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POLYNOMIALLY BOUNDED SOLUTIONS OF THE LOEWNER DIFFERENTIAL EQUATION IN SEVERAL COMPLEX VARIABLES

A. EBADIAN, S. RAHROVI, S. SHAMS AND J. SOKÓŁ*

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Abstract. We determine the form of polynomially bounded solutions to the Loewner differential equation that is satisfied by univalent subordination chains of the form $f(z,t)=e^{\int_0^t A(\tau)\mathrm{d}\tau}z+\cdots$, where $A:[0,\infty]\to L(\mathbb{C}^n,\mathbb{C}^n)$ is a locally Lebesgue integrable mapping and satisfying the condition

$$\sup_{s\geq 0}\int_0^\infty \left\|\exp\left\{\int_s^t [A(\tau)-2m(A(\tau))I_n]\mathrm{d}\tau\right\}\right\| \mathrm{d}t<\infty,$$

and $m(A(t))>0$ for $t\geq 0$, where $m(A)=\min\{\mathfrak{Re}\,\langle A(z),z\rangle : \|z\|=1\}$. We also give sufficient conditions for $g(z,t)=M(f(z,t))$ to be polynomially bounded, where $f(z,t)$ is an $A(t)$-normalized polynomially bounded Loewner chain solution to the Loewner differential equation and M is an entire function. On the other hand, we show that all $A(t)$-normalized polynomially bounded solutions to the Loewner differential equation are Loewner chains.
Keywords: Biholomorphic mapping, Loewner differential equation, polynomially bounded, subordination chain, parametric representation.
MSC(2010): Primary 32H02; Secondary 30C45.

1. Introduction and preliminaries

Subordination chains in several complex variables, the associated differential equations and applications have been studied by various authors (see [1,4,8–10, 12,21] and the references therein). Initially it was assumed that the generator of the subordination chains satisfies the normalization $Dh(0,t)=I_n$ and hence the chains satisfy $Df(0,t)=e^tI_n$. Recently, there has been interest in working a more general normalization ([9,10]) or no normalization at all ([1,4]).

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Becker ([2, 3]) investigated the general form of solutions to the Loewner differential equation in one complex variable

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \qquad t \geq 0, |z| < 1, \tag{1.1}$$

where $p(\cdot, t)$ for any fixed $0 \leq t < \infty$ is in the well-known Carathéodory class $\mathcal{P}$ of the holomorphic functions q with $q(0) = 1$ and $\mathfrak{Re}\{q(z)\} > 0$ for $|z| < 1$, and $p(z, \cdot)$ is measurable on $[0, \infty)$ for $|z| < 1$. In one complex variable there exists a unique univalent solution $f(z,t) = e^t z + \cdots$ of (1.1) (called the canonical solution). Also, Becker proved that the any other solution $g(z,t)$ of (1.1) that is holomorphic on $|z| < 1$ and locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $|z| < 1$, has the form $g(z,t) = L(f(z,t))$, where $f(z,t)$ is the canonical solution and L is an entire function (compare also [23]). In particular, if $g(\cdot, t)$ is univalent on $|z| < 1$ and $g(0,t) = g'(0,t) - e^t = 0$ for $t \geq 0$, then $g(z,t) \equiv f(z,t)$.

In recent years, the general form of solutions to the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \qquad a.e\ t \geq 0, z \in B^n, \tag{1.2}$$

which have the normalization $h(z,t) = A(t)z + \cdots$, where $A : [0, \infty] \to L(\mathbb{C}^n, \mathbb{C}^n)$ is a measurable mapping such that $m(A(t)) > 0$ for $t \geq 0$, has been studied. The case in which $A(t) \equiv A \in L(\mathbb{C}^n, \mathbb{C}^n)$ was studied by Hamada [17] and the case in which $A(t) \equiv A = I_n$ was considered by Graham, Kohr and Pfaltzgraff [14] and the case in which $k_+(A) < 2m(A)$ was considered by Duren, Graham, Hamada and Kohr [6]. One of the results in [6] is as follows. Any bounded solution $g(z,t)$ to the Loewner differential equation (1.2) has the form $g(z,t) = M(f(z,t))$, where $M \in L(\mathbb{C}^n, \mathbb{C}^n)$ and $f(z,t)$ is the unique A-normalized bounded solution to the Loewner differential equation (1.2). The proof of the above result requires a generalization to higher dimensions of the well-known Carathéodory kernel convergence result for univalent functions [6, 19].

On the other hand, Hamada [17] determines the form of arbitrary polynomially bounded univalent solution $g(z,t)$ to the Loewner differential equation (1.2) for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$. Also, Voda [31] finds an A-normalized polynomially bounded solution to the Loewner differential equation (1.2) for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $m(A) > 0$.

Any solution $f(z,t)$ to the Loewner differential equation (1.2) that is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$ is a subordination chain (see Proposition 3.1). In one complex variable, if $a : [0, \infty) \to \mathbb{C}^n$ is a function such that $a(t) \neq 0$ for $t \geq 0$ and $|a(t)|$ is strictly increasing on $[0, \infty)$ and if $f(z,t) = a(t)z + \ldots$ is a non-normalized univalent subordination chain on the unit disk $|z| < 1$, then $f_*(z, t^*) = f(e^{-i\theta(t)}z, t)/a(0)$ is a normalized univalent subordination chain, where $t^* = \log(|a(t)/a(0)|)$ and

$\theta(t) = \arg(a(t)/a(0))$. That is, there exists a normalized univalent subordination chain with essentially the same geometric properties as a original one. But, the situation is different in higher dimension (in dimension $n \geq 2$). There exists non-normalized subordination chain $f(z,t) = e^{\int_0^t A(\tau)\mathrm{d}\tau} z + \cdots$ which can not be normalized by an analogous change of variable. On the other hand, there exists biholomorphic mappings f which have useful embedding in non-normalized subordination chain. For example, if $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition (2.4) and the assumptions of Definition 2.3 and f is generalized spirallike mapping with respect to A, then $f(z,t) = e^{\int_0^t A(\tau)\mathrm{d}\tau} f(z)$ is a univalent subordination chain. In connection with this observation, S. Rahrovi, A. Ebadian and S. Shams [28] introduced the class $S^0_{A(t)}(B^n)$ of mappings which have $A(t)$-parametric representation, i.e. the subclass of $S(B^n)$ which consists of those mappings f that can be embedded in univalent subordination chains $f(z,t) = e^{\int_0^t A(\tau)\mathrm{d}\tau} z + \cdots$ such that $\{e^{-\int_0^t A(\tau)\mathrm{d}\tau} f(z,t)\}_{t\geq 0}$ is a normal family on B^n. Therefore, it is of interest and important to consider subordination chains which do not have the standard normalization in the study of univalent mappings on B^n.

