

GENERALIZED COVERING GROUPS AND DIRECT LIMITS

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ABSTRACT. M. R. R. Moghaddam [Monatsh. Math. 90 (1980) 37-43.] showed that the Baer invariant commutes with the direct limit of a directed system of groups. In this paper, using the generalization of Schur's formula for the structure of a $\mathcal{V}$ -covering group for a Schur-Baer variety $\mathcal{V}$, we show that the structure of a $\mathcal{V}$ -covering group commutes with the direct limit of a directed system, in some senses. It has a useful application in order to extend some known structures of $\mathcal{V}$ -covering groups for several famous products of finitely many to an arbitrary family of groups.

1. Introduction

Historically, there have been several papers from the beginning of the twentieth century trying to find some structures for the well-known notion the covering group and its varietal generalization, the $\mathcal{V}$ -covering group of some famous groups and products of groups, such as the direct product, the nilpotent and the regular product [4, 5, 8, 11, 15, 21, 24].

It is known that any group has at least a covering group [21, 25]. Also the number of covering groups for an arbitrary group has been studied by Schur [22]. Moreover, it is proved that any group has a $\mathcal{V}$ -covering group, where $\mathcal{V}$ is a Schreier variety [8, 10, 15].

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In 1971, Wiegold [24] found a structure of a covering group for the direct product $G = A \times B$ so that the second nilpotent product of the covering groups A^* and B^* is a covering group for G . In 1972, Haebich [4] constructed a covering group for a finite regular product. Moreover the structure of a covering group for the verbal wreath product of two groups has been studied by Haebich [5], in 1977.

Naturally, it is of interest to know which class of groups does *not* have a $\mathcal{V}$ -covering group. The first author [9, 10] gave some examples of groups which do not have any generalized covering group with respect to the variety $\mathcal{N}_c$ of nilpotent groups of class at most $c \geq 2$. More precisely, he [10] proved that every nilpotent group of class n with nontrivial c -nilpotent Schur multiplier does not have any $\mathcal{N}_c$ -covering group for $c > n$. Thus the results of Wiegold and Haebich mentioned above cannot be generalized to an arbitrary variety. Moreover the first author [10] has given a complete answer to the existence of $\mathcal{N}_c$ -covering group for finite abelian groups. Also in 2003, he in a joint paper [11] found a structure for the $\mathcal{N}_c$ -covering group of a nilpotent product of a family of cyclic groups.

Now in this paper, we intend to prove that the structure of $\mathcal{V}$ -covering group commutes with the direct limit of a directed system of groups in some senses (see Theorem 3.5). Furthermore, we give an example showing that the hypothesis of being directed for the system of groups is an essential condition (see Example 3.6). Finally, as an application, we extend some of the previous formulas for the structure of $\mathcal{V}$ -covering groups for finite direct and regular products of groups to infinite ones.

2. Notation and preliminaries

We shall assume that the reader is familiar with the notion of the verbal subgroup $V(G)$ and the marginal subgroup $V^*(G)$ of a group G , associated with a variety $\mathcal{V}$. Whenever varieties of groups are discussed, we refer to Neumann [17] for notation and basic results.

Definition 2.1. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V , and let G be a group with a free presentation

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1,$$

where F is a free group. Then the *Baer invariant* of G , denoted by $\mathcal{VM}(G)$, is defined to be

$$R \cap V(F)/[RV^*F],$$

where $V(F)$ is the verbal subgroup of F and $[RV^*F]$ is the least normal subgroup N of F contained in R such that $R/N \subseteq V^*(F/N)$. Thus it is the subgroup generated by the following set:

$$[RV^*F] = \langle v(f_1, \dots, f_{i-1}, f_i r, f_{i+1}, \dots, f_n) v(f_1, \dots, f_i, \dots, f_n)^{-1} \mid$$

