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NONLINEAR $*$-LIE HIGHER DERIVATIONS ON FACTOR VON NEUMANN ALGEBRAS

F. ZHANG*, X. QI AND J. ZHANG

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Abstract. Let $\mathcal{M}$ be a factor von Neumann algebra. It is shown that every nonlinear $*$-Lie higher derivation $D = \{\phi_n\}_{n\in\mathbb{N}}$ on $\mathcal{M}$ is additive. In particular, if $\mathcal{M}$ is infinite type I factor, a concrete characterization of D is given.

Keywords: Von Neumann algebra, nonlinear $*$-Lie higher derivation, additive $*$-higher derivation.

MSC(2010): Primary: 47B49; Secondary: 15A78, 16W25.

1. **Introduction**

Let $\mathcal{A}$ be any ring. Recall that an additive map $\delta : \mathcal{A} \to \mathcal{A}$ is a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$; in particular, δ is called an *inner derivation* if there exists some $T \in \mathcal{A}$ such that $\delta(A) = AT - TA$ for all $A \in \mathcal{A}$. More generally, δ is said to be a *Lie derivation* if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$. The question of characterizing Lie derivations and revealing the relationship between Lie derivations and derivations have been studied by many authors (see [1–4, 8, 9, 12, 19, 20]).

Let $\mathcal{A}$ be a $*$-ring. An additive map $\delta : \mathcal{A} \to \mathcal{A}$ is a $*$-*derivation* if it is a derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$; is a $*$-*Lie derivation* if $\delta([A, B]_*) = [\delta(A), B]_* + [A, \delta(B)]_*$ for all $A, B \in \mathcal{A}$, where $[A, B]_* = AB - BA^*$. In addition, if the additivity of δ is deleted, then δ is called a *nonlinear* $*$-*derivation* and *nonlinear* $*$-*Lie derivation*, respectively. Yu and Zhang [18] proved that every nonlinear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation.

On the other hand, many different kinds of higher derivations also have been studied in commutative and noncommutative rings. Let $\mathcal{A}$ be an associative

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$*$-algebra over a commutative ring $\mathcal{R}$. Denote by $\mathbb{N}$ the set of all non-negative integers and let $D = \{\phi_n\}_{n\in\mathbb{N}}$ be a family of $\mathcal{R}$-linear mappings on $\mathcal{A}$ such that $\phi_0 = id_{\mathcal{A}}$. D is called:

(a) a *higher derivation* if for each $n \in \mathbb{N}$,

$$\phi_n(xy) = \sum_{i+j=n} \phi_i(x)\phi_j(y)$$

for all $x, y \in \mathcal{A}$;

(b) a *Lie higher derivation* if for each $n \in \mathbb{N}$,

$$\phi_n([x, y]) = \sum_{i+j=n} [\phi_i(x), \phi_j(y)]$$

for all $x, y \in \mathcal{A}$.

(c) an *inner higher derivation* if $\mathcal{A}$ is unital and there exist two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ satisfying the conditions $a_0 = b_0 = 1$ and $\sum_{i=0}^{n} a_i b_{n-i} = \delta_{n0} = \sum_{i=0}^{n} b_i a_{n-i}$ such that

$$\phi_n(x) = \sum_{i=0}^{n} a_i x b_{n-i}$$

for all $x \in \mathcal{A}$ and for each $n \in \mathbb{N}$, where δ_{n0} is the Kronecker sign.

If $n = 1$, then higher derivations, Lie higher derivations and inner higher derivations are usual derivations, Lie derivations and inner derivations, respectively. In addition, D is called a *nonlinear Lie higher derivation* if the $\mathcal{R}$-linearity of D in the above (b) is removed. The structure of Lie higher derivations also had been discussed by many authors. Qi and Hou [13] gave a characterization of Lie higher derivations on nest algebras. Xiao [17] proved that every nonlinear Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a nonlinear functional vanishing on all commutators. For other results, see [5–7, 10, 11, 14, 16] and the references therein.

Motivated by $*$-Lie derivation, we here can introduce a concept of $*$-Lie higher derivations. We say that D is a *$*$-Lie higher derivation* if for each $n \in \mathbb{N}$,

$$[\phi_n([x, y]_*) = \sum_{i+j=n} [\phi_i(x), \phi_j(y)]_*$$

holds for all $x, y \in \mathcal{A}$. If D have no any linearity, then D is called a *nonlinear $*$-Lie higher derivation*. Obviously, $*$-Lie higher derivations are $*$-Lie derivation if $n = 1$.

The purpose of this paper is to consider nonlinear $*$-Lie higher derivations on factor von Neumann algebras.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. Let $\mathcal{H}$ be a complex Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Recall that

$\mathcal{M}$ is a factor if its center is $\mathbb{C}I$, where I is the identity of $\mathcal{M}$. Let $\mathcal{M}_{sa}$ be the subspace of all self-adjoint operators in $\mathcal{M}$.

2. **Main result**

The following is our main result.

Theorem 2.1. *Let $\mathcal{M}$ be factor von Neumann algebras acting on a complex Hilbert space $\mathcal{H}$ with dim $\mathcal{H} \geq 2$. Suppose that $D = \{\phi_n\}_{n\in\mathbb{N}}$ is a nonlinear $*$-Lie higher derivation on $\mathcal{M}$, then $D = \{\phi_n\}_{n\in\mathbb{N}}$ is an additive $*$-higher derivation.*

To prove Theorem 2.1, we need some lemmas. The following three lemmas can be found in [18].

Lemma 2.2. *Let $A \in \mathcal{M}$. Then $AB = BA^*$ for every $B \in \mathcal{M}$ implies that $A \in \mathbb{R}I$.*

Lemma 2.3. *Let $B \in \mathcal{M}$. Then $AB = BA^*$ for every $A \in \mathcal{M}$ implies that $B = 0$.*

Lemma 2.4. *Let $P \in \mathcal{M}$ be a nontrivial projection and $A \in \mathcal{M}$. Then $AB = BA^*$ for every $B \in P\mathcal{M}(I-P)$ implies that $A = \mu P + \overline{\mu}(I-P)$ for some $\mu \in \mathbb{C}$.*

Now we chose a nontrivial projection $P_1 \in \mathcal{M}$ and set $P_2 = I - P_1$. Write $\mathcal{M}_{ij} = P_i\mathcal{M}P_j, i, j = 1, 2$.

Let $D = \{\phi_n\}_{n\in\mathbb{N}}$ be a nonlinear $*$-Lie higher derivation on $\mathcal{M}$. We define $\Delta_n : \mathcal{M} \to \mathcal{M}$ by

$$\Delta_n(A) = \phi_n(A) - [A, U_n],$$

where $U_n = P_1\phi_n(P_1)P_2 - P_2\phi_n(P_1)P_1$. One can verify that $L = \{\Delta_n\}_{n\in\mathbb{N}}$ is also a nonlinear $*$-Lie higher derivation on $\mathcal{M}$.

By [18, Lemmas 2.4, 2.6, Remark 2.1], we can get the following lemma.

Lemma 2.5. Δ_1 *has the following properties:*

(1) $\Delta_1(0) = 0$;
(2) $\Delta_1(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$;
(3) $\Delta_1(\mathbb{C}I) \subseteq \mathbb{C}I$;
(4) $\Delta_1(\frac{1}{2}iI) = 0$;
(5) $\Delta_1(iA) = i\Delta_1(A)$ *for all* $A \in \mathcal{M}$;
(6) $\Delta_1(P_i) \in \mathbb{C}I$ *for* $i = 1, 2$;
(7) $\Delta_1(\mathcal{M}_{ij}) \subseteq \mathcal{M}_{ij}$ *for* $i, j = 1, 2$.