For several results on subordination chains in several complex variables, the readers may consult [1, 4, 8, 10–13, 21, 22, 24–27] and the references therein.

Theorem 3.2 and Corollary 3.3 will show that a polynomial bounded solution of (1.2) can be recovered from its first n_0 coefficients and the solution of (2.5). These results generalize Poreda ([27] Theorem 4.1). Also, we determine the form of arbitrary polynomially bounded univalent solutions $g(z,t)$ to the Loewner differential equation (1.2) for any $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ which is a locally Lebesgue integrable mapping and satisfies in the condition (2.4) and the assumptions of Definition 2.3. The proof is elementary and we do not need the Carathéodory kernel convergence result for univalent mappings. We also give sufficient conditions for $g(z,t) = M(f(z,t))$ to be polynomially bounded, where $f(z,t)$ is an $A(t)$-normalized polynomially bounded Loewner chain solution to the Loewner differential equation (1.2). On the other hand, we show that all $A(t)$-normalized polynomially bounded solutions of (1.2) are Loewner chains.

2. Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{\infty} z_j \bar{w}_j$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B^n_r and the unit ball B^n_1 by B^n. The closed ball $\{z \in \mathbb{C}^n : \|z\| \leq r\}$ is denoted by $\bar{B}^n_r$. In the case of one complex variable, B^1 is denoted by $|z| < 1$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of linear and continuous operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ with the standard operator norm and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. Let Ω be a domain in $\mathbb{C}^n$ which contains the origin and $f \in H(\Omega)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$.

If $f \in H(B^n)$, we say that f is locally biholomorphic on B^n if $J_f(z) \neq 0, z \in B^n$, where $J_f(z) = \det Df(z)$ and $Df(z)$ is the complex Jacobian matrix of f at z. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n, the set $S(B^1)$ is denoted by S. Also, let LS_n be the set of normalized locally biholomorphic mappings on B^n. In the case of one variable, the set LS_1 is denoted by LS.

If $f \in H(B^n)$, then f can be expanded in a power series of homogenous polynomials

$$f(z) = \sum_{k=0}^{\infty} A_k(z^k), \qquad z \in B^n,$$

where $A_k(z^k) = \frac{1}{k!} D^k f(0)(z^k)$. Here, for $h \in \mathbb{C}^n$, $D^0 f(0)(h^0) = f(0)$ and for $k \geq 1$, $D^k f(0)(h^k) = D^k f(0)(\underbrace{h, \dots, h}_{k-times})$.

Several notation from operator theory play a role in studying special classes of holomorphic mappings on B^n or in proving estimates or existing the theorems for the general Loewner differential equation. These notions involve properties of the numerical radius or the spectrum or a linear operator.

If $A \in L(\mathbb{C}^n, \mathbb{C}^n)$, let

$$m(A) = \min\{\mathfrak{Re}\,\{\langle A(z), z\rangle\} : \|z\| = 1\}$$

and

$$k(A) = \max\{\mathfrak{Re}\,\{\langle A(z), z\rangle\} : \|z\| = 1\}.$$

Also, let

$$|V(A)| = \max_{\|z\|=1} |\langle A(z), z\rangle|,$$

be the numerical radius of operator A. Then $\|A\| \leq 2|V(A)|$ by ([16], Theorem 1.3.1). The upper exponential index of A is defined by

$$k_+(A) = \max\{\mathfrak{Re}\lambda : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A. It is known that $k_+(A) = \lim_{t\to\infty} \frac{\ln \|e^{tA}\|}{t}$ and for each $w > k_+(A)$, there exists a positive number $\delta = \delta(w)$ such that

$$\|e^{tA}\| \leq \delta e^{wt}, \qquad t \geq 0. \tag{2.1}$$

by [5], see also [29, p. 311] .

The following classes of mappings in $H(B^n)$ play a key role in our discussion (see [8, 12, 13, 21, 22]):

$$\mathcal{N} = \{h \in H(B^n) : h(0) = 0, \mathfrak{Re}\,\{\langle h(z), z\rangle\} > 0, z \in H(B^n)\backslash\{0\}\},$$

and

$$\mathcal{M} = \{h \in \mathcal{N} : Dh(0) = I_n\}.$$

In one complex variable, we have $f \in \mathcal{M}$ if and only if $f(z)/z \in \mathcal{P}$, where

$$\mathcal{P} = \{p \in H(|z| < 1) : \ p(0) = 1, \mathfrak{Re}\,\{p(z)\} > 0, \ |z| < 1\},$$

is the Carathéodory class.

We need the following result, whose proof is similar to in ([8] Theorem 1.2). The proof uses the fact that if $h \in \mathcal{N}$ has the homogeneous expansion

$$h(z) = A(z) + \sum_{m=2}^{\infty} P_m(z), \qquad z \in B^n,$$

then $\|P_m\| \leq 4mk(A)$ for $m \geq 2$.