$$r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in \mathbf{N} \rangle.$$

Note that the Baer invariant of a group G is always abelian and independent of the choice of the presentation of G and it might be possible to regard $\mathcal{VM}(-)$ as the first left derived functor of the functor from all groups to $\mathcal{V}$ taking G to $G/V(G)$, see [8]. In particular, if $\mathcal{V}$ is the variety of abelian groups, then the Baer invariant of the group G will be $R \cap F'/[R, F]$, which is the well-known notion, the Schur multiplier $M(G)$ of G . Also, if $\mathcal{V} = \mathcal{N}_c$ is the variety of nilpotent groups of class at most c , then the Baer invariant of the group G will be $R \cap \gamma_{c+1}(F)/[R, {}_c F]$, where $\gamma_{c+1}(F)$ is the $(c+1)$ -st term of the lower central series of F and $[R, {}_1 F] = [R, F]$, $[R, {}_c F] = [[R, {}_{c-1} F], F]$, inductively. It is also called the c -nilpotent multiplier of G .

Definition 2.2. A variety $\mathcal{V}$ is called a *Schur-Baer variety* if for any group G for which the marginal factor group $G/V^*(G)$ is finite, then the verbal subgroup $V(G)$ is also finite and $|V(G)|$ divides a power of $|G/V^*(G)|$.

Schur [21] proved that the variety of abelian group is a Schur-Baer variety and Baer [2] proved that a variety defined by some outer commutator words, for instance the variety $\mathcal{N}_c$, has the above property. The following theorem gives a useful property of Schur-Baer varieties.

Theorem 2.3 ([8]). *A variety $\mathcal{V}$ is a Schur-Baer variety if and only if for every finite group G , its Baer invariant $\mathcal{VM}(G)$ is of order dividing a power of $|G|$.*

Definition 2.4. Let $\mathcal{V}$ be a variety of groups and let G be a group. Then, by definition, a $\mathcal{V}$ -covering group of G (a generalized covering group of G with respect to $\mathcal{V}$) is a group G^* with a normal subgroup A such that $G^*/A \cong G$, $A \subseteq V(G^*) \cap V^*(G^*)$, and $A \cong \mathcal{V}M(G)$ (see [8]).

Note that if $\mathcal{V}$ is the variety of abelian groups, then a $\mathcal{V}$ -covering group of G will be an ordinary covering group (sometimes it is called a representing group) of G . Also if $\mathcal{V} = \mathcal{N}_c$, then an $\mathcal{N}_c$ -covering group of G is a group G^* with a normal subgroup A such that

$$\frac{G^*}{A} \cong G, \quad A \subseteq Z_c(G) \cap \gamma_{c+1}(G), \quad \text{and} \quad A \cong \mathcal{N}_c M(G) .$$

It is well-known that every group G has a covering group (see [7, 21, 25]). In general, for the existence of $\mathcal{V}$ -covering groups, we have the following concepts and results. Let $\mathcal{V}$ be a variety of groups. Then a group G is called $\mathcal{V}$ -free if it is a free object in the category of all groups in the variety $\mathcal{V}$. It is known that if F is a free group, then $F/V(F)$ is a $\mathcal{V}$ -free group. By a well-known theorem of Schreier, every subgroup of a free group is also free. Thus it is natural to define the following notion.

Definition 2.5. Let $\mathcal{V}$ be a variety of groups. Then $\mathcal{V}$ is called a *Schreier variety* if and only if every subgroup of a $\mathcal{V}$ -free group is also $\mathcal{V}$ -free.

It has been proved by P. M. Neumann and J. Wiegold [8, 18] that the only Schreier varieties are the variety $\mathcal{G}$ of all groups, the variety $\mathcal{A}$ of abelian groups, and the variety $\mathcal{A}_p$ of all abelian group of exponent p , where p is a prime.

Note that the notion of Schreier varieties can be generalized to varieties in which every subgroup of a $\mathcal{V}$ -free group is a $\mathcal{V}$ -splitting group, i.e., a group G in the variety $\mathcal{V}$ which splits every short exact sequence in $\mathcal{V}$ of the following form

$$1 \rightarrow A \rightarrow B \rightarrow G \rightarrow 1 .$$

Clearly every Schreier variety has the above property. In fact the only varieties with the above property are $\mathcal{G}$, $\mathcal{A}$ and $\mathcal{A}_m$, where m is square free (see [8, 18]).

