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ON STRONGLY DENSE SUBMODULES

E. GHASHGHAEI* AND M. NAMDARI

(Communicated by Rahim Zaare-Nahandi)

Dedicated to Professor O.A.S. Karamzadeh on the occasion of his 70th birthday

Abstract. The submodules with the property of the title (a submodule N of an R-module M is called strongly dense in M, denoted by $N \leq_{sd} M$, if for any index set I, $\prod_I N \leq_d \prod_I M$) are introduced and fully investigated. It is shown that for each submodule N of M there exists the smallest subset $D' \subseteq M$ such that $N + D'$ is a strongly dense submodule of M and $D' \bigcap N = 0$. We also introduce a class of modules in which the two concepts of strong essentiality and strong density coincide. It is also shown that for any module M, dense submodules in M are strongly dense if and only if $M \leq_{sd} \tilde{E}(M)$, where $\tilde{E}(M)$ is the rational hull of M. It is proved that R has no strongly dense left ideal if and only if no nonzero-element of every cyclic R-module M has a strongly dense annihilator in R. Finally, some properties and new concepts related to strong density are studied.
Keywords: Strongly essential submodule, strongly dense submodule, singular submodule, special submodule, column submodule.
MSC(2010): Primary: 16D10; Secondary: 16S90, 16U20.

1. Introduction

It is well-known that the concept of dense submodules in an R-module M (i.e., a submodule N of M is called dense if for any $y \in M$ and $0 \neq x \in M$, there exists $r \in R$ such that $rx \neq 0$ and $r \in (N : y)$), due to Findlay-Lambek plays an important role in the context of algebra (commutative or not). For example, if R is a commutative ring such that it has ACC on annihilator ideals, then an ideal I is dense in R if and only if it contains a regular element (i.e., a non zero-divisor); see [7, Theorem 8.31]. A semiprime commutative ring R (which is called reduced) is Goldie if and only if every dense ideal I of R contains a regular element. We should remind the reader that in a semiprime

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ring R (not necessarily commutative) dense ideals and essential ideals coincide; see Proposition 2.7. We should also recall that a ring R is semiprime left Goldie if and only if for any left ideal I in R, $I \leq_e R$ if and only if I contains a regular element; see for example [7, Theorem 11.13]. For a left module M and $x \in M$, x is in the torsion submodule with respect to Lambek torsion theory if and only if $\mathrm{Ann}(x)$ is a dense left ideal. These objects have this important property of being self-generating, in the sense that each dense left ideal generates other dense objects. For example, if I is a dense left ideal in a ring R and $x \in R$, then $(I : x)$ is a dense left ideal of R. But if N is a dense submodule of an R-module M, similar to essential submodules, $\prod_{i \in I} N$ may not be dense in $\prod_{i \in I} M$, where $\prod_{i \in I} N$ denotes the direct product of copies of N over the index set I, see also [4]. As an example, $\mathbb{Z}$, is dense in $\mathbb{Q}$ as a $\mathbb{Z}$-module, but $\prod_{i \in I} \mathbb{Z}$ is not dense in $\prod_{i \in I} \mathbb{Q}$. We recall that $\oplus_{i \in I} \mathbb{Z}$ is dense in $\oplus_{i \in I} \mathbb{Q}$ as a $\mathbb{Z}$-module.

An outline of this article is as follows. In Section 2, we establish the basic properties of strongly dense submodules. Section 3, is devoted to the study of strongly polyform and strongly monoform modules. In Section 4, we introduce and study submodules corresponding to the concept of dense submodules.

Throughout this article, as usual, unless otherwise specified all rings are associative with identity, and all modules are assumed to be left unitary. $\{E_{ij}\}_{i,j}$ denotes the set of unit matrices. Let X be a subset of an R-module M and let N be a submodule of M, by $(N : X)$ we mean the left ideal $\{r \in R \,|\, rX \subseteq N\}$. If X is a subset of the ring R, then $\mathrm{Ann}_l(X)$ and $\mathrm{Ann}_r(X)$ denote respectively the left and right annihilator of X. According to Definition 1.1 in [4], whenever N is strongly essential in M, we write $N \leq_{se} M$. The strong singular submodule of M, denoted by $SZ(M)$, is defined by $SZ(M) = \{m \in M \,|\, \mathrm{Ann}_l(m) \leq_{se} R\}$; see [4, Definition 2.1]. The socle type of M, denoted by $ST(M)$, is the intersection of all strongly essential submodules of M; see [4, Definition 2.2]. If M is an R-module, then $\tilde{E}(M)$ and $E(M)$ denote respectively the rational hull (unique maximal rational extension) and the injective envelope of M.

2. **Strongly dense submodules**

In [4], the concept of strongly essential submodules is introduced and studied. If M is an R-module and N is a dense submodule of M, then M is called a rational extension of N and we use the notation $N \leq_d M$. An R-module M is called rationally complete if it has no proper rational extensions.

We begin with the following definition.

Definition 2.1. A submodule N of an R-module M is called strongly dense in M and M is called strongly rational extension of N if it satisfies one of the following equivalent conditions:
1. For any index set I, $\prod_I N \leq_d \prod_I M$;

2. For any two subsets $Y \subseteq M$ and $0 \neq X \subseteq M$, there exists $r \in R$ such that $rX \neq 0$ and $r \in (N : Y)$ (i.e., $(N : Y) \setminus \mathrm{Ann}_l(X) \neq \emptyset$).

We use the notation $N \leq_{sd} M$ to indicate that N is a strongly dense submodule of M. In the above definition, in part 1, we may assume that $|I| = |M|$ and in part 2, without loss of generality, we always assume that X properly contains zero. In contrast with part 2 of our definition, dense submodules enjoy a weaker property.

Lemma 2.2. *Let N be a submodule of an R-module M, if $N \leq_{sd} M$ then $N \leq_{se} M$.*

Proof. It is clear. □

Remark 2.3. The converse of Lemma 2.2, is not true in general. For example, for any prime number $p \in \mathbb{Z}$ and $n \geq 1$, $p\mathbb{Z}/p^{n+1}\mathbb{Z}$ is strongly essential in $\mathbb{Z}/p^{n+1}\mathbb{Z}$ but it is not strongly dense.