Lemma 2.1. *Let $h : B^n \to \mathbb{C}^n$ be a mapping such that $h \in \mathcal{N}$, $Dh(0) = A$ and $m(A) > 0$ for $t \geq 0$. Then $\|h(z)\| \leq \frac{4r}{(1-r)^2}|V(A)|$ for all $\|z\| \leq r < 1$.*

We next consider the notation of subordination and subordination chains on B^n. If $f, g \in H(B^n)$, we say that f is subordinate to g ($f \prec g$) if there exists a Schwarz mapping υ (i.e., $\upsilon \in H(B^n)$ and $\|\upsilon(z)\| \leq \|z\|, z \in B^n$) such that $f = go\upsilon$.

Definition 2.2. A mapping $f : B^n \times [0, \infty] \to \mathbb{C}^n$ is called a subordination chain if $f(\cdot, t)$ is holomorphic on B^n, $f(0, t) = 0$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$, $0 \leq s \leq t < \infty$. In addition, if $f(\cdot, t)$ is biholomorphic on B^n for $t \geq 0$, we say that $f(z, t)$ is a Loewner chain. Also, if $f(z, t)$ is a subordination (Loewner) chain such that $Df(0, t) = e^{\int_0^t A(\tau)\mathrm{d}\tau}$ for $t \geq 0$, we say that $f(z, t)$ is an $A(t)$-normalized subordination (Loewner) chain.

The above subordination implies the existence of the transition mapping $v(z, s, t)$ associated with $f(z, t)$, such that $f(z, t) = f(v(z, s, t), t)$ for $z \in B^n$ and $t \geq s \geq 0$.

In this paper, we use liner operators which depended measurably on t and which satisfy the assumptions of Definition 2.3. We remark that condition (2.2) is satisfied if $A(t)$ is constant or if $A(t)$ is diagonal (for details, see [9]).

Definition 2.3. Let $A : [0, \infty] \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping such that $m(A(t)) > 0$ for $t \geq 0$ and $\int_o^{\infty} m(A(t))\mathrm{d}t = \infty$. Moreover, assume that $\|A(\cdot)\|$ is uniformly bounded on $[0, \infty]$ and

$$\int_s^t A(\tau)\mathrm{d}\tau \ o \int_r^s A(\tau)\mathrm{d}\tau = \int_r^s A(\tau)\mathrm{d}\tau \ o \int_s^t A(\tau)\mathrm{d}\tau, \qquad t \geq s \geq r \geq 0. \tag{2.2}$$

Definition 2.4. Let $A : [0, \infty] \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping such that $m(A(t)) > 0$ for $t \geq 0$. Also let Ω be a domain in $\mathbb{C}^n$ which contains the origin, we say that $\mathbb{C}^n$ is generalized spirallike with respect to A if $e^{-\int_s^t A(\tau)\mathrm{d}\tau}(w) \in \Omega$ for all $w \in \Omega$ and $t \geq s \geq 0$.

A mapping $f \in S(B^n)$ is called generalized spirallike with respect to A if $f(B^n)$ is a generalized spirallike domain with respect to A. This condition is characterized by $[Df(z)]^{-1}A(t)f(z) \in \mathcal{N}$.

Remark 2.5. A mapping f is called spirallike of type $\alpha \in (\frac{-\pi}{2}, \frac{\pi}{2})$ if f is spirallike with respect to $A(t) \equiv A = e^{-i\alpha}I_n$ (see [18];cf. [22]) Hence $f \in S^*(B^n)$ if and only if f is spirallike of type zero.

We remark that if $A(t)$ is a constant linear operator in $\mathbb{C}^n$, then Definition 2.4 reduce to the usual definition of spirallikeness (see [30]. Also, if $A(t) = I_n$ we obtain the usual notion of spirallikeness (see [12, 15, 30]). Various results concerning spirallike mapping with respect to constant linear operators may be found in [7, 15, 18, 27, 29, 30].

Example 2.6. Let $n = 2$ and $f : B^2 \to \mathbb{C}^2$ be given by $f(z) = (z_1, z_2 + \alpha z_1^2)$ for $z = (z_1, z_2) \in B^2$, where $\alpha \in \mathbb{C}^2\backslash\{0\}$. Also, let $b > 0$ and $a : [0, \infty) \to \mathbb{C}$ be a measurable function such that $\mathfrak{Re}a(t) > 0$, $|a(t)| \le b$ for $t \ge 0$. Suppose that $A : [0, \infty) \to L(\mathbb{C}^2, \mathbb{C}^2)$ be defined by $A(t) = diag(a(t), 2a(t))$ for $t \ge 0$. Also, let $f(z, t) = e^{\int_0^t A(\tau)\mathrm{d}\tau} f(z)$. On the other hand, since $|a(t)| \le b$ for $t \ge 0$, it follows that $\|A(\cdot)\|$ is uniformly bounded on $[0, \infty)$. Since

$$\mathfrak{Re}\left\{\left\langle [Df(z)]^{-1}A(t)f(z), z\right\rangle\right\} = \mathfrak{Re}\ a(t)(|z_1|^2 + 2|z_2|^2) > 0, \qquad z \in B^2\backslash\{0\},$$

it follows that f is a generalized spirallike mapping with respect to A. then $f(\cdot, t)$ is biholomorphic on B^2, $f(0, t) = 0$, $Df(0, t) = e^{\int_0^t A(\tau)\mathrm{d}\tau}$ for $t \ge 0$ and $f(z, \cdot)$ is differential on $[0, \infty)$ for $z \in B^2$. Hence $f(z, t)$ is a univalent subordination chain.

Definition 2.7. Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a measurable mapping which satisfies the conditions in Definition 2.3 and let $f \in H(B^n)$ be a normalized mapping. We say that f has $A(t)$-parametric representation if there exists a mapping $h : B^n \times [0, \infty] \to \mathbb{C}^n$ such that $h(\cdot, t) \in \mathcal{N}$, $Dh(0, t) = A(t)$ for a.e. $t \ge 0$ and $h(\cdot, t)$ is measurable function on $[0, \infty)$ for $z \in B^n$ and $f(z) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} \upsilon(z, t)$ locally uniformly on B^n, where $\upsilon = \upsilon(z, t)$ is the unique locally absolutely continuous solution of the initial value problem

$$\frac{\partial \upsilon}{\partial t} = -h(\upsilon, t) \quad a.e.\ t \ge 0,\ \upsilon(z, 0) = z, \tag{2.3}$$

for all $z \in B^n$.