Leedham-Green and McKay [8], by a homological method, proved that a sufficient condition for the existence of a $\mathcal{V}$ -covering group of G

is that $G/V(G)$ is a $\mathcal{V}$ -splitting group. Also in the following theorem we give a similar sufficient condition for the existence of a $\mathcal{V}$ -covering group of a group.

Theorem 2.6 ([8, 10, 15]). *Let $\mathcal{V}$ be a variety in which every subgroup of a $\mathcal{V}$ -free group is $\mathcal{V}$ -splitting. Then every group has a $\mathcal{V}$ -covering group. In particular, if $\mathcal{V}$ is a Schreier variety, then every group has a $\mathcal{V}$ -covering group.*

Also the first author [10] showed that if G is a nilpotent group of class n such that $\mathcal{N}_c M(G) \neq 1$ and $c > n$, then G has no $\mathcal{N}_c$ -covering group. Moreover, this fact has been extended to the variety of polynilpotent groups in some senses [12].

We next attend to some concepts as well-known categorical objects, the direct system and the direct limit which are defined [19, 20] for any arbitrary category, if any. Note that in some famous categories, specially the category of all groups, every direct system has a direct limit (see [19, 20]). In particular, in the category of all groups, the free product $\prod_{i \in I}^* G_i$ as a coproduct of $\{G_i\}_{i \in I}$ is a direct limit of this family which is the trivial direct system and so is not directed (see [20]).

In our main results we deal with the particular case of direct systems which are indexed with a directed set, called directed system. In fact, we use one of the equivalent form of the definition of direct limit of a directed system in the category of all groups, which is more simple and useful for our goal, as we mention in following preliminaries. Note that in all following points, we refer the reader to [14, 19, 20] for further details.

Definition 2.7. Let $\{G_i\}$ be a direct system of groups indexed by a partially ordered set I , which is also directed, that is, for every $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$. For $i \leq j$, let there exists a homomorphism $\lambda_i^j : G_i \rightarrow G_j$ such that

- (i) $\lambda_i^i : G_i \rightarrow G_i$ is the identity map of G_i , for all $i \in I$;

(ii) if $i \leq j \leq k$, then $\lambda_i^j \lambda_j^k = \lambda_i^k$, as the following commutative diagram:

$$\begin{array}{ccc} G_i & \xrightarrow{\lambda_i^j} & G_j \\ \lambda_i^k \searrow & & \downarrow \lambda_j^k \\ & & G_k. \end{array}$$

In this case, we call the system $\{G_i; \lambda_i^j, I\}$ a *directed system*. Now we define an equivalence relation on the disjoint union $\bigcup_{i \in I} G_i$ by: if $x \in G_i$ and $y \in G_j$, then

$$x \sim y \text{ if and only if } x\lambda_i^k = y\lambda_j^k \text{ for } k \geq i, j.$$

Let G denote the quotient set $\bigcup_{i \in I} G_i / \sim$ and use $\{x\}$ for the equivalence class of x . Also we define a multiplication on G as follows: if $\{x\}, \{y\}$ are elements of G , we choose $i, j \in I$ such that $x \in G_i$ and $y \in G_j$; then

$$\{x\}\{y\} = \{(x\lambda_i^k)(y\lambda_j^k)\}, \text{ for } k \geq i, j.$$

Clearly this is a well-defined multiplication, which makes G into a group and it is called the *direct limit* of the directed system $\{G_i; \lambda_i^j, I\}$. It will be denoted by

$$\varinjlim G_i = \bigcup_{i \in I} G_i / \sim = G.$$

We need only the following well-known results of direct limits.