Lemma 2.6. Δ_n *has the following properties:*

(1) $\Delta_n(0) = 0$ *for each* $n \in \mathbb{N}$;
(2) $\Delta_n(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$ *for each* $n \in \mathbb{N}$;

(3) $\Delta_n(\mathbb{C}I) \subseteq \mathbb{C}I$ *for each* $n \in \mathbb{N}$;
(4) $\Delta_n(\frac{1}{2}iI) = 0$ *for each* $n \in \mathbb{N}$ *with* $n \geq 1$;
(5) $\Delta_n(iA) = i\Delta_n(A)$ *for all* $A \in \mathcal{M}$ *and for each* $n \in \mathbb{N}$;
(6) $\Delta_n(P_i) \in \mathbb{R}I$ *for* $i = 1, 2$ *and for each* $n \in \mathbb{N}$ *with* $n \geq 1$;
(7) $\Delta_n(\mathcal{M}_{ij}) \subseteq \mathcal{M}_{ij}$ *for* $i, j = 1, 2$ *and for each* $n \in \mathbb{N}$.

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. If $n = 1$, by Lemma 2.5, it is true.

Now we assume that Lemma 2.6 holds for $k < n \in \mathbb{N}$, that is,

$$\Delta_k(0) = 0, \Delta_k(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}, \Delta_k(\mathbb{C}I) \subseteq \mathbb{C}I, \Delta_k(\frac{1}{2}iI) = 0 (k \neq 0),$$

$$\Delta_k(iA) = i\Delta_k(A), \Delta_k(P_i) \in \mathbb{C}I (k \neq 0), \Delta_k(\mathcal{M}_{ij}) \subseteq \mathcal{M}_{ij}, i, j = 1, 2.$$

Our aim is to show that Δ_n satisfies the similar properties. We will prove it by using similar arguments as used in [18].

(1) By the induction hypothesis,

$$\Delta_n(0) = \Delta_n([0, 0]_*) = \sum_{p+q=n} [\Delta_p(0), \Delta_q(0)]_* = [\Delta_n(0), 0]_* + [0, \Delta_n(0)]_* = 0.$$

(2) It follows from $\Delta_k(\mathbb{C}I) \subseteq \mathbb{C}I$ and $\Delta_k(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$ that $\Delta_k(I) = \Delta_k(I)^* \in \mathbb{R}I$. Let $T \in \mathcal{M}$, then

$$[\Delta_n(I), T]_* + [I, \Delta_n(T)]_* = \sum_{p+q=n} [\Delta_p(I), \Delta_q(T)]_* = \Delta_n([I, T]_*) = 0.$$

This implies that $\Delta_n(I)T = T\Delta_n(I)^*$ for all $T \in \mathcal{M}$. By Lemma 2.2, $\Delta_n(I) = \Delta_n(I)^* \in \mathbb{R}I$, and so we have for $A \in \mathcal{M}_{sa}$,

$$\begin{aligned} \Delta_n(A) - \Delta_n(A)^* &= [\Delta_n(A), I]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A), \Delta_q(I)]_* + [A, \Delta_n(I)]_* \\ &= \Delta_n([A, I]_*) = 0. \end{aligned}$$

Thus $\Delta_n(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$.

(3) Let $\lambda \in \mathbb{C}$, we have for any $A \in \mathcal{M}_{sa}$,

$$\begin{aligned} & [\Delta_n(A), \lambda I]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A), \Delta_q(\lambda I)]_* + [A, \Delta_n(\lambda I)]_* \\ = \ & \Delta_n([A, \lambda I]_*) = 0. \end{aligned}$$

It follows from $\Delta_n(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$ and the induction hypothesis that

$$A\Delta_n(\lambda I) = \Delta_n(\lambda I)A$$

for all $A \in \mathcal{M}_{sa}$. Hence $\Delta_n(\mathbb{C}I) \subseteq \mathbb{C}I$.

(4) Since $\Delta_n(\mathbb{C}I) \subseteq \mathbb{C}I$ and $\Delta_n(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$, we have $\Delta_n(-\frac{1}{2}I) \in \mathbb{R}I$. It follows from $[\frac{1}{2}iI, \frac{1}{2}iI]_* = -\frac{1}{2}I$ and $\Delta_k(\frac{1}{2}iI) = 0$ that

$$\begin{aligned} & i\Delta_n(\frac{1}{2}iI) + \frac{1}{2}i(\Delta_n(\frac{1}{2}iI) - \Delta_n(\frac{1}{2}iI)^*) \\ = \quad & [\frac{1}{2}iI, \Delta_n(\frac{1}{2}iI)]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(\frac{1}{2}iI), \Delta_q(\frac{1}{2}iI)]_* + [\Delta_n(\frac{1}{2}iI), \frac{1}{2}iI]_* \\ = \quad & \Delta_n([\frac{1}{2}iI, \frac{1}{2}iI]_*) = \Delta_n(-\frac{1}{2}I) \in \mathbb{R}I. \end{aligned}$$

We have from above equation that $\Delta_n(\frac{1}{2}iI)^* = -\Delta_n(\frac{1}{2}iI)$. Hence $\Delta_n(-\frac{1}{2}I) = 2i\Delta_n(\frac{1}{2}iI)$. Similarly, we can obtain from the fact $[-\frac{1}{2}iI, -\frac{1}{2}iI]_* = -\frac{1}{2}I$ that $\Delta_n(-\frac{1}{2}iI)^* = -\Delta_n(-\frac{1}{2}iI)$ and $\Delta_n(-\frac{1}{2}I) = -2i\Delta_n(-\frac{1}{2}iI)$. Thus $\Delta_n(-\frac{1}{2}iI) = -\Delta_n(\frac{1}{2}iI)$. It follows from $\Delta_k(\frac{1}{2}iI) = 0$ that

$$\begin{aligned} & \Delta_n(\frac{1}{2}iI) = \Delta_n([-\frac{1}{2}iI, -\frac{1}{2}I]_*) \\ = \quad & [\Delta_n(-\frac{1}{2}iI), -\frac{1}{2}I]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(-\frac{1}{2}iI), \Delta_q(-\frac{1}{2}I)]_* \\ + \quad & [-\frac{1}{2}iI, \Delta_n(-\frac{1}{2}I)]_* \\ = \quad & -\Delta_n(-\frac{1}{2}iI) - i\Delta_n(-\frac{1}{2}I) = \Delta_n(\frac{1}{2}iI) - i\Delta_n(-\frac{1}{2}I). \end{aligned}$$

This implies that $\Delta_n(-\frac{1}{2}I) = 0$, and so $\Delta_n(\frac{1}{2}iI) = 0$.

(5) For every $A \in \mathcal{M}$, we have

$$\begin{aligned} & \Delta_n(iA) = \Delta_n([\frac{1}{2}iI, A]_*) \\ = \quad & [\Delta_n(\frac{1}{2}iI), A]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(\frac{1}{2}iI), \Delta_q(A)]_* + [\frac{1}{2}iI, \Delta_n(A)]_* \\ = \quad & i\Delta_n(A). \end{aligned}$$

(6) Since $A_{12} = [P_1, A_{12}]_*$ for all $A_{12} \in \mathcal{M}_{12}$, it follows from $\Delta_n(\mathcal{M}_{sa}) \subseteq \mathcal{M}_{sa}$ that

$$\Delta_n(A_{12}) = [\Delta_n(P_1), A_{12}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(P_1), \Delta_q(A_{12})]_* + [P_1, \Delta_n(A_{12})]_*.$$

By the induction hypothesis, the above relation implies

$$\Delta_n(A_{12}) = \Delta_n(P_1)A_{12} - A_{12}\Delta_n(P_1) + P_1\Delta_n(A_{12}) - \Delta_n(A_{12})P_1.$$

Then

$$P_1\Delta_n(P_1)A_{12} = A_{12}\Delta_n(P_1)P_2.$$

So for any $A_{12} \in \mathcal{M}_{12}$,

$$[P_1\Delta_n(P_1)P_1 + P_2\Delta_n(P_1)P_2, A_{12}]_* = 0.$$

Hence by Lemma 2.2, $P_1\Delta_n(P_1)P_1 + P_2\Delta_n(P_1)P_2 \in \mathbb{R}I$. From the definition of Δ_n, we get

$$\begin{aligned}\Delta_n(P_1) &= \phi_n(P_1) - [P_1, U_n] = P_1\phi_n(P_1)P_1 + P_2\phi_n(P_1)P_2 \\ &= P_1\Delta_n(P_1)P_1 + P_2\Delta_n(P_1)P_2 \in \mathbb{R}I.\end{aligned}$$

Since $[P_1, A]_* = -[P_2, A]_*$ for all $A \in \mathcal{M}$, we have

$$\begin{aligned}&[\Delta_n(P_1), A]_* + \sum_{\substack{p+q=n\\0<p,q<n}} [\Delta_p(P_1), \Delta_q(A)]_* + [P_1, \Delta_n(A)]_* \\ = \quad & -[\Delta_n(P_2), A]_* - \sum_{\substack{p+q=n\\0<p,q<n}} [\Delta_p(P_2), \Delta_q(A)]_* - [P_2, \Delta_n(A)]_*.\end{aligned}$$

Considering the induction hypothesis, the above equation becomes

$$[\Delta_n(P_2), A]_* = 0$$

for all $A \in \mathcal{M}$, by lemma 2.2, $\Delta_n(P_2) \in \mathbb{R}I$.