Note that if $A(t) = A$ is a constant operator, then Definition 2.7 reduce to ([10] Definition 1.5). If $A(t) = I_n$ and f has I_n-parametric representation, then f has parametric representation in the usual sense (see [8, 12]; cf. [24]). Denote by $S^0(B^n)$ the class of mappings which have parametric representation and by $S^0_{A(t)}(B^n)$ the class of mapping which have $A(t)$-parametric representation, also we write $S^0_{I_n}(B^n) = S^0(B^n)$. If $n = 1, S^0(U) = S$ (see [23]), but $S^0(B^n) \subsetneqq S(B^n)$ for $n \ge 2$ (see [8] and [12]). However, important subclasses of $S(B^n)$ are also subclasses of $S^0(B^n)$. For example $S^*(B^n) \subsetneqq S^0(B^n)$ and any spirallike mapping of type $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ belongs to $S^0(B^n)$, too (see [18]).

We next introduce the notion of asymptotic spirallikeness, a natural generalization of spirallikeness.

Definition 2.8. Let $\Omega \subseteq \mathbb{C}^n$ be a domain which contains the origin, and $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ satisfy the condition in Definition 2.3. We say that Ω is $A(t)$-asymptotically spirallike if there exists a mapping $Q = Q(z, t) : \Omega \times [0, \infty) \to \mathbb{C}^n$ which satisfies the following conditions

(i) $Q(\cdot, t)$ is a holomorphic mapping on Ω, $Q(0, t) = 0$, $DQ(0, t) = A(t)$, $t \geq 0$ and the family $\{Q(\cdot, t)\}_{t\geq 0}$ is locally uniformly bounded on Ω;

(ii) $Q(z, \cdot)$ is measurable on $[0, \infty)$ for all $z \in \Omega$;

(iii) the initial value problem

$$\frac{\partial \omega}{\partial t} = -Q(\omega, t) \quad a.e.\ t \geq s,\ \omega(z, s, s) = z,$$

has a unique solution $\omega = \omega(z, s, t)$ for each $z \in \Omega$ and $s \geq 0$, such that $\omega(., s, t)$ is a holomorphic mapping of Ω in to Ω for $t \geq s$, $\omega(z, s, t)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in \Omega$ for $s \geq 0$, and $\lim_{t\to\infty} e^{\int_0^t A(\tau) d\tau}$ $\omega(z, 0, t) = z$ locally uniformly on Ω.

A domain $\Omega \subseteq \mathbb{C}^n$ which contains the origin is called asymptotically spirallike if there exists the mapping $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ which satisfies the conditions in Definition 2.3 and $m(A(t)) > 0$ for $t \geq 0$ such that Ω is $A(t)$-asymptotically spirallike.

Note that if $A = I_n$ in Definition 2.8, then is asymptotically starlike ([11] Definition 2.1 and [25] Definition 3) in the case of the maximum norm).

Definition 2.9. Let $f : B^n \to \mathbb{C}^n$ be a normalized holomorphic mapping, and let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ satisfies the condition Definition 2.3, we say that f is $A(t)$-asymptotically spirallike (asymptotically spirallike) if f is biholomorphic on B^n and $f(B^n)$ is an $A(t)$-asymptotically spirallike (asymptotically spirallike) domain. In particular, we say that f is asymptotically starlike if f is biholomorphic on B^n and $f(B^n)$ is an asymptotically starlike domain.

We need the following existence result for the initial value problem (2.5) ([28] Theorem 2.1, [10] Theorem 2.1; cf. [12] Theorem 8.1.3).

Lemma 2.10. *Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ satisfy the conditions in Definition 2.3. Assume that*

$$\sup_{s\geq o} \int_s^\infty \| e^{\int_s^t [A(\tau) - 2m(A(\tau)) I_n] d\tau} \| dt < \infty, \tag{2.4}$$

Also let $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ *be a mapping which satisfies the following conditions:*

(*i*) $h(\cdot,t) \in \mathcal{N}$ *and* $Dh(0,t) = A(t)$ *for* $t \geq 0$*;*

(*ii*) $h(z,\cdot)$ *is measurable on* $[0,\infty)$ *for* $z \in B^n$*.*

Then for each $s \geq 0$ *and* $z \in B^n$*, the initial value problem*

$$\frac{\partial \upsilon}{\partial t} = -h(\upsilon,t) \quad a.e.\ t \geq s,\ \upsilon(z,s,s) = z, \tag{2.5}$$

has a unique locally absolutely continuous solution $\upsilon_{s,t}(z) = \upsilon(z,s,t)$*. Furthermore, for fixed* s *and* t*,* $0 \leq s \leq t < \infty$*,* $\upsilon_{s,t}(z) = \upsilon(z,s,t)$ *is a univalent Schwarz mapping and for fixed* $s \geq 0$ *and* $z \in B$*, it is a Lipschitz function on* $[s,\infty)$ *locally uniformly with respect to* $z \in B^n$*,* $D\upsilon(0,s,t) = \exp(-\int_s^t A(\tau)\mathrm{d}\tau)$ *for* $t \geq s \geq 0$*.*

Definition 2.11. A mapping $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ which satisfies the assumptions (i) and (ii) of Lemma 2.10 will be called a generating vector field.

We next mention the following growth result that is satisfied by the solution $\upsilon(z,s,t)$ of the initial value problem (2.5) (see [28] Theorem 2.1, [10] Theorem 2.1).