Example 2.4. Let X be a set of commuting indeterminates over the ring R. Then $I \leq_{sd} R$ if and only if $I[X] \leq_{sd} R[X]$ as a left ideal.

Example 2.5. For any family of rings $\{R_i\}_{i \in I}$, $\oplus_{i \in I} R_i \leq_{sd} \prod_{i \in I} R_i$ both as a left and a right ideal in $\prod_{i \in I} R_i$.

Lemma 2.6. *Let I be a left ideal in a ring R, $I \leq_d R$ if and only if $\mathrm{Ann}_r(I : y) = 0$ for any $y \in R$.*

Proof. It is clear. □

Proposition 2.7. *Let R be a semiprime ring, strongly dense ideals, dense ideals, strongly essential ideals and essential ideals are all the same.*

Proof. We note that the right and the left annihilator of each ideal in a semiprime ring are the same, and if I is an essential ideal, then $\mathrm{Ann}(I) = 0$. Hence I is strongly essential by [4, Example 3]. Now suppose that I is a dense ideal in R. Let Y and $0 \neq X$ be subsets of R. Since I is essential in the semiprime ring R, we infer that for $0 \neq x \in X$ we have $0 \neq (I \bigcap Rx)^2 \subseteq IRx$. Thus there exists $i \in I$ such that $iY \subseteq I$ and $0 \neq iX$. Hence I is strongly dense. Now we may show that the essentiality and the density of ideals coincide. To see this, let I be an essential ideal and $y \in R$, so $\mathrm{Ann}(I : y) \leq \mathrm{Ann}(I) = 0$ (i.e., $\mathrm{Ann}(I : y) = 0$), therefore I is a dense ideal by Lemma 2.6. □

Example 2.8. Consider the split-null extension $\mathbb{Q} * \mathbb{Q}$ of the rational numbers $\mathbb{Q}$ (the addition in $\mathbb{Q} * \mathbb{Q}$ is defined componentwise and the multiplication by the rule $(a, b)(c, d) = (ad + bc, bd)$). The split-null extension, which is a ring, is not an integral domain, and $I = (\mathbb{Q}, 0)$ is the unique strongly essential minimal ideal in $\mathbb{Q} * \mathbb{Q}$ while it is not dense.

Proposition 2.9. *Let I be a left ideal in a ring R, $I \leq_{se} R$ if and only if $\mathrm{Ann}_l(X) \subset (I : X)$ for any subset $X \subseteq R$.*

Proof. By [4, Definition 1.1]. □

Proposition 2.10. *Let I be a left ideal in a ring R, $I \leq_{sd} R$ if and only if $\mathrm{Ann}_r(I : X) = 0$ for any subset $X \subseteq R$.*

Proof. Suppose that I is a strongly dense left ideal and there exists a subset X of R such that $0 \neq y \in \mathrm{Ann}_r(I : X)$ and seek a contradiction. Since $I \leq_{sd} R$ there exists $r \in R$ such that $0 \neq ry$ and $r \in (I : X)$. But $0 \neq y \in \mathrm{Ann}_r(I : X)$, hence $0 = ry$ which is a contradiction. Conversely, suppose that I is not strongly dense, then for some subsets $X \subseteq R$ and $0 \neq Y \subseteq R$ there does not exist $r \in R$ such that $0 \neq rY$ and $r \in (I : X)$ (i.e., if $r \in (I : X)$ then $0 = rY$). Hence $0 \neq Y \subseteq \mathrm{Ann}_r(I : X)$ that is a contradiction. □

Corollary 2.11. *Let R be a ring such that $\mathrm{Ann}_r(\mathrm{Ann}_l(X)) = 0$ for every subset $X \subseteq R$. Then every strongly essential left ideal is strongly dense.*

Proof. By Proposition 2.9 and Proposition 2.10, we are done. □

Motivated by the above Corollary, in Section 3, we investigate a class of modules (namely strongly polyform), in which, we observe that the concepts of strong essentiality and strong density coincide.

Corollary 2.12. *Let R be a ring. Then the following statements hold.*

(1) *If $a \in R$ is a central element that is not a zero-divisor, then $Ra \leq_{sd} R$.*
(2) *Let I be a left ideal in R, and $X \subseteq R$. Then $I \leq_{sd} R$ implies that $(I : X) \leq_{sd} R$.*
(3) *If R is prime then any ideal of R is strongly dense in ${}_RR$ and R_R.*
(4) *Let I and J be left ideals of R. If $I \leq_{sd} R$ and for every subset $X \subseteq I$, we have $(J : X) \leq_{sd} R$, then $J \leq_{sd} R$.*

Proof. (1) Let $X \subseteq R$, since $a \in R$ is a central element then $a \in (Ra : X)$. But a is not a zero-divisor, thus $Ann_r(Ra : X) = 0$. Hence $Ra \leq_{sd} R$ by Proposition 2.10.

(2) Let $Y \subseteq R$, then $\mathrm{Ann}_r((I : X) : Y) = \mathrm{Ann}_r(I : YX)$. Since $I \leq_{sd} R$ we have $\mathrm{Ann}_r(I : YX) = 0$. Thus $\mathrm{Ann}_r((I : X) : Y) = 0$ and $(I : X) \leq_{sd} R$.

(3) It is clear by Proposition 2.10.

