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TOTAL PERFECT CODES, OO-IRREDUNDANT AND TOTAL SUBDIVISION IN GRAPHS

H. HOSSEINZADEH AND N. SOLTANKHAH*

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ABSTRACT. Let $G = (V(G), E(G))$ be a graph, $\gamma_t(G)$. Let $ooir(G)$ be the total domination and OO-irredundance number of G, respectively. A total dominating set S of G is called a *total perfect code* if every vertex in $V(G)$ is adjacent to exactly one vertex of S. In this paper, we show that if G has a total perfect code, then $\gamma_t(G) = ooir(G)$. As a consequence, we determine the value of $ooir(G)$ for some classes of graphs. Finally, we prove some new bounds for the total subdivision number.

Keywords: Total domination number, OO- irredundance number, total subdivision number.
MSC(2010): Primary 05; Secondary C69.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph and $u \in X \subseteq V(G)$. The *open neighborhood* of u is the set of all vertices adjacent to u which is denoted by $N(u)$ and, the *close neighborhood* of u is $N[u] = N(u) \cup \{u\}$. The open (closed) neighborhood of X is $N(X) = \cup_{u\in X} N(u)$ $(N[X] = \cup_{u\in X} N[u])$. A subset $S \subseteq V(G)$ is a *dominating* set of G if every vertex of $V(G)/S$ is adjacent to some vertices of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of all dominating sets of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. If every vertex of $V(G)/S$ is adjacent to exactly one vertex of S and S is also an independent set, then S is called a *perfect code* or an *efficient dominating set* [5]. Domination is one of the major and well studied areas in graph theory. For more details on this concept see [5]. Among many types of dominating sets, total dominating sets have been investigated extensively [8]. A subset S of vertices in a graph G is a *total dominating* set if each vertex of $V(G)$ is adjacent to some vertices of S. The *total domination number* is the minimum cardinality of all minimal total dominating sets of G and is

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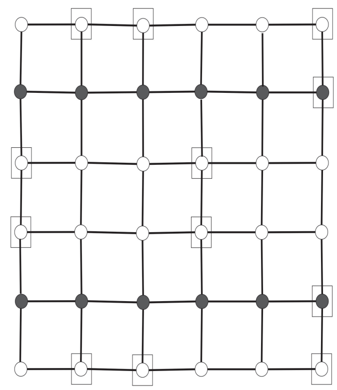

FIGURE 1. Minimum total dominating sets for $P_6 \Box P_6$

denoted by $\gamma_t(G)$. The total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. If every vertex in $V(G)$ is adjacent to exactly one vertex of S, then S is called a *total perfect code* or an *efficient open dominating set* [3]. Not necessarily all graphs have a perfect or total perfect code. For example, the cycle C_5 has neither a perfect code nor a total perfect code, C_6 has a perfect code but not any total perfect code and C_4 has total perfect code but not any perfect code. In this paper we focus on total perfect codes. At the beginning we state some basic results which are used in our proofs.

Theorem 1.1. *[3] If a graph G has a total perfect code S, then $|S| = \gamma_t(G)$ and all total perfect codes have the same cardinality.*

Theorem 1.2. *[3] For a graph G the following are equivalent:*
(a) $C = \{v_1, v_2, \ldots, v_k\}$ is a total perfect code of G.
(b) $\{N(v_1), N(v_2), \ldots, N(v_k)\}$ is a partition of $V(G)$.
(c) $|V(G)| = \sum_{v \in C} \deg(v)$

We note that not necessarily all $\gamma_t(G)$-sets are perfect. For example, in Figure 1, squared vertices is the total perfect code of $P_6 \Box P_6$ and black vertices form the total dominating set of minimum size which is not perfect.

Determining whether an arbitrary graph G has a total perfect code is an NP-complete problem [5] but the existence of total perfect codes for some classes of graphs has been examined. A total perfect code allows a graph to enjoy several properties. For example, as an easy observation it can lead to a better bound for the matching number. We recall that a *matching* in a graph is a set of pairwise nonadjacent edges. Number of edges in the maximum matching of graph G is the matching number $\alpha'(G)$ of G.

It is not hard to see that for every graph G with no isolated vertex, $\gamma(G) \leq \alpha'(G)$. Since $\gamma(G) \leq \gamma_t(G)$, Henning naturally asked in his survey [7] if the inequality $\gamma_t(G) \le \alpha'(G)$ holds for every graph G with sufficiently large minimum degree? This inequality was proved for the family of claw-free graphs with minimum degree at least three and for the family of k-regular graphs when $k \ge 3$ (see [7]). We will show that this is true for all graphs which have total perfect code.