Lemma 2.12. *Suppose that* $h(z,t)$ *satisfies the hypotheses of Theorem 2.10, and let* $\upsilon(z,s,t)$ *be the solution of the initial value problem* (2.5)*. Then*

$$\frac{\|\upsilon(z,s,t)\|}{(1-\|\upsilon(z,s,t)\|)^2} \leq e^{-\int_s^t m(A(\tau))\mathrm{d}\tau}\frac{\|z\|}{(1-\|z\|)^2}, \qquad z \in B^n,\ t \geq s \geq 0, \tag{2.6}$$

$$e^{-\int_s^t k(A(\tau))\mathrm{d}\tau}\frac{\|z\|}{(1+\|z\|)^2} \leq \frac{\|\upsilon(z,s,t)\|}{(1+\|\upsilon(z,s,t)\|)^2}, \qquad z \in B^n,\ t \geq s \geq 0. \tag{2.7}$$

Definition 2.13. (i) A standard solution $f : B^n \times [0,\infty) \to \mathbb{C}^n$ to (1.2) is said to be polynomial bounded (bounded) if $\{e^{-\int_0^t A(\tau)\mathrm{d}\tau} f(z,t)\}_{t\geq 0}$ is locally polynomially bounded (locally bounded), i.e. for any compact set $K \subset B^n$, there exists a constant C_k and a polynomial (constant polynomial) P such that

$$\|e^{-\int_0^t A(\tau)\mathrm{d}\tau} f(z,t)\| \leq C_k P(t), \qquad z \in K,\ t \geq 0;$$

(ii) A function $F_k : [0,\infty) \to \mathcal{P}^k(\mathbb{C}^n)$, where $\mathcal{P}^k(\mathbb{C}^n)$ denote the Banach space of homogenous polynomial mappings of degree k from $\mathbb{C}^n$ to $\mathbb{C}^n$, is said to be polynomial bounded (bounded) if there exists a polynomial (constant polynomial) P such that $\|F_k(t)\| \leq P(t)$, for $t \geq 0$.

If $k_+(A(t)) \leq 2m(A(t))$, then we obtain the existence and uniqueness of $A(t)$-normalized bounded solution to the Loewner differential equations (1.2) [6] Corollary 4.4 (cf. [10] Theorem 2.3).

Lemma 2.14. *Let $A : [0, \infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition (2.4) and the assumptions of Definition 2.3, also let h be as in Theorem 2.10, and let $\upsilon(t) = \upsilon(z, s, t)$ be the unique Lipschitz continuous solution on $[s, \infty)$ of the initial value problem (2.5). If $k_+(A(t)) \le 2m(A(t))$, then the limit*

$$\lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau}\upsilon(z, s, t) = f(z, s), \qquad s \ge 0,$$

exists locally uniformly on B^n. Moreover, $f(z, t)$ is the unique $A(t)$-normalized bounded Loewner chain solution to the Loewner differential equations (1.2).

3. Main results

We begin this section with the following proposition.

Proposition 3.1. *Let $g(z, t)$ be a standard solution to the Loewner differential equations (1.2). Then $g(z, s) = g(\upsilon(z, s, t), t)$ holds for $z \in B^n$ and $t \ge s \ge 0$, where $\upsilon(t) = \upsilon(z, s, t)$ is the unique solution of the initial value problem (2.5). Moreover, $Dg(0, t) = Dg(0, 0)e^{\int_0^t A(\tau)\mathrm{d}\tau}$ holds.*

Proof. Let $g(z, s, t) = g(\upsilon(z, s, t), t)$ for $z \in B^n$ and $t \ge s \ge 0$. We show that $g(z, s, t) = g(z, s, s)$, i.e. $g(\upsilon(z, s, t), t) = g(z, s)$ for $z \in B^n$ and $t \ge s \ge 0$. Fix $\rho \in (0, 1)$ and $T > 0$. Since $g(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$, we deduce that $g(z, t)$ is continuous on $B^n \times [0, \infty)$. Then there exists a constant $L(\rho, T) > 0$ (similar to [28] Theorem 2.9) such that

$$\|g(z, t)\| \le L(\rho, T), \qquad \|z\| \le \rho,\ t \in [0, T].$$

In view of the Cauchy integral formula for holomorphic mappings in several complex variables, we deduce that there exists a constant $L^*(\rho, T) > 0$ such that

$$\|g(z, t)\| \le L^*(\rho, T), \qquad \|z\| \le \rho,\ t \in [0, T]. \tag{3.1}$$

On the other hand, letting $b = |V(A(t))|$ and taking into account the relations (1.2) and (3.1) and Lemma 2.1, we deduce that there exists a constant $M(\rho, T, b) > 0$ such that

$$\|\frac{\partial g}{\partial t}(z, t)\| \le M(\rho, T, b), \qquad \|z\| \le \rho,\ a.e.\ t \in [0, T].$$

Therefore, in view of the local absolute continuous $g(z, \cdot)$ locally uniformly with respect to $z \in B^n$, we deduce that

$$\|g(z, t_1) - g(z, t_2)\| = \|\int_{t_1}^{t_2} \frac{\partial g}{\partial t}(z, t)\mathrm{d}t\| \le M(\rho, T, b)(t_2 - t_1),$$

for $\|z\| \le \rho$ and $0 \le t_1 \le t_2 \le T$. Hence, $g(z, t)$ is locally Lipschitz continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Since, $\upsilon(z, s, \cdot)$ is Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to $z\in B^n$ in view of Lemma 2.10, it is easy to deduce that the same is true for $g(z,s,\cdot)$. Then $g(z,s,t)$ is differentiable for almost all $t\in[s,\infty)$ and in view of (1.2) and (2.5), we deduce that $\frac{\partial g}{\partial t}(z,s,t)=0$ for a.e. $t\geq s$ and for all $z\in B^n$. Hence, $g(\upsilon(z,s,t),t)=g(z,s)$ for $z\in B^n$ and $t\geq s\geq 0$.