(4) Let $X \subseteq R$. We claim that $\mathrm{Ann}_r(J : X) = 0$. Suppose that $0 \neq y \in \mathrm{Ann}_r(J : X)$. Since I is strongly dense, we have $(I : X)y \neq 0$, this means that there exists $r \in R$ such that $rX \subseteq I$ and $ry \neq 0$. But $rX \subseteq I$, hence $(J : rX) \leq_{sd} R$ and $(J : rX)ry \neq 0$. However, $(J : rX)r \subseteq (J : X)$, hence $(J : X)y \neq 0$. Since this holds for all $X \subseteq R$, we infer that $J \leq_{sd} R$. □

Proposition 2.13. *Let M be an R-module. Then:*

(1) *If* $N_1 \leq_{sd} M$, $N_2 \leq_{sd} M$, *then* $N_1 \bigcap N_2 \leq_{sd} M$.
(2) *If* $N_1 \leq_{sd} M_1 \subseteq M$ *and* $N_2 \leq_{sd} M_2 \subseteq M$, *then* $N_1 \bigcap N_2 \leq_{sd} M_1 \bigcap M_2$.
(3) *Let* $N_1 \leq N_2 \leq M$, *Then* $N_1 \leq_{sd} M$ *if and only if* $N_1 \leq_{sd} N_2$ *and* $N_2 \leq_{sd} M$.
(4) *If* $f : M' \to M$ *is an* R*-homomorphism,* $M' \subseteq M$ *and* $N \leq_{sd} M$, *then* $f^{-1}(N) \leq_{sd} M'$.
(5) *Assume that* M *is a t-nonsingular module (i.e.,* $SZ(M) = 0$*). Then* $N \leq_{sd} M$ *if and only if* $N \leq_{se} M$.

Proof. (1) Let Y and $0 \neq X$ be subsets of M. There exists $r_1 \in R$ such that $0 \neq r_1X$ and $r_1 \in (N_1 : Y)$. Since $N_2 \leq_{sd} M$ there exists $r_2 \in R$ such that $0 \neq r_2r_1X$ and $r_2 \in (N_2 : r_1Y)$. Thus $0 \neq r_2r_1X$ and $r_2r_1 \in (N_1 \bigcap N_2 : Y)$. Hence $N_1 \bigcap N_2 \leq_{sd} M$.
(2) It is clear by part (1).
(3) It is clear that if $N_1 \leq_{sd} M$ then $N_1 \leq_{sd} N_2$ and $N_2 \leq_{sd} M$. Conversely, let $N_1 \leq_{sd} N_2$ and $N_2 \leq_{sd} M$. Now suppose that Y and $0 \neq X$ are two subsets of M. There exists $r_1 \in R$ such that $0 \neq r_1X$ and $r_1 \in (N_2 : Y)$. Since $N_2 \leq_{se} M$, there also exists $r_2 \in R$ such that $0 \neq r_2r_1X \subseteq N_2$ (and $r_2r_1 \in (N_2 : Y)$). Since $N_1 \leq_{sd} N_2$ there exists $r_3 \in R$ such that $0 \neq r_3r_2r_1X$ and $r_3r_2r_1 \in (N_1 : Y)$. Hence $N_1 \leq_{sd} M$.

(4) Let Y and $0 \neq X$ be subsets of M'. Therefore $0 \neq X$ and $f(Y)$ are two subsets of M. Since $N \leq_{sd} M$, there exists $r \in R$ such that $0 \neq rX$ and $r \in (N : f(Y))$. Thus $r \in (f^{-1}(N) : Y)$ and $0 \neq rX$ (i.e., $f^{-1}(N) \leq_{sd} M'$).

(5) If $N \leq_{sd} M$, then by Lemma 2.2, it is clear that $N \leq_{se} M$. Conversely, suppose that Y and $0 \neq X$ are two subsets of M. There exists $0 \neq x \in X$ and since $N \leq_{se} M$ we have $(N : Y) \leq_{se} {}_RR$. But $(N : Y)x \neq 0$, for otherwise $(N : Y) \leq \mathrm{Ann}_l(x)$. Now would give the contradiction $x \in SZ(M)$. Hence $(N : Y)X \neq 0$ and $N \leq_{sd} M$.

□

Definition 2.14. Let N be a submodule of an R-module M, and X a subset of M. A peculiar sum of N and X, denoted by $N \oplus_p X$ is defined as follows: $N \oplus_p X = N + X$, where $N \bigcap X = 0$, $(N : X) = \mathrm{Ann}_l(X)$ and $N + X$ is a submodule of M. In particular, if $N \oplus_p X = M$ then N is called a peculiar summand of M.

We use the notation $N \oplus_p X \leq M$ to indicate that the sum $N + X$ is a peculiar sum.

Definition 2.15. An R-module M is called t-injective if the image of any injective R-homomorphism from M to any R-module is a peculiar summand.

Definition 2.16. An R-module M is called strongly rationally complete if it has no proper strongly rational extensions.

Proposition 2.17. *Every t-injective module is strongly rationally complete.*

Proof. Let M be a t-injective module. Now suppose that M' is a module such that $M \leq_{sd} M'$. Let $i : M \to M'$ be natural homomorphism. By definition M is a peculiar summand of M'. But $M \leq_{se} M'$ thus $M = M'$ and we are done. $\square$

Proposition 2.18. *Let M be a t-nonsingular module. Then M is t-injective if and only if M is strongly rationally complete.*

Proof. Let M be t-nonsingular and strongly rationally complete. Suppose that $f : M \to M'$ is an injective R-homomorphism, thus $Im(f)$ is t-nonsingular and it is a strongly rational complete submodule of M'. Now $Im(f)$ is strongly essentially closed by part (5) of Proposition 2.13, therefore $Im(f)$ is a peculiar summand of M'. Hence M is t-injective. The converse is clear. $\square$

Definition 2.19. An R-module M is called t-semisimple if each proper submodule N of M is a peculiar summand.

We should remind the reader that a ring R is semisimple if and only if R has no proper essential left ideals. Motivated by Ghirati and Karamzadeh [4, Theorem 3.3], a ring R is t-semisimple if and only if R has no proper strongly essential left ideals.

Theorem 2.20. *Let R be a ring. Then:*

(1) *R is t-semisimple.*
(2) *R has no proper strongly essential left ideals.*
(3) *R is isomorphic to a product of a finite number of simple rings.*
(4) *R is a left t-nonsingular ring and every left ideal is strongly rationally complete.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By [4, Theorem 3.3].

(1) $\Rightarrow$ (4) Suppose that R is a t-semisimple ring. Let I and J be two left R-modules such that $I \leq_{sd} J$. Since I is a peculiar summand of J then $I = J$. Therefore every ${}_RI$ is strongly rationally complete. We note that $\mathrm{Ann}_l(x) \leq_{se} R$ for any $x \in SZ({}_RR)$. But R is t-semisimple, therefore $\mathrm{Ann}_l(x) = R$ and $x = 0$. Hence R is a left t-nonsingular ring.