Lemma 2.8. *Suppose that $\{G_i; \lambda_i^j, I\}$ is a directed system of groups and $G = \varinjlim G_i$. Then we have the following statements*

(i) *The group G has the universal property so that for a given group H and homomorphisms $\tau_i : G_i \rightarrow H$, such that $\lambda_i^j \tau_j = \tau_i$ for all $i \leq j$, there exists a unique homomorphism $\tau : G \rightarrow H$ such that all the diagrams*

$$\begin{array}{ccc} & G_i & \\ \lambda_i \downarrow & \searrow \tau_i & \\ G & \xrightarrow{\tau} & H \end{array}$$

commute, that is $\lambda_i \tau = \tau_i$, for all $i \in I$.

(ii) *Direct limit of exact sequences, indexed by a directed set, is exact, and so in this case, the direct limit preserves injections.*

(iii) *if G is an arbitrary group, then G is the direct limit of its finitely generated subgroups, under the obvious directed system arising from the inclusion maps.*

Definition 2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $T_1 : \mathcal{C} \rightarrow \mathcal{D}$, $T_2 : \mathcal{D} \rightarrow \mathcal{C}$ be two functors such that for any $X \in \mathcal{C}$, $Y \in \mathcal{D}$ there is a natural equivalence

$$\text{Hom}_{\mathcal{D}}(T_1 X, Y) \simeq \text{Hom}_{\mathcal{C}}(X, T_2 Y).$$

In this case, T_2 is called a *right adjoint functor* to T_1 and the pair (T_1, T_2) is called an *adjoint pair*.

It is well-known fact that every functor which has a right-adjoint, commutes with direct limits. So we have the following lemma (see [20]).

Lemma 2.10. *Let $\{X_i; \lambda_i^j, I\}$ be a direct system of sets indexed by a partially ordered set I , and let $\mathcal{C}$ and $\mathcal{G}$ denote the categories of sets and groups, respectively. If $F : \mathcal{C} \rightarrow \mathcal{G}$ is the free functor which associates with every set the free group on that set as basis, then F commutes with direct limit, that is,*

$$F(\varinjlim X_i) = \varinjlim F(X_i).$$

Remark 2.11. As a corollary, suppose that $\{G_i; \lambda_i^j, I\}$ is a directed system of groups, and the sequence

$$1 \rightarrow R_i \rightarrow F_i \rightarrow G_i \rightarrow 1$$

is a free presentation for G_i , where $F_i (= F(G_i))$ is the free group on the underlying set of G_i , for all $i \in I$. Now using lemma 2.8.(ii), the direct limit of a directed set is an exact functor, and hence kernel-preserving, so by Lemma 2.10, the sequence

$$1 \rightarrow \varinjlim R_i \rightarrow \varinjlim F_i \rightarrow \varinjlim G_i \rightarrow 1$$

is a free presentation for $\varinjlim G_i$.

Lemma 2.12. *With the above assumption and notation, we have the following relations*

- (i) $(\varinjlim R_i) \cap V(\varinjlim F_i) = \varinjlim (R_i \cap F_i);$
- (ii) $[(\varinjlim R_i)V^*(\varinjlim F_i)] = \varinjlim [R_i V^* F_i].$

Theorem 2.13 ([14]). *Let $\{G_i; \lambda_i^j, I\}$ be a directed system of groups. Then for a given variety $\mathcal{V}$, the Baer invariant commutes with direct limit, that is,*

$$\mathcal{V}M(\varinjlim G_i) = \varinjlim \mathcal{V}M(G_i).$$

3. The main result

In order to deal with $\mathcal{V}$ -covering groups of a group G , it is useful to know more relationship between the Baer invariant $\mathcal{V}M(G)$ and the $\mathcal{V}$ -covering groups of G . In this aspect, to prove our main theorem, first of all we need to point the following notes which are the generalization of some parts of an important theorem of Schur [7, Theorem 2.4.6].