Observation: If G is a graph with $\delta(G) \ge 2$ that has a total perfect code, then $\gamma_t(G) \le \alpha'(G)$.

Proof. Let $C = \{v_1, v_2, \ldots, v_k\}$ be a total perfect code of G. We may assume that $v_i v_{i+1} \in E(G)$, for all odd i with $1 \le i \le k-1$. By Theorem 1.1, $N(v_1) \cup \ldots \cup N(v_k)$ is a partition of $V(G)$. Since $\delta(G) \ge 2$, for each i, $1 \le i \le k$, there is $u_i \in N(v_i) \setminus C$. The edges $u_i v_i$ for $1 \le i \le k$ form a matching for the graph G. Therefore, $\gamma_t(G) \le \alpha'(G)$. □

In the next section we will establish the exact values of OO-irredundance number for some classes of graphs. In Section 3 we will give some bounds for total subdivision domination number of some graphs.

2. Total perfect codes and OO-irredundance

The set $X \subseteq V(G)$ is OO-irredundance if and only if for each $v \in X$, $N(v) - N(X \setminus \{v\}) \ne \emptyset$. For convenience we let $PN(v) = N(v) \setminus N(X \setminus \{v\})$. The minimum cardinality among all maximal OO-irredundant set denoted by $ooir(G)$ and is called OO-irredundance number of the graph G. To the best of our knowledge *ooir* has only been determined for paths and cycles [1]. The main theorem of this section leads to determination of the exact value of OO-irredundace number for several classes of graphs.

The following proposition is an easy consequence of definitions.

Proposition 2.1. *Any minimal total dominating set of a graph G is also a maximal OO-irredundant set. Therefore, $ooir(G) \le \gamma_t(G)$.*

Proof. Assume that S is a minimal total dominating set of G. Then, for each $u \in S$ there is $v \in V(G)$ such that $N(v) \cap N(S) = \{u\}$. Therefore, $N(v) \setminus N(S \setminus \{v\}) \ne \emptyset$ and S is an OO-irredundant set. On the other hand, for each $x \in V \setminus S$, $S \cup \{x\}$ is not an OO-irredundant set since otherwise there is $v \in V$ that is not adjacent to any vertex in S. □

The following lemma is our main result of this section.

Lemma 2.2. *Let $G = (V, E)$ be a simple graph, $C = \{v_1, v_2, \ldots, v_k\}$ total perfect code and $S \subseteq V$ a maximal OO-irredundant set of G. Then, $|S| \ge |C|$.*

Proof. Without loss of generality we can assume that $v_1 \notin S$. We claim that we can find a vertex $x(v_1) \in (V(G) \setminus C) \cap S$. Since S is a maximal OO-irredundant set, $S \cup \{v_1\}$ is not OO-irredundant. This means that there is at least one vertex

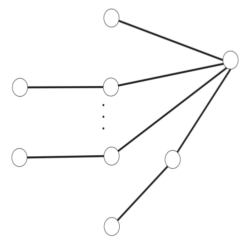

Figure 2. A tree with large total domination number and small OO-irredundance number

in $S \cup \{v_1\}$ that has no private neighborhood. To proceed with the proof, we consider the following two cases:
Case 1) There is $y \in S$ such that $N(y) \setminus N(S \setminus \{y\}) \subseteq N(v_1)$. This means that all private neighbors of y belong to $N(v_1)$. Accordingly, we define $x(v_1) = y$. Furthermore, we define $X_1(v_1) \subseteq S$ to be the set of all vertices $x(v_1)$ (The index 1 for X indicates the Case 1.)
Case 2) $N(v_1) \subseteq N(S)$. Note that this implies $N(v_1) \setminus N(S \cup \{v_1\}) = \emptyset$. In this case we define $x(v_1)$ to be a vertex in S which has a common neighbor with v_1. Likewise, we define the set $X_2(v_1)$ consisting of vertices in S of this type.
Note that in both cases $X_i(v_1) \cap C = \emptyset$ $(i \in \{1, 2\})$. Now it remains to prove that if $v_i, v_j \notin S$ then $x(v_i) \neq x(v_j)$. Since $X_2(v_i) \cap X_2(v_j) = \emptyset$ without loss of generality we can assume $X_2(v_i) = X_2(v_j) = \emptyset$ and $X_1(v_i) = X_1(v_j) = \{y\}$. This means that y has at least one private neighbor in $N(v_i)$ and one is in $N(v_j)$. Now consider $S \bigcup \{v_i\}$. Since it is not an OO-irredundant set, there is $z \in S \bigcup \{v_i\}$ such that $N(z) \setminus N((S \bigcup \{v_i\}) \setminus z) \neq \emptyset$. We know that $z \neq v_i$ since otherwise $z \in X_2(v_i)$. Also $z \neq y$ contradicts the assumption that $X_1(v_i) = \{y\}$. By symmetry this is true for v_j. We conclude that $S \setminus \{y\} \cup \{v_i, v_j\}$ is an OO-irredundant set for a subset of G properly containing S, and this contradiction with maximality of S. Hence either $X_2(v_i) = X_2(v_j) \neq \emptyset$ or $|X_1(v_i) \Delta X_1(v_j)| \geqslant 2$. □