(4) $\Rightarrow$ (1) Assume that R is a left t-nonsingular ring and every left ideal is strongly rationally complete. We show that every left ideal I is a peculiar summand of ${}_RR$. By [4, Proposition 1.6], let X is the largest t-component of I. Then $I \oplus_p X \leq_{se} {}_RR$. Since ${}_RR$ is t-nonsingular, this implies that $I \oplus_p X \leq_{sd} {}_RR$. By hypothesis, $I \oplus_p X$ is strongly rationally complete. Hence $I \oplus_p X = R$, as desired.

$\square$

The following result is now in order.

Proposition 2.21. *Let $S \subseteq R$ be rings such that ${}_SS \leq_{se} {}_SR$, and let $N \leq M$ be left R-modules.Then the following statements hold.*

(1) *Let $_SM$ be t-nonsingular. Then $_RN \leq_{sd} {_RM}$ if and only if $_SN \leq_{sd} {_SM}$.*
(2) *Let $_SN$ be t-nonsingular. Then $_RN \leq_{se} {_RM}$ if and only if $_SN \leq_{se} {_SM}$.*
(3) *Let $_SN$ be t-nonsingular. If $_SN$ is t-injective then $_RN$ is t-injective.*
(4) *If $_SN$ is t-nonsingular and strongly rationally complete, then $_RN$ is also strongly rationally complete.*

Proof. (1) If $_SN \leq_{sd} {_SM}$ then it is clear that $_RN \leq_{sd} {_RM}$. Now suppose that $_RN \leq_{sd} {_RM}$. Let Y be a subset of M and $0 \neq x \in M$. There exists $r_1 \in R$ such that $0 \neq r_1x$ and $r_1Y \subseteq N$. Since $_SS \leq_{se} {_SR}$, then R/S is t-singular and there exists a left ideal $J \leq_{se} {_SS}$ such that $Jr_1 \subseteq S$. Now $r_1x \notin SZ(_RM) = 0$, hence $0 \neq jr_1x$ for some $j \in J$. For $r_2 = jr_1 \in S$, we have $0 \neq r_2x$ and $r_2Y = jr_1Y \subseteq JN \subseteq N$. This means that $_SN \leq_{sd} {_SM}$.

(2) If $_SN \leq_{se} {_SM}$ then it is clear that $_RN \leq_{se} {_RM}$. Now assume that $_RN \leq_{se} {_RM}$, repeat the argument above with $0 \neq Y = X$. Hence there exists $0 \neq r$, with $0 \neq rY = rX \subseteq N$. Now by assumption $SZ(_SN) = 0$, we are done.

(3) Suppose $_SN$ is t-injective. If $_RM$ is any strongly essential extension of $_RN$ then by part (2), $_SN \leq_{se} {_SM}$, and hence $N = M$. This means that $_RN$ is also t-injective.

(4) If $_RN \leq_{sd} {_RM}$, then $_RN \leq_{se} {_RM}$. Since by assumption $SZ(_SN) = 0$, part (2) implies that $_SN \leq_{se} {_SM}$. Thus we have $SZ(_SM) = 0$, and by part (1) $_SN \leq_{sd} {_SM}$. Hence $N = M$ as desired. □

Definition 2.22. If N is a submodule of an R-module M, then the t-dense complement of N in M, denoted by $D_M(N)$, is the union of all subsets $Y \subseteq M$ such that $0 \in Y$ and $(N : Y) \subseteq \mathrm{Ann}(x)$ for some $0 \neq x \in M$. When there is no ambiguity we just write $D(N)$ for $D_M(N)$.

Now let us put $D' = \cup\{Y \subseteq M | (N : Y) \subseteq \mathrm{Ann}_l(X)$ for some $0 \neq X \subseteq M$ and $Y \bigcap N = 0\}$, and $D'(N)$ is called the t-dense component of N in M. Consequently for a submodule N of M, we have $N + D' = \{n + t | n \in N$ and $t \in D'\} = N + D_M(N)$.

Proposition 2.23. *Let N be a submodule of an R-module M. Then $D(N)$ is a submodule of M and $N + D' = N + D(N) \leq_{sd} M$.*

Proof. Similar to [4, Proposition 1.6]. □

3. Strongly polyform and strongly monoform modules

In this section, strongly polyform modules, strongly uniform modules and strongly monoform modules are defined and studied.

Zelmanowitz in [13] termed a module M polyform if every essential submodule of M is dense.

By Lemma 2.2, the following definition is natural.

Definition 3.1. A module M is called strongly polyform if every strongly essential submodule is strongly dense in M.

By part (5) of Proposition 2.13, the next corollary is an immediate consequence.

Corollary 3.2. *Let ${}_RM$ be a t-nonsingular R-module, then M is strongly polyform.*

We should remind the reader that a commutative ring R is reduced if and only if it is nonsingular if and only if it is polyform; see [7, Corollary 8.9].

Theorem 3.3. *For a ring R the following statements are equivalent:*

(1) *R is a t-nonsingular ring.*
(2) *R is a strongly polyform ring.*

Proof. (1) $\Rightarrow$ (2) If R is a left t-nonsingular ring, then by Proposition 2.13, every strongly essential left ideal $I \leq R$ is strongly dense in ${}_RR$.

(2) $\Rightarrow$ (1) Let $0 \neq x \in SZ({}_RR)$ and we obtain a contradiction. We note that $\mathrm{Ann}_l(x) \leq_{se} {}_RR$ and $(\mathrm{Ann}_l(x) : 1)x = 0$, then $\mathrm{Ann}_l(x)$ is not strongly dense in ${}_RR$ which is a contradiction. □

Corollary 3.4. *Every semiprime ring R is a strongly polyform ring.*

Proof. In view of [4, Proposition 2.1] and Theorem 3.3, we are done. □

The following definition is in order.

Definition 3.5. A nonzero R-module M is called strongly uniform if every nonzero submodule of M is strongly essential in M.

Note that all nonzero submodules and all strongly essential extensions of strongly uniform modules are strongly uniform.