Lemma 3.1. *Let $\mathcal{V}$ be a variety of groups and G be a group with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$. If S is a normal subgroup of F such that*

$$\frac{R}{[RV^*F]} \cong \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]}, \quad (3.1)$$

then $G^ = F/S$ is a $\mathcal{V}$ -covering group of G .*

Proof. Setting $A = R/S$, so $G^*/A \cong F/R \cong G$ and using (3.1), we have

$$A = \frac{R}{S} \cong \frac{R/[RV^*F]}{S/[RV^*F]} \cong \frac{R \cap V(F)}{[RV^*F]} \cong \mathcal{V}M(G).$$

Since $[RV^*F] \subseteq S$, we have $A = R/S \subseteq V^*(F/S) = V^*(G^*)$. Also we have

$$A = \frac{R}{S} = \frac{RS}{S} \stackrel{\text{by } (*)}{\subseteq} \frac{V(F)S}{S} = V\left(\frac{F}{S}\right) = V(G^*).$$

Hence G^* is a $\mathcal{V}$ -covering group of G . □

Lemma 3.2. *Let $\mathcal{V}$ be a variety of groups, let G be a group with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$, and let G^* be a $\mathcal{V}$ -covering group of G . Then G^* is a homomorphic image of $F/[RV^*F]$.*

Proof. Let F be free on X and let $\pi : F \rightarrow G$ be an epimorphism such that $R = \ker(\pi)$. Since G^* is a $\mathcal{V}$ -covering group of G , we have the

following exact sequence

$$1 \rightarrow A \longrightarrow G^* \xrightarrow{\Phi} G \rightarrow 1 ,$$

where $A \subseteq V^*(G^*) \cap V(G^*)$ and $A \cong \mathcal{VM}(G)$. Since Φ is surjective, there exists l_x in G^* such that $\Phi(l_x) = \pi(x)$, for all $x \in X$. Therefore we have

$$G^* = \langle A, l_x | x \in X \rangle = AN,$$

where $N = \langle l_x | x \in X \rangle$. Now, by a result of Hekster [6],

$$A \subseteq V(G^*) = V(AN) = V(N)[AV^*G^*] \subseteq V(N) \subseteq N .$$

(Note that since $A \subseteq V^*(G^*)$, we have $[AV^*G^*] = 1$.) Hence we have

$$G^* = N = \langle l_x | x \in X \rangle .$$

Now consider the homomorphism $\Psi : F \rightarrow G^*$ defined by $\Psi(x) = l_x$, $x \in X$. Then Ψ is surjective and $\pi = \Phi \circ \Psi$. Since $1 = \pi(R) = \Phi(\Psi(R))$, we have $\Psi(R) \subseteq A$, so that

$$\Psi([RV^*F]) \subseteq [\Psi(R)V^*G^*] \subseteq [AV^*G^*] = 1 .$$

It follows that Ψ induces an epimorphism $\bar{\Psi} : F/[RV^*F] \longrightarrow G^*$. $\square$

Theorem 3.3. *Let $\mathcal{V}$ be a Schur-Baer variety and let G be a finite group with a free presentation $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$. If G^* is $\mathcal{V}$ -covering group of G , then there exists a normal subgroup S of F such that*

$$\frac{R}{[RV^*F]} \cong \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]} ,$$

and so $G^* \cong F/S$.

Proof. By the proof of Lemma 3.2 and its notation, for every $a \in A$, there exists $x \in F$ such that $a = \Psi(x)$. Hence $\pi(x) = \Phi \circ \Psi(x) = \Phi(\Psi(x)) = \Phi(a) = 1$, so $x \in R$ and thus $A \subseteq \Psi(R)$. Also in the proof of Lemma 3.2 we showed that $\Psi(R) \subseteq A$, so $A = \Psi(R)$. Next we observe that

$$\Psi(R \cap V(F)) \subseteq \Psi(R) \cap \Psi(V(F)) = A \cap V(G^*) = A .$$

To prove the other inclusion, assume that $z = \Psi(x) = \Psi(y)$ for some $x \in V(F)$ and $y \in R$. Then $x^{-1}y \in \ker \Psi$, so $\pi(x^{-1}y) = 1$, and therefore $x^{-1}y \in R$. It follows that $x \in R$ and $z \in \Psi(R \cap V(F))$. Thus