(7) Let $A_{12} \in \mathcal{M}_{12}$, it follows from (6) that

$$\Delta_n(A_{12}) = \Delta_n([P_1, A_{12}]_*) = [P_1, \Delta_n(A_{12})]_*$$

This yields

$$P_2\Delta_n(A_{12})P_1 = P_1\Delta_n(A_{12})P_1 = P_2\Delta_n(A_{12})P_2 = 0.$$

Then $\Delta_n(A_{12}) \in \mathcal{M}_{12}$ for all $A_{12} \in \mathcal{M}_{12}$. We can similarly prove $\Delta_n(A_{21}) \in \mathcal{M}_{21}$ is valid by considering $\Delta_n([P_2, A_{21}]_*)$.

Let $X \in \mathcal{M}_{11} \cup \mathcal{M}_{22}$. It follows from the fact $[P_i, X]_* = 0$ and $\Delta_n(P_i) \in \mathbb{R}I$ that

$$0 = \Delta_n([P_i, X]_*) = [P_i, \Delta_n(X)]_*.$$

This implies that $P_i\Delta_n(X)P_j = 0$ for $i, j \in \{1, 2\}$ with $i \neq j$. Let $A_{11} \in \mathcal{M}_{11}, B_{22} \in \mathcal{M}_{22}$. We have from the induction hypothesis and the fact $[A_{11}, B_{22}]_* = [B_{22}, A_{11}]_* = 0$ that

$$\begin{aligned}\Delta_n([A_{11}, B_{22}]_*) &= [\Delta_n(A_{11}), B_{22}]_* + [A_{11}, \Delta_n(B_{22})]_* \\ &= \sum_{\substack{p+q=n\\0<p,q<n}} [\Delta_p(A_{11}), \Delta_q(B_{22})]_* \\ &= [\Delta_n(A_{11}), B_{22}]_* + [A_{11}, \Delta_n(B_{22})]_* \\ &= [P_2\Delta_n(A_{11})P_2, B_{22}]_* + [A_{11}, P_1\Delta_n(B_{22})P_1]_* = 0.\end{aligned}$$

and

$$\Delta_n([B_{22}, A_{11}]_*) = [P_1\Delta_n(B_{22})P_1, A_{11}]_* + [B_{22}, P_2\Delta_n(A_{11})P_2]_* = 0.$$

This implies that

$$[A_{11}, P_1\Delta_n(B_{22})P_1]_* = 0$$

for all $A_{11} \in \mathcal{M}_{11}$ and

$$[B_{22}, P_2\Delta_n(A_{11})P_2]_* = 0$$

for all $B_{22} \in \mathcal{M}_{22}$. By Lemma 2.3, then

$$P_1\Delta_n(B_{22})P_1 = 0, \quad P_2\Delta_n(A_{11})P_2 = 0.$$

Hence $\Delta_n(\mathcal{M}_{ii}) \subseteq \mathcal{M}_{ii}$ for $i = 1, 2$. The proof is completed. □

In order to obtain Theorem 2.1, we proceed by induction on $n \in \mathbb{N}$. When $n = 1$, Δ_1 is a nonlinear $*$-Lie derivation on $\mathcal{M}$. By [18, Theorem 2.1], Δ_1 is an additive $*$-derivation. Now we assume that Δ_m is an additive higher $*$-derivation for $m < n \in \mathbb{N}$. Our aim is to show that Δ_n is an additive higher $*$-derivation.

Lemma 2.7. *Let $i, j \in \{1, 2\}$ with $i \neq j$. Then*

(1) $\Delta_n(A_{ii} + A_{ij}) = \Delta_n(A_{ii}) + \Delta_n(A_{ij})$ *for all* $A_{ii} \in \mathcal{M}_{ii}$ *and* $A_{ij} \in \mathcal{M}_{ij}$;
(2) $\Delta_n(A_{ii} + A_{ji}) = \Delta_n(A_{ii}) + \Delta_n(A_{ji})$ *for all* $A_{ii} \in \mathcal{M}_{ii}$ *and* $A_{ji} \in \mathcal{M}_{ji}$;
(3) $\Delta_n(A_{11} + A_{22}) = \Delta_n(A_{11}) + \Delta_n(A_{22})$ *for all* $A_{11} \in \mathcal{M}_{11}$ *and* $A_{22} \in \mathcal{M}_{22}$;
(4) $\Delta_n(A_{12} + A_{21}) = \Delta_n(A_{12}) + \Delta_n(A_{21})$ *for all* $A_{12} \in \mathcal{M}_{12}$ *and* $A_{21} \in \mathcal{M}_{21}$.

Proof. (1) Let $X_{jj} \in \mathcal{M}_{jj}$. It follows from $[X_{jj}, A_{ij}]_* = [X_{jj}, A_{ii} + A_{ij}]_*$, Lemma 2.6 and the induction hypothesis that

$$\begin{aligned}
& [\Delta_n(X_{jj}), A_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{jj}), \Delta_q(A_{ij})]_* + [X_{jj}, \Delta_n(A_{ij})]_* \\
= \; & [\Delta_n(X_{jj}), A_{ii} + A_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{jj}), \Delta_q(A_{ii} + A_{ij})]_* \\
+ \; & [X_{jj}, \Delta_n(A_{ii} + A_{ij})]_* \\
= \; & [\Delta_n(X_{jj}), A_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{jj}), \Delta_q(A_{ij})]_* + [X_{jj}, \Delta_n(A_{ii} + A_{ij})]_*.
\end{aligned}$$

Hence

$$X_{jj}(\Delta_n(A_{ii} + A_{ij}) - \Delta_n(A_{ij})) = (\Delta_n(A_{ii} + A_{ij}) - \Delta_n(A_{ij}))X_{jj}^* \tag{2.1}$$

for all $X_{jj} \in \mathcal{M}_{jj}$. Taking $X_{jj} = P_j$ in Eq. (2.1), we have from the fact $\Delta_n(A_{ij}) \in \mathcal{M}_{ij}$ that

$$P_j(\Delta_n(A_{ii} + A_{ij})P_i = P_j(\Delta_n(A_{ii} + A_{ij}) - \Delta_n(A_{ij}))P_i = 0. \tag{2.2}$$

Also, we have from Eq. (2.1) and Lemma 2.3 that

$$P_j(\Delta_n(A_{ii} + A_{ij})P_j = P_j(\Delta_n(A_{ii} + A_{ij}) - \Delta_n(A_{ij}))P_j = 0. \tag{2.3}$$

Clearly, it follows from Eq. (2.1) that $P_i(\Delta_n(A_{ii}+A_{ij})-\Delta_n(A_{ij}))X_{jj}^*=0$ for all $X_{jj}\in\mathcal{M}_{jj}$. This implies that

$$P_i\Delta_n(A_{ii}+A_{ij})P_j=\Delta_n(A_{ij}). \tag{2.4}$$

On the other hand, we have from Lemma 2.6, the fact $[A_{ii},X_{ii}]_*=[A_{ii}+A_{ij},X_{ii}]_*$ for all $X_{ii}\in\mathcal{M}_{ii}$ and the induction hypothesis that

$$\begin{aligned}
& [\Delta_n(A_{ii}),X_{ii}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii}),\Delta_q(X_{ii})]_*+[A_{ii},\Delta_n(X_{ii})]_*\\
= & [\Delta_n(A_{ii}+A_{ij}),X_{ii}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii}+A_{ij}),\Delta_q(X_{ii})]_*\\
+ & [A_{ii}+A_{ij},\Delta_n(X_{ii})]_*\\
= & [\Delta_n(A_{ii}+A_{ij}),X_{ii}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii}),\Delta_q(X_{ii})]_*+[A_{ii},\Delta_n(X_{ii})]_*.
\end{aligned}$$