Moreover, we have

$$Dg(0,0)=Dg(0,t)D\upsilon(0,0,t)=Dg(0,t)e^{-\int_0^t A(\tau)\mathrm{d}\tau},$$

by Lemma 2.10. Thus, $Dg(0,t)=Dg(0,0)e^{-\int_0^t A(\tau)\mathrm{d}\tau}$. This completes the proof. □

In the following Theorem, we determine the form of locally biholomorphic polynomial bounded solutions of the Loewner differential equation, the following theorem is due to [17] Theorem 3.2 (see also [31] Proposition 2.1).

Theorem 3.2. *Let $g(z,t)$ be a polynomially bounded solution to the Loewner differential equations (1.2), such that $M=Dg(0,0)$ is a non-singular matrix. Then*

$$g(z,s)=M(f(z,s)),\qquad z\in B^n,\ s\geq 0, \tag{3.2}$$

where $f(z,s)$ is an $A(t)$-normalized subordination chain solution to the Loewner differential equations (1.2). Also $f(z,s)$ can be written as follows:

$$f(z,s)=\lim_{t\to\infty}e^{\int_0^t A(\tau)\mathrm{d}\tau}(\upsilon(z,s,t)+\sum_{k=2}^{n_0}G_k(t)(\upsilon(z,s,t)^k)), \tag{3.3}$$

locally uniformly in z, here $\upsilon(z,s,t)$ is the unique solution of the initial value problem (2.5), $n_0=[k_+(A(t))/m(A(t))]$ and

$$f(z,t)=e^{\int_0^t A(\tau)\mathrm{d}\tau}(z+\sum_{k=2}^{\infty}G_k(t)(z^k)).$$

Moreover, if $Dg(0,0)$ commutes with $A(t)$, then $f(z,t)$ is an $A(t)$-normalized polynomially bounded Loewner chain solution to the Loewner differential equations (1.2).

Proof. Let $e^{-\int_0^t A(\tau)\mathrm{d}\tau}f(z,t)=z+\sum_{k=2}^{n_0}G_k(t)(z^k)+R(t)(z)$ be the power series expansion of $e^{-\int_0^t A(\tau)\mathrm{d}\tau}f(z,t)$ on B^n for $t\geq 0$, where

$$G_k(t)(z^k)=e^{-\int_0^t A(\tau)\mathrm{d}\tau}\frac{1}{k!}D^kf(0,t)(\underbrace{z,\ldots,z}_{k-times}),$$

and $R(t)(z)=\sum_{n_0+1}^{\infty}G_k(t)(z^k)$.

Now, let $f(,t)=[Dg(0,0)]^{-1}g(z,t)$. Then

$$f(z,s)=f(\upsilon(z,s,t),t)=e^{\int_0^t A(\tau)\mathrm{d}\tau}(\upsilon(z,s,t)+\sum_{k=2}^{n_0}G_k(t)(\upsilon(z,s,t)^k))$$

$$+e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(\upsilon(z,s,t)),$$

where $R(t)(z) = \sum_{n_0+1}^{\infty} G_k(t)(z^k)$. Since g is polynomially bounded, using the formula for the reminder of the Taylor series and Cauchy formula, we obtain

$$\|e^{-\int_0^t A(\tau)\mathrm{d}\tau} Dg(0,0) e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(z)\| \leq C_\rho P(t)\|z\|^{n_0+1}, \qquad \|z\| \leq \rho \leq 1,$$

where C_ρ is a constant which depends only on ρ and $P(t)$ is a polynomial in t. From (2.4), (2.6) and this inequality, we have

$$\begin{aligned}
&\left\| e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(\upsilon(z,s,t)) \right\| \\
= &\left\| [Dg(0,0)]^{-1} e^{\int_0^t A(\tau)\mathrm{d}\tau} e^{-\int_0^t A(\tau)\mathrm{d}\tau} Dg(0,0) e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(\upsilon(z,s,t)) \right\| \\
\leq &\ \|[Dg(0,0)]^{-1}\|\ C_\rho P(t)\ \|e^{\int_0^t A(\tau)\mathrm{d}\tau}\|\ \|\upsilon(z,s,t)\|^{n_0+1} \\
\leq &\ C_{\rho,s} \|e^{\int_0^t (A(\tau)-(n_0+1)m(A(\tau))I_n))\mathrm{d}\tau}\|\ P(t), \qquad \|z\| \leq \rho \leq 1,
\end{aligned}$$

where $C_{\rho,s}$ is a constant which depends only on ρ and s. Since the relation (1.1) is true we conclude that $e^{\int_0^t A(\tau)\mathrm{d}\mathrm{d}\tau} R(t)(\upsilon(z,s,t)) \to 0$ locally uniformly in z. Thus we obtain (3.2) and (3.3).

Next, assume that $Dg(0,0)$ commutes with $A(t)$. Since g is polynomially bounded $G_k(t)(z) = [Dg(0,0)]^{-1} e^{\int_0^t A(\tau)\mathrm{d}\tau} e^{-\int_0^t A(\tau)\mathrm{d}\tau} Dg(0,0) G_k(t)(z)$, $2 \leq k \leq n_0$, are polynomially bounded locally uniformly on B^n. Thus, by [17, Theorem 3.2] (also see [31, Theorem 2.8]), $[Dg(0,0)]^{-1} g(z.t) = f(z,t)$ is an $A(t)$-normalized polynomially bounded Loewner chain solution to the Loewner differential equations (1.2). This completes the proof. □

Corollary 3.3. *Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition (2.4) and the assumptions of Definition 2.3. If $f(z,t)$ is a polynomially bounded solution to the Loewner differential equations (1.2) such that*

$$f(z,t) = e^{\int_0^t A(\tau)\mathrm{d}\tau}(z + \sum_{k=2}^{\infty} G_k(t)(z^k)),$$

then

$$f(z,s) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau}(\upsilon(z,s,t) + \sum_{k=2}^{n_0} G_k(t)(\upsilon(z,s,t)^k)),$$

and the limit is locally uniformly in z.

