) = \phi_{i_0,l}(a'_{i_0}) \otimes f_2(x)$ in $A_l \otimes Y$. Since A_l is equalizer flat, the equalizer of $1_{A_l} \otimes f_1$ and $1_{A_l} \otimes f_2$ is $A_l \otimes E$ and $\phi_{i_0,l}(a'_{i_0}) \otimes x \in A_l \otimes E$. Then $x \in E$ and $a \otimes x \in A \otimes E$. □

Remark 2.8. In [[3, 7]], except equalizer flatness, by systematically varying the requirements on φ and the types of pullbacks considered, the author obtains all of the known flatness conditions in figure 1. For example, let A_S be a right S-act, they prove:
(1)A_S satisfies condition (P) if and only if the corresponding φ is surjective for every pullback diagram P(M, N, f, g, Q).
(2)A_S is strongly flat if and only if the corresponding φ is bijective for every pullback diagram P(M, N, f, g, Q).
(3)S-act A_S satisfies condition (PWP) if and only if the corresponding φ is surjective for every pullback diagram P(Ss, Ss, f, f, S), where $s \in S$.

In the following Theorem 2.9, if we say that a right S-act A has flatness property, it means A has one of the properties in the Figure 1. If we say a class of S-acts having a flatness property, it means every object of the class has one of the flatness property.

Now we can prove the main result of this paper.

Theorem 2.9. *Every class of S-acts having a flatness property is closed under directed colimits.*

Proof. By Lemma 2.3, Lemma 2.4, Proposition 2.7 and Remark 2.8, the result is clear. □

Remark 2.10. By Theorem 1.1 and Theorem 2.9, when we investigate the flatness covers of S-acts, we only need to consider their precovers. Furthermore, these results imply that if an S-act A has a flatness precover, then A has a flatness cover.

Corollary 2.11 ([[10], Proposition 5.2]). *Let S be a monoid. Every directed colimit of a direct system of strongly flat acts is strongly flat.*

Corollary 2.12 ([[1], Proposition 2.9]). *Let S be a monoid. Every directed colimit of a direct system of acts that satisfy condition (P), satisfy condition (P).*

Let $\mathcal{T}_{\mathcal{F}}$ be the class of torsion free S-acts, then we also have

Corollary 2.13 ([[2], Lemma 5.3]). *$\mathcal{T}_{\mathcal{F}}$ is closed under directed colimits*

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