Hence

$$(\Delta_n(A_{ii}+A_{ij})-\Delta_n(A_{ii}))X_{ii}=X_{ii}(\Delta_n(A_{ii}+A_{ij})-\Delta_n(A_{ii}))^*.$$

By Lemmas 2.2 and 2.6, there exists a scalar $\lambda\in\mathbb{R}$ such that

$$P_i\Delta_n(A_{ii}+A_{ij})P_i=\Delta_n(A_{ii})+\lambda P_i. \tag{2.5}$$

Combining Eqs. (2.2)-(2.5), we obtain that

$$\Delta_n(A_{ii}+A_{ij})=\Delta_n(A_{ii})+\Delta_n(A_{ij})+\lambda P_i. \tag{2.6}$$

For each $X_{ij}\in\mathcal{M}_{ij}$, we have from Eq. (2.6) that there exists a scalar $\alpha\in\mathbb{R}$ such that

$$\begin{aligned}
& \Delta_n(-X_{ij}A_{ij}^*)+\Delta_n(A_{ii}X_{ij})+\alpha P_i\\
= & \Delta_n(-X_{ij}A_{ij}^*+A_{ii}X_{ij})=\Delta_n([A_{ii}+A_{ij},X_{ij}]_*)\\
= & [\Delta_n(A_{ii}+A_{ij}),X_{ij}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii}+A_{ij}),\Delta_q(X_{ij})]_*\\
+ & [A_{ii}+A_{ij},\Delta_n(X_{ij})]_*\\
= & [\Delta_n(A_{ii})+\Delta_n(A_{ij})+\lambda P_i,X_{ij}]_*\\
+ & \sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii})+\Delta_p(A_{ij}),\Delta_q(X_{ij})]_*+[A_{ii}+A_{ij},\Delta_n(X_{ij})]_*\\
= & \Delta_n([A_{ij},X_{ij}]_*)+\Delta_n([A_{ii},X_{ij}]_*)+\lambda X_{ij}\\
= & \Delta_n(-X_{ij}A_{ij}^*)+\Delta_n(A_{ii}X_{ij})+\lambda X_{ij}.
\end{aligned}$$

Then $\lambda X_{ij}=\alpha P_i$ for each $X_{ij}\in\mathcal{M}_{ij}$. This implies that $\lambda=0$, and so by Eq. (2.6) we have $\Delta_n(A_{ii}+A_{ij})=\Delta_n(A_{ii})+\Delta_n(A_{ij})$.

(2) Let $X_{ji} \in \mathcal{M}_{ji}$, Then by the induction hypothesis,

$$\begin{aligned}
&\Delta_n([A_{ii}+A_{ji},X_{ji}]_*)\\
&=[\Delta_n(A_{ii}+A_{ji}),X_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii}+A_{ji}),\Delta_q(X_{ji})]_*\\
&+[A_{ii}+A_{ji},\Delta_n(X_{ji})]_*\\
&=[\Delta_n(A_{ii}+A_{ji}),X_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii})+\Delta_p(A_{ji}),\Delta_q(X_{ji})]_*\\
&+[A_{ii}+A_{ji},\Delta_n(X_{ji})]_*. \qquad (2.7)
\end{aligned}$$

On the other hand, it follows from (1) that

$$\begin{aligned}
&\Delta_n([A_{ii}+A_{ji},X_{ji}]_*)\\
=\;&\Delta_n(-X_{ji}A_{ji}^*-X_{ji}A_{ii}^*)=\Delta_n(-X_{ji}A_{ji}^*)+\Delta_n(-X_{ji}A_{ii}^*)\\
=\;&\Delta_n([A_{ji},X_{ji}]_*)+\Delta_n([A_{ii},X_{ji}]_*)\\
=\;&[\Delta_n(A_{ji}),X_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ji}),\Delta_q(X_{ji})]_*+[A_{ji},\Delta_n(X_{ji})]_*\\
+\;&[\Delta_n(A_{ii}),X_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii}),\Delta_q(X_{ji})]_*+[A_{ii},\Delta_n(X_{ji})]_*\\
=\;&[\Delta_n(A_{ii})+\Delta_n(A_{ji}),X_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(A_{ii})+\Delta_p(A_{ji}),\Delta_q(X_{ji})]_*\\
+\;&[A_{ii}+A_{ji},\Delta_n(X_{ji})]_*.
\end{aligned}$$

Hence by Eq. (2.7)

$$[\Delta_n(A_{ii}+A_{ji}),X_{ji}]_*=[\Delta_n(A_{ii})+\Delta_n(A_{ji}),X_{ji}]_*.$$

for all $X_{ji} \in \mathcal{M}_{ji}$. By Lemma 2.4,

$$\Delta_n(A_{ii}+A_{ji})-\Delta_n(A_{ii})-\Delta_n(A_{ji})=\mu P_j+\overline{\mu}P_i. \qquad (2.8)$$

for some $\mu \in \mathbb{C}$. Since $[X_{jj}, A_{ji}]_* = [X_{jj}, A_{ii} + A_{ji}]_*$ for all $X_{jj} \in \mathcal{M}_{jj}$, we have from Lemma 2.6, the induction hypothesis and Eq. (2.8) that

$$\begin{aligned}
&[\Delta_n(X_{jj}),A_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(X_{jj}),\Delta_q(A_{ji})]_*+[X_{jj},\Delta_n(A_{ji})]_*\\
=\;&[\Delta_n(X_{jj}),A_{ii}+A_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(X_{jj}),\Delta_q(A_{ii}+A_{ji})]_*\\
+\;&[X_{jj},\Delta_n(A_{ii}+A_{ji})]_*\\
=\;&[\Delta_n(X_{jj}),A_{ji}]_*+\sum_{\substack{p+q=n\\0<p,q<n}}[\Delta_p(X_{jj}),\Delta_q(A_{ji})]_*+[X_{jj},\Delta_n(A_{ji})+\mu P_j]_*
\end{aligned}$$

Then $\mu X_{jj} = \mu X_{jj}^*$ for all $X_{jj} \in \mathcal{M}_{jj}$, and so $\mu = 0$. By Eq. (2.8), hence $\Delta_n(A_{ii} + A_{ji}) = \Delta_n(A_{ii}) + \Delta_n(A_{ji})$.

(3) Let $X_{11} \in \mathcal{M}_{11}$. It follows from $[X_{11}, A_{11}]_* = [X_{11}, A_{11} + A_{22}]_*$, Lemma 2.6 and the induction hypothesis that

$$\begin{aligned} & [\Delta_n(X_{11}), A_{11}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{11}), \Delta_q(A_{11})]_* + [X_{11}, \Delta_n(A_{11})]_* \\ = & [\Delta_n(X_{11}), A_{11} + A_{22}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{11}), \Delta_q(A_{11} + A_{22})]_* \\ + & [X_{11}, \Delta_n(A_{11} + A_{22})]_* \\ = & [\Delta_n(X_{11}), A_{11}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{11}), \Delta_q(A_{11})]_* + [X_{11}, \Delta_n(A_{11} + A_{22})]_* \end{aligned}$$

Then

$$X_{11}(\Delta_n(A_{11} + A_{22}) - \Delta_n(A_{11})) = (\Delta_n(A_{11} + A_{22}) - \Delta_n(A_{11}))X_{11}^*$$

for every $X_{11} \in \mathcal{M}_{11}$. Applying the same argument as in (1), we can show that

$$P_1\Delta_n(A_{11} + A_{22})P_2 = P_2\Delta_n(A_{11} + A_{22})P_1 = 0 \tag{2.9}$$

and

$$P_1\Delta_n(A_{11} + A_{22})P_1 = \Delta_n(A_{11}). \tag{2.10}$$

From the fact $[X_{22}, A_{22}]_* = [X_{22}, A_{11} + A_{22}]_*$ for all $X_{22} \in \mathcal{M}_{22}$, similarly, we can obtain that

$$P_2\Delta_n(A_{11} + A_{22})P_2 = \Delta_n(A_{22}). \tag{2.11}$$

Combining Eqs. (2.9)-(2.11), we see that $\Delta_n(A_{11}+A_{22}) = \Delta_n(A_{11})+\Delta_n(A_{22})$.