Definition 3.6. Let X be a multiplicatively closed set in a ring R ($1 \in X$, $ab \in X$ for all $a, b \in X$). Then X satisfies the strongly left Ore condition provided that, for each $0 \neq r_1 \in R$, there exists $r_2 \in R$ such that $0 \neq r_2R \subseteq Xr_1$. A multiplicatively closed set satisfying the strongly left Ore condition is called a strongly left Ore set. The strongly right Ore condition and strongly right Ore sets are defined symmetrically. A strongly Ore set is a multiplicatively closed set which is both a strongly right and a strongly left Ore set. A strongly left Ore domain is any domain R in which the nonzero elements from a strongly left Ore set (note, domains in this article are not necessarily commutative).

Proposition 3.7. *The following conditions on a domain R are equivalent.*

(1) *R is a strongly left Ore set.*
(2) *${}_RR$ is strongly uniform.*

(3) *Every principal left ideal in R contains a principal right ideal.*

Proof. (1) $\Leftrightarrow$ (3) It is clear.

(2) $\Rightarrow$ (3) Let I be a nonzero principal left ideal in R, since I is a strongly essential left ideal, there exists $r \in R$ such that $rR \subseteq I$.

(3) $\Rightarrow$ (2) Let I be a nonzero left ideal in R and $X \subseteq R$ is a subset. Since $0 \neq I$ then there exists $0 \neq b \in I$. Now by our assumption the left ideal Rb contains a principal right ideal rR say. Therefore $rX \subseteq rR \subseteq Rb \subseteq I$, this means that I is a strongly essential left ideal.

□

We recall that an R-module M is called monoform if every nonzero submodule of M is dense in M.
The following definition is now in order.

Definition 3.8. A nonzero R-module M is called strongly monoform if every nonzero submodule of M is strongly dense in M.

Note that all nonzero submodules and all strongly rational extensions of strongly monoform modules are strongly monoform.

Remark 3.9. We observe that every strongly monoform module is strongly uniform. However there are examples of strongly uniform modules that are not strongly monoform. To see this, let $M = \mathbb{Z}/4\mathbb{Z}$, considered as $\mathbb{Z}$-module. Then M is strongly uniform which is not strongly monoform.

Corollary 3.10. *For any t-nonsingular R-module M, the concepts of strongly uniform and strongly monoform coincide.*

Proposition 3.11. *Let I be a left ideal in a ring R. Then the following statements hold.*

(1) *$I \leq_{sd} R$ if and only if $M_n(I) \leq_{sd} M_n(R)$.*
(2) *$I \leq_{d} R$ if and only if $M_n(I) \leq_{d} M_n(R)$.*

Proof. (1) First suppose that $M_n(I) \leq_{sd} M_n(R)$. Let $Y = \{y_k\}_{k\in K}$ and $0 \neq X = \{x_k\}_{k\in K}$ be two arbitrary subsets in R. Now put $X' = \{A_k = (a_{ij})_{n\times n} | a_{11} = x_k$ and $a_{ij} = 0$ for $(i,j) \neq (1,1)\}_{k\in K}$ and $Y' = \{B_k = (b_{ij})_{n\times n} | b_{11} = y_k$ and $b_{ij} = 0$ for $(i,j) \neq (1,1)\}_{k\in K}$ which are two subsets in $M_n(R)$. Since $M_n(I) \leq_{sd} M_n(R)$ there exists $C = (c_{ij})_{n\times n} \in M_n(I)$ such that $CY' \subseteq M_n(I)$ and $0 \neq CX'$. Hence there exists $c_{m1} \in R$ $(1 \leq m \leq n)$ such that $c_{m1}Y \subseteq I$ and $0 \neq c_{m1}X$. This checks $I \leq_{sd} R$. Conversely, suppose that $I \leq_{sd} R$. Let $Y = \{B_k = (b_{ij})_{n\times n}\}_{k\in K}$ and $0 \neq X = \{A_k = (a_{ij})_{n\times n}\}_{k\in K}$ be two arbitrary subsets in $M_n(R)$. We put $Y' = \{b_{ij} | b_{ij}$ is an array in B_k for $k \in K\}$ and $0 \neq X' = \{a_{ij} | a_{ij}$ is an array in A_k for $k \in K\}$ which are two subsets in R. So there exist $r \in R$ and (s,t) such that $rY' \subseteq I$ and $0 \neq ra_{st} \in rX'$. Hence $rE_{1s}Y \subseteq M_n(I)$ and $0 \neq rE_{1s}X$. This means that $M_n(I) \leq_{sd} M_n(R)$.

(2) The proof is similar to part (1).

□

Corollary 3.12. *Let R be a ring. Then R is strongly monoform (resp., monoform) if and only if and only if $M_n(R)$ is strongly monoform (resp., monoform).*

The following definition is borrowed from [7], see also [2].

Definition 3.13. An R-module M is called prime if, whenever N is a non-zero submodule of M and I is an ideal of R such that $IN = 0$, then $IM = 0$.

Proposition 3.14. *Let R be a commutative ring. Then an R-module M is strongly monoform if and only if M is a strongly uniform prime module.*

Definition 3.15. A left ideal I of a ring R is called strongly co-monoform if R/I is strongly monoform.

Proposition 3.16. (1) *Every maximal left ideal is strongly co-monoform.*

(2) *If R is a commutative ring, then the strongly co-monoform ideals of R are exactly the prime ideals.*

(3) *An ideal I of an arbitrary ring R is strongly co-monoform as a left ideal if and only if R/I is a strongly left Ore domain.*

Proof. (1) and (2) are clear.

(3) It is clear by [4, Proposition 2.11], Proposition 3.7 and Corollary 3.10. □

4. Column and special submodules

In this section, the column, the column type, the special and the strong special submodules of an R-module are defined and studied.

Definition 4.1. Let M be an R-module. Then the special (resp., strong special) submodule of M, denoted by $S(M)$ (resp., $SS(M)$) is defined by $S(M) = \{m \in M | \operatorname{Ann}_l(m) \leq_d R\}$ (resp., $SS(M) = \{m \in M | \operatorname{Ann}_l(m) \leq_{sd} R\}$).
It is clear that $S(M)$ and $SS(M)$ are two submodules of M. We say that M is special (resp., nonspecial) if $S(M) = M$ (resp., $S(M) = 0$). It is t-special (resp., t-nonspecial) if $SS(M) = M$ (resp., $SS(M) = 0$).