$$A = \Psi(R \cap V(F)) .$$

Now $\bar{\Psi}$ defines an epimorphism

$$\bar{\Psi}_1 : \frac{R \cap V(F)}{[RV^*F]} \longrightarrow A .$$

Since $\mathcal{V}$ is a Schur-Baer variety, $A \cong \mathcal{V}M(G) = R \cap V(F)/[RV^*F]$, and G is finite, and hence by Theorem 1.3, A is also finite. Thus the above epimorphism is isomorphism. Now, put $S = \ker(\Psi) \cap R$. Then clearly $S \trianglelefteq F$ and $S/[RV^*F]$ is the kernel of the restriction of $\bar{\Psi}$ to $R/[RV^*F]$, i.e.,

$$\bar{\Psi}_2 : \frac{R}{[RV^*F]} \longrightarrow A .$$

Now we may consider the following short exact sequence

$$1 \rightarrow \frac{S}{[RV^*F]} \longrightarrow \frac{R}{[RV^*F]} \xrightarrow{\bar{\Psi}_2} A \rightarrow 1. \quad (3.2)$$

Since $\bar{\Psi}_1$ is an isomorphism, we have $\bar{\Psi}_2 \circ \bar{\Psi}_1^{-1} = 1_A$, and hence the above short exact sequence splits. Therefore we have

$$\frac{R}{[RV^*F]} \cong A \rtimes \frac{S}{[RV^*F]} \cong \frac{R \cap V(F)}{[RV^*F]} \rtimes \frac{S}{[RV^*F]} .$$

But clearly $R \cap V(F)/[RV^*F] \trianglelefteq R/[RV^*F]$, so the above semidirect product is actually a direct product. Now, by Lemma 3.1, F/S is a $\mathcal{V}$ -covering group of G .

Let $\theta : F/S \longrightarrow G^*$ be the homomorphism induced by Ψ . Since Ψ is surjective and $\Psi(R) = A$, θ is surjective and $\theta(R/S) = A$. However $|F/S| = |G^*|$, so θ becomes an isomorphism, i.e., $G^* \cong F/S$. $\square$

Note that this generalization of the Schur theorem, has been posed and proved by Moghaddam and Salemkar [16], but it seems that there are some missing points in their proof, specially the splitting of the exact sequence (3.2), and so the condition of being Schur-Baer for the variety $\mathcal{V}$.

Lemma 3.4. *The direct limit with a directed index set, as we mentioned in Definition 2.5, preserves the finite direct product, that is, for any two directed systems of groups $\{A_i; \lambda_i^j, I\}$ and $\{B_i; \mu_i^j, I\}$, we have*

$$\varinjlim (A_i \times B_i) = \varinjlim A_i \times \varinjlim B_i.$$

Proof. Firstly, for any $i \in I$, we have the following split exact sequence with natural homomorphisms

$$1 \rightarrow A_i \rightarrow A_i \times B_i \rightarrow B_i \rightarrow 1.$$

Now using Lemma 2.8(ii), the direct limit preserves exactness and so we have the following exact sequence which is also split

$$1 \rightarrow \varinjlim A_i \rightarrow \varinjlim (A_i \times B_i) \rightarrow \varinjlim B_i \rightarrow 1.$$

On the other hand, we know that the direct limit is kernel-preserving and so preserves normal subgroups. Hence $\varinjlim B_i$ is a normal subgroup of $\varinjlim (A_i \times B_i)$ and so the result holds. $\square$

Now in order to state and prove the main result of the paper, we need to explain the concept of an induced directed system of $\mathcal{V}$ -covering groups which we use in the main theorem. Let $\mathcal{V}$ be a Schur-Baer variety, and let $\{G_i; \lambda_i^j, I\}$ be a directed system of finite groups. suppose that G_i^* is a $\mathcal{V}$ -covering group for G_i , for all $i \in I$. Now if we consider the sequence

$$1 \rightarrow R_i \rightarrow F_i \rightarrow G_i \rightarrow 1$$

as a free presentation, then using Theorem 3.3 there exists a normal subgroup S_i of F_i in such a way that $G_i^* = F_i/S_i$ and specially satisfies the following relation

$$\frac{R_i}{[R_i V^* F_i]} \cong \frac{R_i \cap V(F_i)}{[R_i V^* F_i]} \times \frac{S_i}{[R_i V^* F_i]}. \quad (3.3)$$