(4) Let $X_{12} \in \mathcal{M}_{12}$. By the induction hypothesis

$$\begin{aligned} & \Delta_n([A_{12} + A_{21}, X_{12}]_*) \\ & = [\Delta_n(A_{12} + A_{21}), X_{12}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{12} + A_{21}), \Delta_q(X_{12})]_* \\ & + [A_{12} + A_{21}, \Delta_n(X_{12})]_* \\ & = [\Delta_n(A_{12} + A_{21}), X_{12}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{12}) + \Delta_p(A_{21}), \Delta_q(X_{12})]_* \\ & + [A_{12} + A_{21}, \Delta_n(X_{12})]_*. \end{aligned} \tag{2.12}$$

On the other hand, we have from (3) that

$$\begin{aligned}
& \Delta_n([A_{12}+A_{21},X_{12}]_*) \\
= & \Delta_n(A_{21}X_{12}-X_{12}A_{12}^*) = \Delta_n(A_{21}X_{12}) + \Delta_n(-X_{12}A_{12}^*) \\
= & \Delta_n([A_{21},X_{12}]_*) + \Delta_n([A_{12},X_{12}]_*) \\
= & [\Delta_n(A_{21}),X_{12}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{21}),\Delta_q(X_{12})]_* + [A_{21},\Delta_n(X_{12})]_* \\
+ & [\Delta_n(A_{12}),X_{12}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{12}),\Delta_q(X_{12})]_* + [A_{12},\Delta_n(X_{12})]_* \\
= & [\Delta_n(A_{12})+\Delta_n(A_{21}),X_{12}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{12})+\Delta_p(A_{21}),\Delta_q(X_{12})]_* \\
+ & [A_{12}+A_{21},\Delta_n(X_{12})]_*
\end{aligned}$$

This and Eq. (2.12) show that

$$[\Delta_n(A_{12}+A_{21}),X_{12}]_* = [\Delta_n(A_{12})+\Delta_n(A_{21}),X_{12}]_*$$

for all $X_{12} \in \mathcal{M}_{12}$. By Lemma 2.4, there exists a scalar $\mu \in \mathbb{C}$ such that

$$\Delta_n(A_{12}+A_{21}) = \Delta_n(A_{12}) + \Delta_n(A_{21}) + \mu P_1 + \overline{\mu} P_2. \tag{2.13}$$

We see from Eq. (2.13) that for each $X_{11} \in \mathcal{M}_{11}$ there exists a scalar $\alpha \in \mathbb{C}$ such that

$$\begin{aligned}
\Delta_n([X_{11},A_{12}+A_{21}]_*) &= \Delta_n(X_{11}A_{12}-A_{21}X_{11}^*) \\
&= \Delta_n(X_{11}A_{12}) + \Delta_n(-A_{21}X_{11}^*) + \alpha P_1 + \overline{\alpha} P_2.
\end{aligned}$$

On the other hand, it follows from Eq. (2.13) again

$$\begin{aligned}
& \Delta_n([X_{11},A_{12}+A_{21}]_*) \\
= & [\Delta_n(X_{11}),A_{12}+A_{21}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{11}),\Delta_q(A_{12}+A_{21})]_* \\
+ & [X_{11},\Delta_n(A_{12}+A_{21})]_* \\
= & [\Delta_n(X_{11}),A_{12}+A_{21}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(X_{11}),\Delta_q(A_{12})+\Delta_q(A_{21})]_* \\
+ & [X_{11},\Delta_n(A_{12})+\Delta_n(A_{21})+\mu P_1+\overline{\mu} P_2]_* \\
= & \Delta_n([X_{11},A_{12}]_*) + \Delta_n([X_{11},A_{21}]_*) + [X_{11},\mu P_1+\overline{\mu} P_2]_* \\
= & \Delta_n(X_{11}A_{12}) + \Delta_n(-A_{21}X_{11}^*) + \mu(X_{11}-X_{11}^*).
\end{aligned}$$

Hence $\mu(X_{11}-X_{11}^*) = \alpha P_1 + \overline{\alpha} P_2$. This implies that $\overline{\alpha} = 0$, then $\mu(X_{11}-X_{11}^*) = 0$ for all $X_{11} \in \mathcal{M}_{11}$, and so $\mu = 0$. Therefore, we have from Eq. (2.13) that $\Delta_n(A_{12}+A_{21}) = \Delta_n(A_{12}) + \Delta_n(A_{21})$. The proof is completed. □

Lemma 2.8. $\Delta_n(\sum_{i,j=1}^2 A_{ij}) = \sum_{i,j=1}^2 \Delta_n(A_{ij})$ *for all* $A_{ij} \in \mathcal{M}_{ij}$.

Proof. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then

$$[A_{ii} + A_{ji}, T_{ii}]_* = [A_{ii} + A_{ij} + A_{ji}, T_{ii}]_*$$

for all $T_{ii} \in \mathcal{M}_{ii}$, and so by Lemma 2.6 and the induction hypothesis

$$[\Delta_n(A_{ii} + A_{ij} + A_{ji}) - \Delta_n(A_{ii} + A_{ji}), T_{ii}]_* = 0 \tag{2.14}$$

for all $T_{ii} \in \mathcal{M}_{ii}$. It follows from Lemmas 2.2, 2.6 and 2.7(2) that

$$P_i\Delta_n(A_{ii} + A_{ij} + A_{ji})P_i = \Delta_n(A_{ii}) + \lambda_1 P_i \tag{2.15}$$

for some $\lambda_1 \in \mathbb{R}$. By Eq. (2.14), we have

$$P_j(\Delta_n(A_{ii} + A_{ij} + A_{ji}) - \Delta_n(A_{ii} + A_{ji}))T_{ii} = 0$$

for all $T_{ii} \in \mathcal{M}_{ii}$. It follows from Lemmas 2.6 and 2.7(2) that

$$P_j\Delta_n(A_{ii} + A_{ij} + A_{ji})P_i = \Delta_n(A_{ji}). \tag{2.16}$$

On the other hand, we have from Lemma 2.6, the fact $[A_{ij}, T_{jj}]_* = [A_{ii} + A_{ij} + A_{ji}, T_{jj}]_*$ for all $T_{jj} \in \mathcal{M}_{jj}$ and the induction hypothesis that

$$[\Delta_n(A_{ii} + A_{ij} + A_{ji}) - \Delta_n(A_{ij}), T_{jj}]_* = 0.$$

for all $T_{jj} \in \mathcal{M}_{jj}$. Then by Lemmas 2.2 and 2.6, there is a $\lambda_2 \in \mathbb{R}$ such that

$$P_j\Delta_n(A_{ii} + A_{ij} + A_{ji})P_j = \lambda_2 P_j. \tag{2.17}$$

Also, we have $P_i(\Delta_n(A_{ii} + A_{ij} + A_{ji}) - \Delta_n(A_{ij}))T_{jj} = 0$ for all $T_{jj} \in \mathcal{M}_{jj}$. This implies that

$$P_i\Delta_n(A_{ii} + A_{ij} + A_{ji})P_j = \Delta_n(A_{ij}). \tag{2.18}$$

Combining Eqs. (2.15)-(2.18), we obtain that

$$\Delta_n(A_{ii} + A_{ij} + A_{ji}) = \Delta_n(A_{ii}) + \Delta_n(A_{ij}) + \Delta_n(A_{ji}) + \lambda_1 P_i + \lambda_2 P_j. \tag{2.19}$$