Remark 4.2. Let N be a submodule of an R-module M and $N \leq_{sd} M$ (resp., $N \leq_d M$), then M/N is t-special (resp., special).

Proof. Let $N \leq_{sd} M$ be a proper submodule N of M. Consider any $x + N \in M/N$. Recall that $(N : x)$ is a strongly dense left ideal of R and $(N : x)(x + N) = N$. Thus $x + N \in SS(M/N)$ and hence M/N is t-special. □

Note that any proper factor module of a strongly monoform (resp., monoform) module is t-special (resp., special).

Lemma 4.3. *Let $M = \oplus_{i \in I} M_i$ be an R-module. Then $S(M) = \oplus_{i \in I} S(M_i)$ and $SS(M) = \oplus_{i \in I} SS(M_i)$.*

In the following results, up to Proposition 4.8, some appropriate and useful facts concerning the above concepts are given.

Proposition 4.4. (1) *All submodule, factor module, and sum (direct or not) of t-special (resp., special) modules are t-special (resp., special).*

(2) *All submodule, direct sum, and strongly rational extension (resp., rational extension) of t-nonspecial (resp., nonspecial) module are t-nonspecial (resp., nonspecial).*

(3) *Let N be a submodule of module M. If N and M/N are both t-nonspecial (resp., nonspecial), then M is t-nonspecial (resp., nonspecial).*

Proposition 4.5. *Let M be an R-module. Then M is t-nonspecial (resp., nonspecial) if and only if for any t-special (resp., special) R-module N, $Hom_R(N, M) = 0$.*

Proof. First assume that M is t-nonspecial. Consider any $R-$homomorphism $f : N \to M$, where N is a t-special R-module. Since $f(N) = f(SS(N)) \subseteq SS(M) = 0$, we have $f = 0$. Conversely, if M is not t-nonspecial, then $N := SS(M)$ is a nonzero t-special module, and the inclusion map $N \to M$ is a nonzero element in $Hom_R(N, M)$. □

Proposition 4.6. *Let M be a t-nonspecial (resp., nonspecial) left R-module and I, J be two strongly dense (resp., dense) left ideals, then $IJM \leq_{sd} M$ (resp., $IJM \leq_d M$).*

Proof. Similar to [4, Proposition 2.8]. □

Theorem 4.7. *Let R be a left nonspecial ring. Then:*

(1) *The factor module $M/S(M)$ is nonspecial;*

(2) *If N is a submodule of R-module M such that N and M/N are both special, then M is special;*

(3) *If M is special then all rational extension of M are special.*

Proof. (1) Let $m \in M$ be such that $\overline{m} \in S(\frac{M}{S(M)})$. Then $Jm \subseteq S(M)$ for some left ideal $J \leq_d {}_RR$. To show that $m \in S(M)$, we must show that $\text{Ann}(m) \leq_d {}_RR$. Let $y \in R$ and $0 \neq x \in R$ then there exists $r \in R$ such that $ry \in J$ and $0 \neq rx$, hence we have $rym \subseteq Jm \subseteq S(M)$, so $Irym = 0$ for some $I \leq_d {}_RR$. But $Irx \neq 0$ for otherwise $0 \neq rx \in S(R)$. Therefore $irx \neq 0$ for some $i \in I$, and $irym = 0$ implies that $iry \subseteq \text{Ann}(m)$. This means that $\text{Ann}(m) \leq_d {}_RR$, and we are done.

(2) Since N is special, $N \leq S(M)$, whence M/N maps onto $M/S(M)$, and so $M/S(M)$ is special. On the other hand $M/S(M)$ is nonspecial by part (1), and hence $M/S(M) = 0$. Thus $M = S(M)$ is special.

(3) Let M' be a rational extension of a left R-module M, and suppose that M is special, by Remark 4.2 M'/M is special, and then part (2) shows that M' is special. □

Proposition 4.8. *Let R be a ring. Then the following statements hold.*

(1) $SS(M_n(R)) = M_n(SS(R))$.
(2) $S(M_n(R)) = M_n(S(R))$.

Proof. (1) Put $A = (a_{ij})_{n\times n} \in M_n(SS(R))$. Thus for any $1 \leq i, j \leq n$ we have $a_{ij} \in SS(R)$ (i.e., $\mathrm{Ann}(a_{ij}) \leq_{sd} R$). Now put $I = \bigcap_{i,j} \mathrm{Ann}(a_{ij})$ so $I \leq_{sd} R$. Hence $M_n(I) \leq_{sd} M_n(R)$ by Proposition 3.11. But $M_n(I) \leq \mathrm{Ann}(A) \leq M_n(R)$. Thus $A \in SS(M_n(R))$. Conversely, suppose that $A = (a_{ij}) \in SS(M_n(R))$. Therefore $\mathrm{Ann}(A) \leq_{sd} M_n(R)$. By Proposition 3.11, there exists $I \leq_{sd} R$ such that $\mathrm{Ann}(A) = M_n(I)$. Hence $M_n(I).A = 0$ and for any $1 \leq i, j \leq n$ we have $Ia_{ij} = 0$. Thus $I \leq \mathrm{Ann}(a_{ij})$. This checks $\mathrm{Ann}(a_{ij}) \leq_{sd} R$ (i.e, $a_{ij} \in SS(R)$). Hence $A \in M_n(SS(R))$.

(2) The proof is, word-for-word, similar to part (1). □

A ring R is called a left Kasch if every simple left R-module M can be embedded in ${}_RR$. We recall that R is a left Kasch ring if and only if the only dense left ideal in R is R itself.
The next result seems interesting.

Theorem 4.9. *Let R be a ring. Then:*

(1) *R has no proper strongly dense left ideal if and only if for any cyclic left R-module M, $SS(M) = 0$.*
(2) *R is left Kasch if and only if for any cyclic left R-module M, $S(M) = 0$.*

Proof. (1) First assume that R has no proper strongly dense left ideal. Let ${}_RM$ be an R-module and $m \in SS(M)$. Thus $\mathrm{Ann}(m) \leq_{sd} {}_RR$. But by assumption $\mathrm{Ann}(m) = R$. Hence $m = 0$ and $SS(M) = 0$. Conversely, suppose that R has a strongly dense left ideal I. Since R/I is a cyclic t-special module then $I = R$ by Remark 4.2. Hence the only strongly dense left ideal in R is R itself.