By these notations, for any $i \leq j$ in I , there exists an induced homomorphism $\widehat{\lambda}_i^j$ commutes the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & R_i & \longrightarrow & F_i & \longrightarrow & G_i \longrightarrow 1 \\ & & & & \downarrow \widehat{\lambda}_i^j & & \downarrow \lambda_i^j \\ 1 & \longrightarrow & R_j & \longrightarrow & F_j & \longrightarrow & G_j \longrightarrow 1. \end{array}$$

The commutativity of this diagram implies that the homomorphism $\widehat{\lambda}_i^j$ maps R_i into R_j and so $\widehat{\lambda}_i^j(R_i \cap V(F_i)) \subseteq R_j \cap V(F_j)$. Hence if $\widehat{\lambda}_i^j$ maps S_i into S_j , we will have the following induced homomorphism, for any $i \leq j$

$$\widetilde{\lambda}_i^j : G_i^* = \frac{F_i}{S_i} \longrightarrow G_j^* = \frac{F_j}{S_j},$$

which forms the directed system $\{G_i^*; \tilde{\lambda}_i^j, I\}$, called an *induced directed system of covering groups*.

Note that in general, any family of covering groups of a directed system of groups is not necessarily an induce one. For example we consider the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ with two non-isomorphic covering groups D_8 and Q_8 . So it takes the trivial directed system $\{G_i; \lambda_i^j, \mathbf{N}\}$ which $G_i = \mathbf{Z}_2 \times \mathbf{Z}_2$ and λ_i^j to be identity, for any $i, j \in \mathbf{N}$. Now if we take the family of covering groups $\{G_i^*; i \in \mathbf{N}\}$ such that $G_{2i}^* = D_8$ and $G_{2i+1}^* = Q_8$. Then $\{G_i^*; i \in I\}$ does not form an induced directed system.

Theorem 3.5. *Suppose that $\mathcal{V}$ is a Schur-Baer variety. If $\{G_i; \lambda_i^j, I\}$ is a directed system of finite groups with an induced system of $\mathcal{V}$ -covering groups $\{G_i^*; \tilde{\lambda}_i^j, I\}$, as we mentioned above, then the group $G^* = \varinjlim G_i^*$ is a $\mathcal{V}$ -covering group for $G = \varinjlim G_i$.*

Proof. Using the isomorphism (3.3) and Lemma 3.4, we have

$$\varinjlim \frac{R_i}{[R_i V^* F_i]} \cong \varinjlim \frac{R_i \cap V(F_i)}{[R_i V^* F_i]} \times \varinjlim \frac{S_i}{[R_i V^* F_i]}. \quad (3.4)$$

Also by Lemma 2.8(ii), we have

$$\varinjlim \frac{R_i}{[R_i V^* F_i]} = \frac{\varinjlim R_i}{\varinjlim [R_i V^* F_i]}, \quad (3.5)$$

$$\varinjlim \frac{R_i \cap V(F_i)}{[R_i V^* F_i]} = \frac{\varinjlim (R_i \cap V(F_i))}{\varinjlim [R_i V^* F_i]} \quad \text{and} \quad \varinjlim \frac{S_i}{[R_i V^* F_i]} = \frac{\varinjlim S_i}{\varinjlim [R_i V^* F_i]}. \quad (3.6)$$

Therefore using Lemma 2.12 and (3.4), (3.5), (3.6) we conclude that

$$\frac{(\varinjlim R_i)}{[(\varinjlim R_i) V^* (\varinjlim F_i)]} \cong \frac{(\varinjlim R_i) \cap V(\varinjlim F_i)}{[(\varinjlim R_i) V^* (\varinjlim F_i)]} \times \frac{(\varinjlim S_i)}{[(\varinjlim R_i) V^* (\varinjlim F_i)]}.$$