The previous lemma yields that if the graph G has a total perfect code, then $\gamma_t(G) \leq ooir(G)$. Therefore, by Proposition 2.1 we have the following theorem.

Theorem 2.3. *If a graph G has a total perfect code, then $ooir(G) = \gamma_t(G)$.*

The tree T in Figure 2 shows that the difference between $ooir(G)$ and $\gamma_t(G)$ can be arbitrarily large. Note that T is of order $2k + 2$. The vertex of degree $k + 1$ is called the root of T. The root and one of its neighbors is a maximal OO-irredundant set of minimum size, hence $ooir(T) = 2$. While, we need to pick up all vertices of degree 2 and the root in the minimum total dominating set of T, we have $\gamma_t(T) = k + 1$.

The existence of total perfect codes for some classes of graphs has been investigated in the literatures. For example, P_n has a total perfect code if and only if $n \not\equiv 1 \pmod 4$ and C_n has a total perfect code if and only if $n \equiv 0 (\text{mod } 4)$ ([3]). Also we have $\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & n \equiv 0(\text{mod } 4) \\ \lfloor \frac{n}{2} \rfloor + 1 & \textit{otherwise} \end{cases}$ [5]. This means we can easily compute the value of *ooir* for infinitely many cycles and paths. This has been a challenge in [1]. Grid graphs $P_n \Box P_m$ which have total perfect code are characterized in [9, 12]. In particular, it is proved in [9] that for two positive integers m and n, $n \geq m \geq 2$, $P_n \Box P_m$ has a total perfect code if and only if m is even and $n = m + c(m+1)$ or $n = m + 2 + c(m+1)$ or $n = m - 2 + c(m+1)$, for a nonnegative integer c. All trees which have a total perfect codes are found in [3]. So using our result, we can find the value of *ooir* for these graphs.

In the following we consider the existence of total perfect codes for some other graphs.

Proposition 2.4. *The hypercube Q_n with 2^n vertices has a total perfect code if and only if $n = 2^m$ for some integer m.*

Proof. First assume that $C = \{v_1, \dots, v_k\}$ is a total perfect code of Q_n. Then, by Theorem 1.1, $V(Q_n) = N(v_1) \cup \cdots \cup N(v_k)$. Since for each $1 \leqslant i \leqslant k$, $|N(v_i)| = n$, thus $2^n = |V(Q_n)| = k \times n$. So $n = 2^m$ for an integer m. Also $Q_{2^m} = Q_{2^{(m-1)}} \Box P_2$, where $\Box$ shows the Cartesian product. But it is proved that $Q_{2^{m-1}}$ has perfect codes [10]. Let C be a perfect code of $Q_{2^{m-1}}$, then $V(Q_{2^{m-1}})[C] \Box P_2$ is a total perfect code of Q_{2^m}. $\square$

Proposition 2.5. *If both m and n are integer multiples of 4, then $C_n \Box C_m$ has a total perfect code.*

Proof. We know that C_n $(n \equiv 0(\text{ mod } 4))$ has a total perfect code of size $\frac{n}{2}$. So we can assume P and $\bar{P}$ are two disjoint total perfect codes for C_n such that $P \cup \bar{P} = V(C_n)$. Let $C_n^1, C_n^2, \cdots, C_n^m$ be m copies of C_n in $C_n \Box C_m$. For convenience we denote the vertex set of P in jth copy of C_n by $P_j (1 \leqslant j \leqslant m)$. We define two sequences $\{a_i\}_{i=2}^m$ and $\{b_j\}_{j=2}^m$ as follows: $a_{i+1} = a_i + 4$, , $a_1 = 1$ and $b_{j+1} = b_j + 4$, $b_1 = 3$. Also we define the set $T = \{\cup_{i=1}^m P_{a_i} \bigcup \cup_{j=1}^m \bar{P}_{b_j}\}$. The condition $m \equiv 0 \pmod 4$, guarantees that T is a total perfect code of $C_n \Box C_m$.