If $f(z,t)$ is an $A(t)$-normalized Loewner chain satisfying (1.2), then we can write

$$e^{-\int_0^t A(\tau)\mathrm{d}\tau} f(z,t) = z + \sum_{k=2}^{\infty} F_k(t)(z^k) \quad \text{or} \quad f(z,t) = e^{\int_0^t A(\tau)\mathrm{d}\tau}(z + \sum_{k=2}^{\infty} F_k(t)(z^k)),$$

and

$$F_k(t)(z^k) = e^{-\int_0^t A(\tau)\mathrm{d}\tau} \frac{1}{k!} D^k f(0,t)(\underbrace{z,\dots,z}_{k-times}),$$

where $F_k(t)(z^k)$ is a homogenous polynomial mapping of degree k.

Theorem 3.4. *Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition (2.4) and the assumptions of Definition 2.3. Assume that $n_0 = 1$. Let $g(z,t)$ be a polynomially bounded solution to the Loewner differential equations (1.2). Then*

$$g(z,s) = L(f(z,s)), \qquad z \in B^n,\ s \geq 0, \tag{3.4}$$

where $L = Dg(0,0)$ and

$$f(z,s) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} \upsilon(z,s,t),$$

is the unique $A(t)$-normalized bounded Loewner chain solution to the Loewner differential equations(1.2).

Proof. Let

$$g(z,t) = Dg(0,0)e^{\int_0^t A(\tau)\mathrm{d}\tau} z + e^{\int_0^t A(\tau)\mathrm{d}\tau} \sum_{k=2}^{\infty} G_k(t)(z^k),$$

be a polynomially bounded Loewner chain solution to the Loewner differential equations (1.2). Then

$$g(z,s) = g(\upsilon(z,s,t),t) = Dg(0,0)\upsilon(z,s,t)e^{\int_0^t A(\tau)\mathrm{d}\tau} + e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(\upsilon(z,s,t)),$$

where $R(t)(z) = \sum_{k=2}^{\infty} G_k(t)(z^k)$. Since g is polynomially bounded, using the formula for the reminder of the Taylor series and Cauchy formula, we obtain

$$\|R(t)(z)\| \leq C_\rho P(t)\|z\|^2, \qquad \|z\| \leq \rho,$$

where C_ρ is a constant which depends only on ρ and $P(t)$ is a polynomial in t. From (2.4), (2.6) and this inequality, we have

$$\begin{aligned} \|e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(\upsilon(z,s,t))\| &\leq C_\rho P(t)\ \|e^{\int_0^t k(A(\tau))\mathrm{d}\tau}\|\ \|\upsilon(z,s,t)\|^2 \\ &\leq C_{\rho,s}\ \|e^{\int_0^t (A(\tau)-2m(A(\tau))I_n))\mathrm{d}\tau}\|\ P(t), \qquad \|z\| \leq \rho, \end{aligned}$$

where $C_{\rho,s}$ is a constant which depends only on ρ and s. Since the relation (2.4) is true we conclude that $e^{\int_0^t A(\tau)\mathrm{d}\tau} R(t)(\upsilon(z,s,t)) \to 0$ locally uniformly in z. Thus, we obtain (3.4). By Lemma 2.14, $f(z,s) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau}\upsilon(z,s,t)$ is the unique $A(t)$-normalized bounded Loewner chain solution to the Loewner differential equations (1.2). This completes the proof. □

As a corollary of above theorem, we obtain the following uniqueness of polynomially bounded solution to the Loewner differential equations (1.2) ([6, Corollary 4.4]).

Corollary 3.5. *Assume that $n_0 = 1$. Let $f(z,t)$ be a polynomially bounded solution to the Loewner differential equations* (2.20) *such that $Df(0,0) = e^{\int_0^t A(\tau)\mathrm{d}\tau}$. Then*

$$f(z,s) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} \upsilon(z,s,t),$$

and it is the unique $A(t)$-normalized bounded Loewner chain solution to the Loewner differential equations (1.2).

Lemma 3.6. *Let $A : [0,\infty) \to L(\mathbb{C}^n, \mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition* (2.4) *and the assumptions of Definition 2.3. If P is a polynomial such that $P(t) \geq 0$ for $t \geq s$ then*

$$\begin{aligned} &\int_s^\infty P(t)\ \|e^{-\int_s^t A(\tau)\mathrm{d}\tau}\|\ \frac{\|\upsilon(z,s,t)\|^{n_0+1}}{(1-\|\upsilon(z,s,t)\|)^2}\mathrm{d}t \\ \leq\ & \frac{Q_{\varepsilon,A(t),P}(s)}{(1-\|z\|)^{2\frac{k_+(A(t))}{m(A(t))}+\varepsilon}}, \qquad \varepsilon > 0, \end{aligned}$$

where $Q_{\varepsilon,A(t),P}$ is a polynomial of the same degree as f.

Proof. Let

$$\alpha = \frac{k_+(A(t))}{m(A(t))} + \frac{\varepsilon}{2}.$$

We can restrict to the case when ε is small enough so that $\alpha < n_0 + 1$. Using (2.6) we see that

$$\begin{aligned} \frac{\|\upsilon(z,s,t)\|^{n_0+1}}{(1-\|\upsilon(z,s,t)\|)^2} &\leq \frac{\|\upsilon(z,s,t)\|^{\alpha}}{(1-\|\upsilon(z,s,t)\|)^2} \\ &\leq \frac{e^{-\alpha\int_s^t m(A(\tau))\mathrm{d}\tau}\|z\|^{\alpha}}{(1-\|z\|)^{2\alpha}} \\ &\leq \frac{e^{-\alpha\int_s^t m(A(\tau))\mathrm{d}\tau}}{(1-\|z\|)^{2\alpha}}. \end{aligned}$$