It follows from Eq. (2.19) that for each $T_{ii} \in \mathcal{M}_{ii}$ there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\begin{aligned}
& \Delta_n(T_{ii}A_{ii} - A_{ii}T_{ii}^*) + \Delta_n(T_{ii}A_{ij}) + \Delta_n(-A_{ji}T_{ii}^*) + \alpha_1 P_i + \alpha_2 P_j \\
= \; & \Delta_n(T_{ii}A_{ii} + T_{ii}A_{ij} - A_{ii}T_{ii}^* - A_{ji}T_{ii}^*) = \Delta_n([T_{ii}, A_{ii} + A_{ij} + A_{ji}]_*) \\
= \; & [\Delta_n(T_{ii}), A_{ii} + A_{ij} + A_{ji}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(T_{ii}), \Delta_q(A_{ii}) + \Delta_q(A_{ij}) \\
+ \; & \Delta_q(A_{ji})]_* + [T_{ii}, \Delta_n(A_{ii}) + \Delta_n(A_{ij}) + \Delta_n(A_{ji}) + \lambda_1 P_i + \lambda_2 P_j]_* \\
= \; & \Delta_n([T_{ii}, A_{ii}]_*) + \Delta_n([T_{ii}, A_{ij}]_*) + \Delta_n([T_{ii}, A_{ji}]_*) + \lambda_1(T_{ii} - T_{ii}^*) \\
= \; & \Delta_n(T_{ii}A_{ii} - A_{ii}T_{ii}^*) + \Delta_n(T_{ii}A_{ij}) + \Delta_n(-A_{ji}T_{ii}^*) + \lambda_1(T_{ii} - T_{ii}^*)
\end{aligned}$$

This implies that $\lambda_1 = \alpha_1 = \alpha_2 = 0$. On the other hand, by Lemma 2.7(4), the induction hypothesis and Eq. (2.19), we have for any $T_{jj} \in \mathcal{M}_{jj}$,

$$\begin{aligned}
& \Delta_n(T_{jj}A_{ji}) + \Delta_n(-A_{ij}T_{jj}^*) \\
= \quad & \Delta_n(T_{jj}A_{ji} - A_{ij}T_{jj}^*) = \Delta_n([T_{jj}, A_{ii} + A_{ij} + A_{ji}]_*) \\
= \quad & [\Delta_n(T_{jj}), A_{ii} + A_{ij} + A_{ji}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(T_{jj}), \Delta_q(A_{ii}) + \Delta_q(A_{ij}) \\
+ \quad & \Delta_q(A_{ji})]_* + [T_{jj}, \Delta_n(A_{ii}) + \Delta_n(A_{ij}) + \Delta_n(A_{ji}) + \lambda_1 P_i + \lambda_2 P_j]_* \\
= \quad & \Delta_n([T_{jj}, A_{ii}]_*) + \Delta_n([T_{jj}, A_{ij}]_*) + \Delta_n([T_{jj}, A_{ji}]_*) + \lambda_2(T_{jj} - T_{jj}^*) \\
= \quad & \Delta_n(-A_{ij}T_{jj}^*) + \Delta_n(T_{jj}A_{ji}) + \lambda_2(T_{jj} - T_{jj}^*).
\end{aligned}$$

This implies that $\lambda_2 = 0$. Hence by Eq. (2.19), we have for $i \neq j$,

$$\Delta_n(A_{ii} + A_{ij} + A_{ji}) = \Delta_n(A_{ii}) + \Delta_n(A_{ij}) + \Delta_n(A_{ji}). \tag{2.20}$$

Since $[T_{11}, \sum_{i,j=1}^2 A_{ij}]_* = [T_{11}, A_{11}+A_{12}+A_{21}]_*$ for all $T_{11} \in \mathcal{M}_{11}$, it follows from the fact $\Delta_n(T_{11}) \in \mathcal{M}_{11}$ and the induction hypothesis that

$$[T_{11}, \Delta_n(\sum_{i,j=1}^2 A_{ij}) - \Delta_n(A_{11} + A_{12} + A_{21})]_* = 0.$$

By Lemmas 2.3, 2.6 and Eq. (2.20), then

$$P_1\Delta_n(\sum_{i,j=1}^2 A_{ij})P_1 = \Delta_n(A_{11}). \tag{2.21}$$

Also, we can show that

$$P_1\Delta_n(\sum_{i,j=1}^2 A_{ij})P_2 = \Delta_n(A_{12}), \quad P_2\Delta_n(\sum_{i,j=1}^2 A_{ij})P_1 = \Delta_n(A_{21}). \tag{2.22}$$

Since $[T_{22}, \sum_{i,j=1}^2 A_{ij}]_* = [T_{22}, A_{22}+A_{12}+A_{21}]_*$ for all $T_{22} \in \mathcal{M}_{22}$, it follows from the fact $\Delta_n(T_{22}) \in \mathcal{M}_{22}$ and the induction hypothesis that

$$[T_{22}, \Delta_n(\sum_{i,j=1}^2 A_{ij}) - \Delta_n(A_{22} + A_{12} + A_{21})]_* = 0.$$

By Lemmas 2.3, 2.6 and Eq. (2.20), then

$$P_2\Delta_n(\sum_{i,j=1}^2 A_{ij})P_2 = \Delta_n(A_{22}). \tag{2.23}$$

Combining Eqs. (2.21)-(2.23), we have $\Delta_n(\sum_{i,j=1}^2 A_{ij}) = \sum_{i,j=1}^2 \Delta_n(A_{ij})$. The proof is completed. □

Lemma 2.9. *$\Delta_n(A_{ij}+B_{ij}) = \Delta_n(A_{ij})+\Delta_n(B_{ij})$ for all $A_{ij}, B_{ij} \in \mathcal{M}_{ij}, i, j = 1, 2$.*

Proof. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $[T_{ij}, P_j]_* = T_{ij} - T_{ij}^*$ for all $T_{ij} \in \mathcal{M}_{ij}$, and so by Lemmas 2.7(4) and 2.5,

$$\begin{aligned}\Delta_1(T_{ij}) + \Delta_1(-T_{ij}^*) &= \Delta_1(T_{ij} - T_{ij}^*) = [\Delta_1(T_{ij}), P_j]_* + [T_{ij}, \Delta_1(P_j)]_* \\ &= \Delta_1(T_{ij}) - \Delta_1(T_{ij})^* + T_{ij}\Delta_1(P_j) - \Delta_1(P_j)T_{ij}^*.\end{aligned}$$

Since $\Delta_1(P_j) \in \mathbb{C}I$ and $\Delta_1(-T_{ij}^*), \Delta_1(T_{ij})^* \in \mathcal{M}_{ji}$, we have from above equation that $T_{ij}\Delta_1(P_j) = 0$ for all $T_{ij} \in \mathcal{M}_{ij}$. Hence $\Delta_1(P_j) = 0$ for $j = 1, 2$. We next by induction show that $\Delta_n(P_j) = 0$. Assume that $\Delta_m(P_j) = 0$ holds for $m < n \in \mathbb{N}$, by Lemmas 2.7(4) and 2.6,

$$\begin{aligned}&\Delta_n(T_{ij}) + \Delta_n(-T_{ij}^*) = \Delta_n(T_{ij} - T_{ij}^*) \\ =\ & [\Delta_n(T_{ij}), P_j]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(T_{ij}), \Delta_q(P_j)]_* + [T_{ij}, \Delta_n(P_j)]_* \\ =\ & \Delta_n(T_{ij}) - \Delta_n(T_{ij})^* + T_{ij}\Delta_n(P_j) - \Delta_n(P_j)T_{ij}^*.\end{aligned}$$

Similarly, we can obtain that $\Delta_n(P_j) = 0$ for $j = 1, 2$.