$\square$

Therefore, there are plenty of graphs for which we can compute their OO-irredundance numbers rather easily by Theorem 2.3, while it is too computational to determine them directly.

3. **Total perfect code and total subdivision number**

We say that an edge $uv \in E$ is subdivided if the edge uv is deleted, but a new vertex x is added along with two new edges ux and xv. We only permit an edge to be subdivided once. The *domination subdivision* number $sd_\gamma(G)$ of a graph G is the minimum number of edges that must be subdivided in order to increase the domination number. We also define the *total domination subdivision* number $(sd_{\gamma_t}(G))$ of G in a same way. Since the total domination number of a single edge does not change when it is subdivided, in the study of total domination subdivision numbers we must assume that the graph is of order $n \geq 3$. It is known that these two parameters can be arbitrarily large [4,6]. It is also difficult to find good upper bounds for them. The best known upper bound for the total domination subdivision number of a connected graph G of order $n \geq 3$ is $\lfloor \frac{2n}{3} \rfloor$ [2].

The following theorems shows that if a graph G has a total perfect code, then we can find a new upper bound for the total subdivision number of G.

Theorem 3.1. *Let G be a graph with $\delta(G) \geq 2$ and let $C = \{v_1, v_2, \ldots, v_k\}$ be a total perfect code of G. If there is an edge $v_iv_j \in E(C)$ such that $N(v_i) \cup N(v_j)/\{v_i, v_j\}$ is not an independent set in G, then $sd_{\gamma_t}(G) \leq 2$.*

Proof. We show that there are two edges in the graph G such that the total domination number of G will increase if we subdivide them. Assume $v_iv_j \in E(C)$ and that $N(v_i) \cup N(v_j)/\{v_i, v_j\}$ is not an independent set. Let G' be the graph obtained by subdividing two edges v_iv_j and $ab \in E(G[N(v_i) \cup N(v_j)/\{v_i, v_j\}])$ with two vertices x and y, respectively. Also let S' be a $\gamma_t(G')$-set. We consider the following two cases:

(1) $|S' \cap \{x, y\}| \in \{1, 2\}$. It is not hard to see that vertices x and y can not totally dominate $N(v_i) \cup N(v_j)$ and we need at least two more vertices from $N(v_i) \cup N(v_j)$. So $|S' \cap \{N(v_i) \cup N(v_j)\}| \geqslant 3$. Also we know that $V(G) = N(v_1) \cup \cdots \cup N(v_k)$. So to totally dominate other vertices of $V(G')$ we need at least one vertex of each set. This means we need at least $k + 1$ vertices in S' and hence we have $\gamma_t(G') > \gamma_t(G)$.

(2) $|S' \cap \{x, y\}| = 0$. First note that a unique vertex of $N(v_i) \cup N(v_j)$ can not dominate both x and y, also $|S' \cap \{v_i, v_j\}| \geqslant 1$. Without loss of generality we can assume $v_i \in S'$. Now to totally dominate v_i we need $|S' \cap N(v_i)| \geqslant 1$. But none of the already chosen vertices can dominate v_j. So we need at least one other vertex of $S' \cap N(v_j)$. This means $|S' \cap \{N(v_i) \cup N(v_j)\}| \geqslant 3$. So, $\gamma_t(G') > \gamma_t(G)$.

□

Haynes et al. [6] proved that for any grid graph $P_n \Box P_m$, $1 \leq sd_{\gamma_t}(P_n \Box P_m) \leq 4$. Soltankhah [11] improved this bound and proved that $sd_{\gamma_t}(P_n \Box P_m) \leq 3$ for $m, n \geq 3$. Since total perfect codes of grid graphs satisfy the conditions of

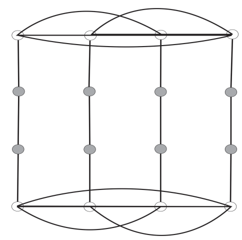

FIGURE 3. A graph with a total perfect code and $sd_{\gamma_t} > 2$

Theorem 3.1, we conclude that, $sd_{\gamma_t}(P_n \Box P_m) \leq 2$ for any grid graphs which has a total perfect code.