Let ε_1 be small enough so that

$$\begin{aligned} \|e^{-\int_s^t A(\tau)\mathrm{d}\tau}\| &\leq e^{-\int_s^t k(A(\tau))\mathrm{d}\tau} \\ &\leq C_{\varepsilon_1} e^{-\int_s^t (k(A(\tau))+\varepsilon_1)\mathrm{d}\tau} \\ &= C_{\varepsilon_1} e^{-\int_s^t (k(A(\tau))\mathrm{d}\tau}) + \varepsilon_1(t-s), \end{aligned}$$

and $\gamma = \alpha m(A(\tau)) - k(A(\tau)) - \varepsilon_1$. Then

$$\begin{aligned}
&\int_s^\infty P(t)\ \|e^{\int_s^t A(\tau)\mathrm{d}\tau}\|\ \frac{\|\upsilon(z,s,t)\|^{n_0+1}}{(1-\|\upsilon(z,s,t)\|)^2}\mathrm{d}t \\
\le\ & \int_s^\infty C_{\varepsilon_1}\frac{e^{\int_s^t (k(A(\tau))+\varepsilon_1)\mathrm{d}\tau}e^{-\alpha\int_s^t m(A(\tau))\mathrm{d}\tau}}{(1-\|z\|)^{2\alpha}}P(t)\mathrm{d}t \\
=\ & \frac{C_{\varepsilon_1}}{(1-\|z\|)^{2\alpha}}\int_s^\infty e^{-\gamma(t-s)}P(t)\mathrm{d}t,
\end{aligned}$$

and it is not hard to see that

$$Q_{\varepsilon,A(t),P}(s) = C_{\varepsilon_1}\int_s^\infty e^{-\gamma(t-s)}P(t)\mathrm{d}t,$$

satisfies our requirements. This completes the proof. □

Theorem 3.7. *Let $A:[0,\infty)\to L(\mathbb{C}^n,\mathbb{C}^n)$ be a locally Lebesgue integrable mapping which satisfies the condition(2.4) and the assumptions of Definition 2.3. If $F_k:[0,\infty)\to\mathcal{P}^k(\mathbb{C}^n)$, $k=2,\dots,m$ are polynomially bounded and the limit*

$$f(z,s) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau}(\upsilon(z,s,t) + \sum_{k=2}^{n_0} F_k(t)(\upsilon(z,s,t)^k)),$$

exists locally uniformly in $\in B^n$ for some $s\ge 0$, then $f(z,s)$ is univalent.

Proof. First note that if $Q\in\mathcal{P}^k(\mathbb{C}^n)$ then

$$\begin{aligned}
\|Q(z^k)-Q(w^k)\| &= \|\sum_{j=0}^{k-1} Q(z-w, z^j, w^{k-1-j}\| \\
&\le \|Q\|\ \|z-w\|\sum_{j=0}^{k-1}\|z\|^j\|w\|^{k-1-j}.
\end{aligned}$$

Using the above and (2.6) we see that

$$\|\sum_{k=2}^m F_k(t)(\upsilon(z_1,s,t)^k) - \sum_{k=2}^m F_k(t)(\upsilon(z_2,s,t)^k)\|$$

$$\le C_r P(t) e^{-\int_s^t m(A(\tau))\mathrm{d}\tau}\|\upsilon(z_1,s,t)-\upsilon(z_2,s,t)\|,\quad \|z_1\|,\|z_2\|\le r,$$

where P is a polynomial bounded on F_k, $k=2,\dots,m$. For sufficiently large t we get

$$\|\sum_{k=2}^m F_k(t)(\upsilon(z_1,s,t)^k) - \sum_{k=2}^m F_k(t)(\upsilon(z_2,s,t)^k)\| \le \|\upsilon(z_1,s,t)-\upsilon(z_2,s,t)\|,$$

for $\|z_1\|, \|z_2\| \leq r$, which implies that for sufficiently large t

$$\upsilon(z,s,t) + \sum_{k=2}^{m} F_k(t)(\upsilon(z,s,t)^k),$$

is univalent on the ball $\|z\| \leq r$. Now the conclusion follows easily. □

From Corollary 3.3 and Theorem 3.7 we have the following consequences.

Corollary 3.8. *All $A(t)$-normalized polynomially bounded solution of (1.2) are Loewner chains.*

Theorem 3.9. *Let $f : B^n \to \mathbb{C}^n$ be a holomorphic mapping and*

$$f(z) = z + \sum_{k=2}^{\infty} G_k(z^k).$$

Then f is $A(t)$-asymptotically spirallike with respect to A if and only if there exists $h \in H(B^n)$, $Dh(0) = A(t)$ such that

$$f(z) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau}(\upsilon(z,t) + \sum_{k=2}^{n_0} G_k(\upsilon(z,t)^k)), \tag{3.5}$$

exists locally uniformly in $\in B^n$, where υ is the solution of (2.5).

Proof. First assume that f is $A(t)$-asymptotically spirallike with respect to A. Hence there exists the mapping $Q : f(B^n) \times [0,\infty) \to \mathbb{C}^n$ satisfying the assumptions of Definition 2.8. Let υ be the solution of the initial value problem (2.5). By definition it will satisfy

$$f(z) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} V(f(z), 0, t),$$

locally uniformly on $\in B^n$.

Let υ be defined by $\upsilon(z,s,t) = f^{-1}(V(f(z),s,t))$, $z \in B^n$ and $t \geq s$. Also, let $h(z,t) = [Df]^{-1}Q(f(z),t)$, $t \geq 0$. With the same proof as in [10, Theorem 3.5], one sees that $h \in H(B^n)$, $Dh(0) = A(t)$ and that υ is the solution of (2.5).

We have

$$f(z) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} V(f(z), 0, t) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} f(\upsilon(z,0,t))$$

locally uniformly on $\in B^n$. Similar to the proof of Theorem 3.2 we also see that

$$\lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau} f(\upsilon(z,0,t)) = \lim_{t\to\infty} e^{\int_0^t A(\tau)\mathrm{d}\tau}(\upsilon(z,0,t) + \sum_{k=2}^{n_0} G_k(\upsilon(z,0,t)^k)),$$

yielding the desired conclusion (the fact that f is univalent follows from Theorem 3.7).

Now assume that (3.5) holds. The conclusion follows exactly as in the proof of [10] Theorem 3.1. □

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