Let $A_{ij}, B_{ij} \in \mathcal{M}_{ij} (i \neq j)$. Then

$$[P_i + A_{ij}, P_j + B_{ij}]_* = A_{ij} + B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*,$$

and so by Lemmas 2.8, 2.7, 2.6 and the induction hypothesis,

$$\begin{aligned}&\Delta_n(A_{ij} + B_{ij}) + \Delta_n(-A_{ij}^*) + \Delta_n(-B_{ij}A_{ij}^*) \\ =\ & \Delta_n(A_{ij} + B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*) = \Delta_n([P_i + A_{ij}, P_j + B_{ij}]_*) \\ =\ & [\Delta_n(P_i) + \Delta_n(A_{ij}), P_j + B_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(P_i) + \Delta_p(A_{ij}), \Delta_q(P_j) \\ +\ & \Delta_q(B_{ij})]_* + [P_i + A_{ij}, \Delta_n(P_j) + \Delta_n(B_{ij})]_* \\ =\ & [\Delta_n(A_{ij}), P_j + B_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{ij}), \Delta_q(B_{ij})]_* \\ +\ & [P_i + A_{ij}, \Delta_n(B_{ij})]_* \\ =\ & \Delta_n([A_{ij}, B_{ij}]_*) + [\Delta_n(A_{ij}), P_j]_* + [P_i, \Delta_n(B_{ij})]_* \\ =\ & \Delta_n(-B_{ij}A_{ij}^*) + \Delta_n(A_{ij}) - \Delta_n(A_{ij})^* + \Delta_n(B_{ij}).\end{aligned}$$

This implies that

$$\Delta_n(A_{ij} + B_{ij}) = \Delta_n(A_{ij}) + \Delta_n(B_{ij}). \tag{2.24}$$

for all $A_{ij}, B_{ij} \in \mathcal{M}_{ij} (i \neq j)$.

Let $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$ and $T_{ij} \in \mathcal{M}_{ij} (i \neq j)$. It follows from Eq. (2.24) and the induction hypothesis that

$$\begin{aligned}
& \Delta_n([A_{ii}+B_{ii}, T_{ij}]_*) = \Delta_n(A_{ii}T_{ij} + B_{ii}T_{ij}) \\
= \ & \Delta_n(A_{ii}T_{ij}) + \Delta_n(B_{ii}T_{ij}) = \Delta_n([A_{ii}, T_{ij}]_*) + \Delta_n([B_{ii}, T_{ij}]_*) \\
= \ & [\Delta_n(A_{ii}), T_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{ii}), \Delta_q(T_{ij})]_* + [A_{ii}, \Delta_n(T_{ij})]_* \\
+ \ & [\Delta_n(B_{ii}), T_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(B_{ii}), \Delta_q(T_{ij})]_* + [B_{ii}, \Delta_n(T_{ij})]_* \\
= \ & [\Delta_n(A_{ii}) + \Delta_n(B_{ii}), T_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{ii}+B_{ii}), \Delta_q(T_{ij})]_* \\
+ \ & [A_{ii}+B_{ii}, \Delta_n(T_{ij})]_*.
\end{aligned}$$

On the other hand, we have from Lemma 2.6 that

$$\begin{aligned}
& \Delta_n([A_{ii}+B_{ii}, T_{ij}]_*) \\
= \ & [\Delta_n(A_{ii}+B_{ii}), T_{ij}]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{ii}+B_{ii}), \Delta_q(T_{ij})]_* \\
+ \ & [A_{ii}+B_{ii}, \Delta_n(T_{ij})]_*.
\end{aligned}$$

Hence $(\Delta_n(A_{ii}+B_{ii}) - \Delta_n(A_{ii}) - \Delta_n(B_{ii}))T_{ij} = 0$ for all $T_{ij} \in \mathcal{M}_{ij}$. This implies that $\Delta_n(A_{ii}+B_{ii}) = \Delta_n(A_{ii}) + \Delta_n(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$. The proof is completed. □

Lemma 2.10. *Let $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$ and $A_{ij}, B_{ij} \in \mathcal{M}_{ij} (i \neq j)$. Then*

(1) $\Delta_n(A_{ii}B_{ij}) = \Delta_n(A_{ii})B_{ij} + A_{ii}\Delta_n(B_{ij}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii})\Delta_q(B_{ij})$;
(2) $\Delta_n(A_{ii}B_{ii}) = \Delta_n(A_{ii})B_{ii} + A_{ii}\Delta_n(B_{ii}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii})\Delta_q(B_{ii})$;
(3) $\Delta_n(A_{ij}B_{ji}) = \Delta_n(A_{ij})B_{ji} + A_{ij}\Delta_n(B_{ji}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ij})\Delta_q(B_{ji})$;
(4) $\Delta_n(A_{ij}B_{jj}) = \Delta_n(A_{ij})B_{jj} + A_{ij}\Delta_n(B_{jj}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ij})\Delta_q(B_{jj})$.

Proof. (1) By Lemma 2.6, we have

$$\begin{aligned}
& \Delta_n(A_{ii}B_{ij}) = \Delta_n([A_{ii}, B_{ij}]_*) \\
= \ & [\Delta_n(A_{ii}), B_{ij}]_* + [A_{ii}, \Delta_n(B_{ij})]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{ii}), \Delta_q(B_{ij})]_* \\
= \ & \Delta_n(A_{ii})B_{ij} + A_{ii}\Delta_n(B_{ij}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii})\Delta_q(B_{ij}).
\end{aligned}$$

(2) Let $X_{ij} \in \mathcal{M}_{ij}$. From (1) and the induction hypothesis, it is easy to see that

$$\begin{aligned}
&\Delta_n(A_{ii}B_{ii}X_{ij}) = \Delta_n((A_{ii}B_{ii})X_{ij}) \\
&= \Delta_n(A_{ii}B_{ii})X_{ij} + A_{ii}B_{ii}\Delta_n(X_{ij}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii}B_{ii})\Delta_q(X_{ij}) \\
&= \Delta_n(A_{ii}B_{ii})X_{ij} + A_{ii}B_{ii}\Delta_n(X_{ij}) \\
&+ \sum_{\substack{p+q+r=n \\ 0<r<n}} \Delta_p(A_{ii})\Delta_q(B_{ii})\Delta_r(X_{ij}) \qquad (2.25)
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_n(A_{ii}B_{ii}X_{ij}) = \Delta_n(A_{ii}(B_{ii}X_{ij})) \\
&= \Delta_n(A_{ii})B_{ii}X_{ij} + A_{ii}\Delta_n(B_{ii}X_{ij}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii})\Delta_q(B_{ii}X_{ij}) \\
&= \Delta_n(A_{ii})B_{ii}X_{ij} + A_{ii}\Delta_n(B_{ii})X_{ij} + A_{ii}B_{ii}\Delta_n(X_{ij}) \\
&+ \sum_{\substack{p+q=n \\ 0<p,q<n}} A_{ii}\Delta_p(B_{ii})\Delta_q(X_{ij}) + \sum_{\substack{p+q+r=n \\ 0<p<n}} \Delta_p(A_{ii})\Delta_q(B_{ii})\Delta_r(X_{ij}) \\
&= \Delta_n(A_{ii})B_{ii}X_{ij} + A_{ii}\Delta_n(B_{ii})X_{ij} + A_{ii}B_{ii}\Delta_n(X_{ij}) \\
&+ \sum_{\substack{p+q=n \\ 0<p<n}} \Delta_p(A_{ii})\Delta_q(B_{ii})X_{ij} + \sum_{\substack{p+q+r=n \\ 0<r<n}} \Delta_p(A_{ii})\Delta_q(B_{ii})\Delta_r(X_{ij}). \qquad (2.26)
\end{aligned}$$

Combining Eqs. (2.25)-(2.26), we have

$$\Delta_n(A_{ii}B_{ii})X_{ij} \quad = \quad (\Delta_n(A_{ii})B_{ii} + A_{ii}\Delta_n(B_{ii}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii})\Delta_q(B_{ii}))X_{ij}.$$

This implies that

$$\Delta_n(A_{ii}B_{ii}) = \Delta_n(A_{ii})B_{ii} + A_{ii}\Delta_n(B_{ii}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ii})\Delta_q(B_{ii}).$$

(3) Since $A_{ij}B_{ji} = [A_{ij}, B_{ji}]_*$, we have

$$\begin{aligned}
&\Delta_n(A_{ij}B_{ji}) = \Delta_n([A_{ij}, B_{ji}]_*) \\
= \quad &[\Delta_n(A_{ij}), B_{ji}]_* + [A_{ij}, \Delta_n(B_{ji})]_* + \sum_{\substack{p+q=n \\ 0<p,q<n}} [\Delta_p(A_{ij}), \Delta_q(B_{ji})]_* \\
= \quad &\Delta_n(A_{ij})B_{ji} + A_{ij}\Delta_n(B_{ji}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{ij})\Delta_q(B_{ji}).
\end{aligned}$$