The Conditions in Theorem 3.1 are in fact necessary. In the graph of Figure 3, black vertices form a total perfect code of the graph and it is easy to check that subdividing any two edges of the graph do not increase the total domination number. However, our next theorem shows that the total subdivision number of this graph is equal to 3.

Theorem 3.2. *Let G be a connected graph with $\delta(G) \geq 2$ and the set $C = \{v_1, v_2, \ldots, v_k\}$ be a total perfect code of G. If for each $v_iv_j \in E(G)$ such that $v_i, v_j \in C$, $N(v_i) \cup N(v_j)/\{v_i, v_j\}$ is an independent set in G, then $sd_{\gamma_t}(G) \leq 3$.*

Proof. According to Theorem 1.1, $V(G) = N(v_1) \cup \ldots \cup N(v_k)$ is a partition of $V(G)$. We choose two integers $i \neq j \in \{1, \ldots, k\}$ such that $v_iv_j \in E(G)$. Let $x \in N(v_i)$ and $y \in N(v_j)$.

We can show that the total domination number increases by subdividing three edges xv_i, v_iv_j and v_jy with three vertices a, b and c, respectively. Let G' be the resulting graph and let S' be a $\gamma_t(G')-$set. First assume that, $|S' \cap \{a, b, c\}| = 3$. To totally dominate vertices in $\bigcup_{l=1, l\neq i,j}^{k} N(v_l)$ we need at least $k-2$ vertices in S', so we conclude $\gamma_t(G') > \gamma_t(G)$. Also if $|S' \cap \{a, b, c\}| = \{1, 2\}$, we need at least two other vertices of $N(v_i) \cup N(v_j)$ to totally dominate the set $\{a, b, c, v_i, v_j\}$. Thus again we have $\gamma_t(G') > \gamma_t(G)$. Finally, if $|S' \cap \{a, b, c\}| = 0$ then $|S' \cap \{v_i, v_j\}| \geq 1$. Let $v_i \in S'$, to totally dominate v_i there must be a vertex $t \in N(v_i) \cap S'$ but neither t nor v_i can dominate v_j. Therefore, S' contains at least one other vertex of $N(v_j)$. Thus $\gamma_t(G') > \gamma_t(G)$. □

REFERENCES

[1] E. J. Cockayne, S. Finbow and J. S. Swarts, OO-irredundance and maximum degree in paths and trees, *J. Combin. Math. Combin. Comput.* **73** (2010) 223–236.

[2] O. Favaron, H. Karami and S. M. Sheikholeslami, Bounding the total domination subdivision number of a graph in terms of its order, *J. Comb. Optim.* **21** (2011), no. 2, 209–218.

[3] H. Gavlas and K. Schultz, Efficient open domination. The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications, 11 pages, Electron. Notes Discrete Math., 11, Elsevier Sci. B. V., Amsterdam, 2002.

[4] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely and L. C. van der Merwe, Domination subdivision numbers, *Discuss. Math. Graph Theory* **21** (2001), no. 2, 239–253.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of domination in graphs, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1998.

[6] T. W. Haynes, S. T. Hedetniemi and L. C. van der Merwe, Total domination subdivision numbers, *J. Combin. Math. Combin. Comput.* **44** (2003) 115–128.

[7] M. A. Henning, A survey of selected recent results on total domination in graphs, *Discrete Math.* **309** (2009), no. 1, 32–63.

[8] M. A. Henning and A. Yeo, Total Domination in Graphs, Monographs in Mathematics, Springer, New York, 2013.

[9] W. F. Klostermeyer and J. L. Goldwasser, Total perfect codes in grid graphs, *Bull. Inst. Combin. Appl.* **46** (2006) 61–68.

[10] J. Lee, Independent perfect domination sets in Cayley graphs, *J. Graph Theory* **37** (2001), no. 4, 213–219.

[11] N. Soltankhah, On the total domination subdivision numbers of grid graphs, *Int. J. Contemp. Math. Sci.* **5** (2010), no. 49-52, 2419–2432.

[12] N. Soltankhah, Results on total domination and total restrained domination in grid graphs, *Int. Math. Forum* **5** (2010), no. 5-8, 319–332.

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