(2) One can easily apply the similar proof of part (1). □

Proposition 4.10. *Let N be a submodule of an R-module M. If M/N is a t-nonspecial (resp., nonspecial) module, then N is strongly rationally complete (resp., rational complete).*

Proof. Let K be a submodule of M such that $N \leq_{sd} K \leq M$ and $x \in K$. Thus $(N : x) \leq_{sd} R$. Hence $x + N \in SS(M/N) = 0$. Thus $x \in N$ and it follows that $N = K$. □

Proposition 4.11. *Let X be a set in a ring R. If R is a left t-nonspecial (resp., nonspecial) ring, then $\mathrm{Ann}_l(X)$ is strongly rationally complete (resp., rationally complete) in R.*

Proof. Consider any left ideal K such that $\mathrm{Ann}_l(X) \leq_{sd} K$ and let $k \in K$. Thus $(\mathrm{Ann}_l(X) : k) \leq_{sd} R$. From $(\mathrm{Ann}_l(X) : k)kX = 0$, we have $kX \subseteq SS({}_RR) = 0$. Hence $k \in \mathrm{Ann}_l(X)$ and $\mathrm{Ann}_l(X) = K$.

□

Definition 4.12. Let M be an R-module, then the column of M (resp., column type), denoted by $Col(M)$ (resp., $CT(M)$), is the intersection of all dense (resp., strongly dense) submodules of M.

In view of Proposition 3.11, we immediately have the next proposition.

Proposition 4.13. *Let R be a ring. Then:*
(1) $CT(M_n(_RR)) = M_n(CT(_RR))$.
(2) $Col(M_n(_RR)) = M_n(Col(_RR))$.

Example 4.14. Let R be the ring of all upper triangular matrices over a field F. Then $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is only proper ideal of R that is strongly dense in $_RR$ so $CT(_RR) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Similarly $CT(R_R) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$.

Example 4.15. Let $p \in \mathbb{Z}$ be a prime number and put $M = \mathbb{Z}/p^2\mathbb{Z}$, considered as $\mathbb{Z}$-module. Then $Soc(M) = p\mathbb{Z}/p^2\mathbb{Z}$ while $Col(M) = M$, and also $Soc(M)$ is an essential submodule in M which is not dense submodule.

Let us recall that in extending the concept of the socle, which is the intersection of essential submodules, the socle series is introduced and consequently the Loewy modules which play an important role in module theory are considered. Similarly, in the following definition, we introduce the column series of an R-module.

Definition 4.16. Let M be a module over an arbitrary ring R. Inductively define a well-ordered sequence of fully invariant submodules $Col_\alpha(M)$ of M as follows:
$Col_0(M) = 0$,
$Col_{\alpha+1}(M)/Col_\alpha(M) = Col(M/Col_\alpha(M))$ for every non-limit ordinal α,
$Col_\beta(M) = \cup_{\alpha \prec \beta} Col_\alpha(M)$ for every limit ordinal β.
Clearly, $Col_0(M) \subseteq Col_1(M) \subseteq Col_2(M) \subseteq ... \subseteq Col_\alpha(M) \subseteq ...$ and we call this chain column series of M.

The module M is said to weakly-Loewy module if there is an ordinal α such that $M = Col_\alpha(M)$, in which case, there exists the least ordinal α such that $M = Col_\alpha(M)$, and we call it the column length of M. Note that the column series is always stationary, that is, for every module M there exists an ordinal α such that $Col_\beta(M) = Col_\alpha(M)$ for every $\beta \geq \alpha$ (note, M is a set). For such an ordinal α, set $\sigma(M) = Col_\alpha(M)$. Then $\sigma(M)$ is the largest weakly-Loewy submodule of M, and $M/\sigma(M)$ has zero column.

Proposition 4.17. *A module M is weakly-Loewy if and only if every non-zero homomorphic image of M has a non-zero column.*

Proof. If M is weakly-Loewy and N is a proper submodule of M, consider the set of all the ordinal numbers α such that $Col_\alpha(M) \subseteq N$. It is easily seen that this set has a greatest element β. Then M/N is a homomorphic image of $M/Col_\beta(M)$, and the image of the column $Col_{\beta+1}(M)/Col_\beta(M)$ of $M/Col_\beta(M)$ in M/N is non-zero. Therefore the column of M/N is non-zero. Conversely, if M is not weakly-Loewy, then $M/\sigma(M)$ is a non-zero homomorphic image of M with zero column.

□

A ring R is called a Loewy (semi-Artinian) ring if every non-zero cyclic R-module has a nonzero socle. It is clear that for every R-module M, $Soc(M) \subseteq Col(M)$, therefore every Loewy module is a weakly-Loewy module.

Remark 4.18. Let M be a Loewy module. By induction, it is clear that if M has finite Loewy length n then M has finite weakly-Loewy length m and $n \geq m$. We note that for a module M, the Loewy length and the weakly-Loewy length may not be equal. For example, let $p \in \mathbb{Z}$ be a prime number and put $M = \mathbb{Z}/p^2\mathbb{Z}$. Then M as an $\mathbb{Z}$-module has Loewy length 2, but M is a module of weakly-Loewy length 1.

We recall that every dense (resp., strongly dense) submodule is essential (resp., strongly essential). Thus DCC (resp., ACC) on essential (resp., strongly essential) submodules in an R-module M implies DCC (resp., ACC) on dense (resp., strongly dense) submodules.
In [1], Armendariz, has shown that an R-module M has DCC on essential submodules if and only if $M/Soc(M)$ is Artinian.

Proposition 4.19. *An R-module M satisfies the DCC on dense (resp., strongly dense) submodules if and only if $M/Col(M)$ (resp., $M/CT(M)$) is Artinian.*

Proof. Similar to [1, Proposition 1.1]. □

The next result is also immediate.