(4) From the above and the induction hypothesis, it is easy to see that

$$\begin{aligned}
&\Delta_n(X_{ji}A_{ij}B_{jj}) = \Delta_n(X_{ji}(A_{ij}B_{jj}))\\
&= \Delta_n(X_{ji})A_{ij}B_{jj} + X_{ji}\Delta_n(A_{ij}B_{jj}) + \sum_{\substack{p+q=n\\0<p,q<n}} \Delta_p(X_{ji})\Delta_q(A_{ij}B_{jj})\\
&= \Delta_n(X_{ji})A_{ij}B_{jj} + X_{ji}\Delta_n(A_{ij}B_{jj})\\
&+ \sum_{\substack{p+q+r=n\\0<p<n}} \Delta_p(X_{ji})\Delta_q(A_{ij})\Delta_r(B_{jj})
\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}
&\Delta_n(X_{ji}A_{ij}B_{jj}) = \Delta_n((X_{ji}A_{ij})B_{jj})\\
&= \Delta_n(X_{ji}A_{ij})B_{jj} + X_{ji}A_{ij}\Delta_n(B_{jj}) + \sum_{\substack{p+q=n\\0<p,q<n}} \Delta_p(X_{ji}A_{ij})\Delta_q(B_{jj})\\
&= \Delta_n(X_{ji})A_{ij}B_{jj} + X_{ji}\Delta_n(A_{ij})B_{jj} + X_{ji}A_{ij}\Delta_n(B_{jj})\\
&+ \sum_{\substack{p+q=n\\0<p,q<n}} \Delta_p(X_{ji})\Delta_q(A_{ij})B_{jj} + \sum_{\substack{p+q+r=n\\0<r<n}} \Delta_p(X_{ji})\Delta_q(A_{ij})\Delta_r(B_{jj})\\
&= \Delta_n(X_{ji})A_{ij}B_{jj} + X_{ji}\Delta_n(A_{ij})B_{jj} + X_{ji}A_{ij}\Delta_n(B_{jj})\\
&+ \sum_{\substack{p+q=n\\0<p,q<n}} X_{ji}\Delta_p(A_{ij})\Delta_q(B_{jj}) + \sum_{\substack{p+q+r=n\\0<p<n}} \Delta_p(X_{ji})\Delta_q(A_{ij})\Delta_r(B_{jj}).
\end{aligned}\tag{2.28}$$

Combining Eqs. (2.27)-(2.28), we have

$$\begin{aligned}
X_{ji}\Delta_n(A_{ij}B_{jj}) &= X_{ji}(\Delta_n(A_{ij})B_{jj} + A_{ij}\Delta_n(B_{jj})\\
&\quad + \sum_{\substack{p+q=n\\0<p,q<n}} \Delta_p(A_{ij})\Delta_q(B_{jj})).
\end{aligned}$$

This implies that

$$\Delta_n(A_{ij}B_{jj}) = \Delta_n(A_{ij})B_{jj} + A_{ij}\Delta_n(B_{jj}) + \sum_{\substack{p+q=n\\0<p,q<n}} \Delta_p(A_{ij})\Delta_q(B_{jj}).$$

The proof is completed. □

Lemma 2.11. *Δ_n is an additive $*$-higher derivation of $\mathcal{M}$.*

Proof. Let $A, B \in \mathcal{M}$. Then $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$ for some $A_{ij}, B_{ij} \in \mathcal{M}_{ij}$. It follows from Lemmas 2.8 and 2.9 that

$$\Delta_n(A+B) = \sum_{i,j=1}^{2} \Delta_n(A_{ij}+B_{ij}) = \sum_{i,j=1}^{2} (\Delta_n(A_{ij})+\Delta_n(B_{ij})) = \Delta_n(A)+\Delta_n(B).$$

By Lemma 2.10,

$$\begin{aligned}
\Delta_n(AB) &= \Delta_n(A_{11}B_{11}) + \Delta_n(A_{11}B_{12}) + \Delta_n(A_{12}B_{21}) + \Delta_n(A_{12}B_{22}) \\
&+ \Delta_n(A_{21}B_{11}) + \Delta_n(A_{21}B_{12}) + \Delta_n(A_{22}B_{21}) + \Delta_n(A_{22}B_{22}) \\
&= \Delta_n(A_{11})B_{11} + A_{11}\Delta_n(B_{11}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{11})\Delta_q(B_{11}) \\
&+ \Delta_n(A_{11})B_{12} + A_{11}\Delta_n(B_{12}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{11})\Delta_q(B_{12}) \\
&+ \Delta_n(A_{12})B_{21} + A_{12}\Delta_n(B_{21}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{12})\Delta_q(B_{21}) \\
&+ \Delta_n(A_{12})B_{22} + A_{12}\Delta_n(B_{22}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{12})\Delta_q(B_{22}) \\
&+ \Delta_n(A_{21})B_{11} + A_{21}\Delta_n(B_{11}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{21})\Delta_q(B_{11}) \\
&+ \Delta_n(A_{21})B_{12} + A_{21}\Delta_n(B_{12}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{21})\Delta_q(B_{12}) \\
&+ \Delta_n(A_{22})B_{21} + A_{22}\Delta_n(B_{21}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{22})\Delta_q(B_{21}) \\
&+ \Delta_n(A_{22})B_{22} + A_{22}\Delta_n(B_{22}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{22})\Delta_q(B_{22}),
\end{aligned}$$

on the other hand, by Lemma 2.6,

$$\begin{aligned}
& \Delta_n(A)B + A\Delta_n(B) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A)\Delta_q(B) \\
= \; & \Delta_n(A_{11})(B_{11} + B_{12}) + \Delta_n(A_{12})(B_{21} + B_{22}) \\
+ \; & \Delta_n(A_{21})(B_{11} + B_{12}) + \Delta_n(A_{22})(B_{21} + B_{22}) \\
+ \; & A_{11}\Delta_n(B_{11} + B_{12}) + A_{12}\Delta_n(B_{21} + B_{22}) \\
+ \; & A_{21}\Delta_n(B_{11} + B_{12}) + A_{22}\Delta_n(B_{21} + B_{22}) \\
+ \; & \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{11})\Delta_q(B_{11}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{11})\Delta_q(B_{12}) \\
+ \; & \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{12})\Delta_q(B_{21}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{12})\Delta_q(B_{22})
\end{aligned}$$

$$+ \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{21})\Delta_q(B_{11}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{21})\Delta_q(B_{12})$$
$$+ \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{22})\Delta_q(B_{21}) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A_{22})\Delta_q(B_{22}).$$

We immediately obtain

$$\Delta_n(AB) = \Delta_n(A)B + A\Delta_n(B) + \sum_{\substack{p+q=n \\ 0<p,q<n}} \Delta_p(A)\Delta_q(B)$$

for all $A, B \in \mathcal{M}$. This shows that each Δ_n is an additive higher derivation on $\mathcal{M}$. Let $A = B + iC$ where $B, C \in \mathcal{M}_{sa}$. By Lemma 2.6, then

$$\Delta_n(A^*) = \Delta_n(B) - \Delta_n(iC) = \Delta_n(B) - i\Delta_n(C) = \Delta_n(A)^*$$

for all $A \in \mathcal{M}$. Hence Δ_n is an additive $*$-higher derivation of $\mathcal{M}$. The proof is completed. □

Note that it is not true that every additive derivation on any factor von Neumann algebras is inner. However, if $\mathcal{H}$ is infinite dimensional, then every additive derivation on type I factor von Neumann algebras is inner ([15]). Nowicki in [11] proved that if every additive (linear) derivation of $\mathcal{A}$ is inner, then every additive (linear) higher derivation of $\mathcal{A}$ is inner (see also [16, Proposition 2.6]). So, by Theorem 2.1, the following corollary is immediate.

Corollary 2.12. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space. Assume that $\mathcal{M}$ is a type I factor von Neumann algebra on $\mathcal{H}$. Then every nonlinear $*$-Lie higher derivation $D = \{\phi_n\}_{n\in\mathbb{N}}$ is inner with $\phi_n(A^*) = \phi_n(A)^*$ for each $A \in \mathcal{M}$ and every $n \in \mathbb{N}$.*

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