Proposition 4.20. *DCC on dense (resp, strongly dense) submodules implies DCC on essential (resp, strongly essential) submodules in R-module M if and only if $Col(M)/Soc(M)$ (resp, $CT(M)/ST(M)$) is Artinian.*

Proof. By [1, Proposition 1.1] and Proposition 4.19, we are done. □

Proposition 4.21. *Let R be a ring satisfying the DCC on dense ideals. Then:*
(1) $S(R)$ is a nil ideal.
(2) $S(R) \subseteq J(R)$, which $J(R)$ is the Jacobson ideal of R.
(3) $(J(R))^n \subseteq Col(R)$ for some integer $n \geq 1$.

Proof. (1) If $x \in S(R)$ then $\mathrm{Ann}_l(x) \leq_d R$. Thus $Col(R) \subseteq \mathrm{Ann}_l(x) \subseteq \mathrm{Ann}_l(x^2) \subseteq ....$ But $R/Col(R)$ is Artinian, hence it is Noetherian and thus there is an integer $k \geq 1$ for which $\mathrm{Ann}_l(x^k) = \mathrm{Ann}_l(x^{k+1})$ for all k. Now if

$rx^k \in Rx^k \bigcap \mathrm{Ann}_l(x^k)$ then $rx^{2k} = 0$ and $rx^k = 0$. Hence $Rx^k \bigcap \mathrm{Ann}_l(x^k) = 0$. Since $\mathrm{Ann}_l(x^k)$ is essential we have $Rx^k = 0$ and $x^k = 0$. Hence $S(R) = 0$ is a nil ideal.

(2) Evident.

(3) Since $(J(R)+Col(R))/Col(R) \subseteq J(R/Col(R))$ we have $(J(R)+Col(R))$ $/Col(R)$ which it is nilpotent because $R/Col(R)$ is Artinian. Then for some integer $n \geq 1$, $(J(R))^n \subseteq Col(R)$. □

The proof of the following result is evident, by the previous observations.

Proposition 4.22. *Let M and N be R-modules.*

1. *$Soc(M) \subseteq Col(M) \subseteq CT(M)$ and $Soc(M) \subseteq ST(M) \subseteq CT(M)$.*
2. *$Col(_RR).S(M) = 0$ and $CT(_RR).SS(M) = 0$.*
3. *If $f : N \to M$ is any $R-$homomorphism, then $f(S(N)) \subseteq S(M)$ and $f(SS(N)) \subseteq SS(M)$.*
4. *If $N \subseteq M$, then $S(N) = S(M) \bigcap N$ and $SS(N) = SS(M) \bigcap N$.*
5. *If $Col(_RR) \leq_d {_RR}$, then $S(_RR) = \mathrm{Ann}_r(Col(_RR))$.*
6. *If $CT(_RR) \leq_{sd} {_RR}$, then $SS(_RR) = \mathrm{Ann}_r(CT(_RR))$.*

Let us assume that whenever we write "$D = SD$ in M" it means that the dense submodules of M are strongly dense.
The next result is similar to [4, Theorem 4.5].

Theorem 4.23. *For a ring R the following statements are equivalent.*

1. *For any left R-module M, $M \leq_{sd} \tilde{E}(M)$.*
2. *For any left R-module M, $D = SD$ in M.*

Proof. (1) ⇒ (2) Let N be a dense submodule of a left R-module M. Since $N \leq_d M \leq_{sd} \tilde{E}(M)$, hence $N \leq_d \tilde{E}(M)$. Thus $\tilde{E}(N) = \tilde{E}(M)$ and $N \leq_d M \leq_{sd} \tilde{E}(N)$, but by assumption in part (1), we have $N \leq_{sd} \tilde{E}(N)$, hence $N \leq_{sd} M$.

(2) ⇒ (1) It is clear. □

We should remind the reader that if R is a left nonsingular left Artinian ring, then $Soc(_RR)$ is the smallest dense left ideal of R; see [7, Corollary 13.25]. But R is a left nonsingular ring, therefore R is a polyform ring (i.e., dense left ideals coincide with essential left ideals); see [7, Corollary 8.9]. Hence $Col(_RR) = Soc(_RR)$ is the smallest dense left ideal. We note that some class of rings have essential socle, for example ue-rings [6] and finitely embedded rings, see [7].

Proposition 4.24. *Let R be a ring. Then the following statements hold.*

(1) $Soc(R) \leq_d R$ if and only if R is nonsingular and $Soc(R) \leq_e R$.

(2) $Soc(R) \leq_{sd} R$ if and only if R is t-nonsingular and $Soc(R) \leq_{se} R$.

(3) Let M be an R-module. Then $Col(M) \leq_{sd} M$ if and only if $Col(M) \leq_d M$ and $D = SD$ in M.

Proof. (1) If $Soc(R) \leq_d R$ then all essential left ideals of R are dense, thus R is nonsingular by [7, Corollary 8.9]. The converse is clear.

(2) The proof is similar to part (1) by Theorem 3.3.

(3) Let $Col(M) \leq_{sd} M$, since every dense submodule contains $Col(M)$, thus density and strong density are the same. The converse is clear. □

The set D of all dense left ideals of a ring R is a Gabriel topology; see [11, pages 144, 146, 149], but the set of all essential or strongly essential left ideals of R is not a Gabriel topology in general, because part (4) of Corollary 2.12 may not be true if we replace dense ideals by essential ideals or strongly essential ideals. The Gabriel topology on the set D consisting of dense left ideals of R is called dense topology; see [11, page 149], and it is also called Lambek topology by some authors; see for example [10]. We observe that the set of all strongly dense left ideals is also a Gabriel topology by Proposition 2.13 and Corollary 2.12, and we called it strongly dense topology.

We also remark that a Gabriel topology is closed under product.
The next corollary is immediate.

Corollary 4.25. *Let R be a ring and I,J be two strongly dense left ideals, then $IJ \leq_{sd} R$.*

We conclude the article with the following result which is an easy consequence of Theorem 4.23.

Theorem 4.26. *For a ring R, strongly dense and dense topologies coincide if and only if $R \leq_{sd} \tilde{E}(R)$.*

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