Now by the above relation, Corollary 2.8 and Lemma 3.1, $\varinjlim F_i / \varinjlim S_i$ and so the group

$$G^* = \varinjlim G_i^* = \varinjlim \frac{F_i}{S_i} = \frac{\varinjlim F_i}{\varinjlim S_i}$$

will be a $\mathcal{V}$ -covering group of $G = \varinjlim G_i$. $\square$

We end this section by an example showing that the condition of being directed for index set of the direct system in our study is essential.

Example 3.6. If we omit the condition of being directed, then the free product of any two groups A, B as a particular direct limit of groups whose index set is not directed (see [20]), should have a covering group with the structure $A^* * B^*$, where A^* and B^* are the covering groups of A and B , respectively. But this is a contradiction, when we choose A with nontrivial Schur multiplier. Since in this case, if $A^* * B^*$ is a covering group of $A * B$, then by Definition 2.4, we will have

$$M(A * B) \cong N \text{ with } N \subseteq Z(A^* * B^*) \cap (A^* * B^*)' = 1.$$

But using a result of Miller [13], we have $M(A * B) \cong M(A) \times M(B) \neq 1$, which is a contradiction.

4. Applications

As we mentioned in the introduction, we have the structure of covering or generalized covering groups for some famous products of finitely many groups, such as finite direct and finite regular products of finite groups [4, 24]. In this section, as an important application of the main result, we present a generalized covering group for the above products when their index sets are arbitrary. Also the main result of this note may have an application in the sense that if one wants to find a generalized covering group for an arbitrary group, one only needs to find a generalized covering group for every finitely generated subgroups of it.

Firstly, by a result of Schur [22], for a finite nilpotent group G , we have

$$M(G) \cong M(S_1) \times M(S_2) \times \cdots \times M(S_n),$$

where $S_1, S_2, \dots, S_n$ are all sylow subgroups of G .

Using this property and the definition of covering group, we deduce the following straightforward fact.

Corollary 4.1. *Let G be a finite nilpotent group and let $S_1, S_2, \dots, S_n$ be its Sylow subgroups. If S_i^* is a covering group for S_i , Then the group $G^* = S_1^* \times S_2^* \times \dots \times S_n^*$ is a covering group of G .*

Now using the main result of this paper, we conclude a generalization of the above note as follows.

Corollary 4.2. *Let G be a torsion nilpotent group and let $\{S_i\}_{i \in I}$ be the set of its Sylow subgroups. If S_i^* is a covering group of S_i , for all $i \in I$, then the group $G^* = \prod_{i \in I}^{\times} S_i^*$ is a covering group for G .*

Proof. First, note that every torsion nilpotent group, is the direct product of its Sylow subgroups and so $G = \prod_{i \in I}^{\times} S_i$. Now if we consider the system $\{\prod_{j \in J}^{\times} S_j; \lambda_J^K\}$, where $J \subseteq K$ are finite subsets of I and λ_J^K is the inclusion map, then the group $G = \prod_{i \in I}^{\times} S_i$ is the direct limit of this system which is obviously a directed system.

Clearly we have the induced directed system on the covering groups of $\prod_{j \in J}^{\times} S_j$'s with the morphisms $\tilde{\lambda}_J^K$ as follows

$$\{\prod_{j \in J}^{\times} S_j^*; \tilde{\lambda}_J^K\}_{J \subseteq I}^{finite}.$$

Note that the morphisms $\tilde{\lambda}_J^K$ are clearly inclusion maps and so the direct product $G^* = \prod_{i \in I}^{\times} S_i^*$ will be considered as the direct limit of this system and hence, using Theorem 3.5, the proof is completed. $\square$

Also by a result of Ellis [3], for a finite nilpotent group G with Sylow subgroups $P_1, P_2, \dots, P_n$, we have

$$\mathcal{N}_c M(G) \cong \mathcal{N}_c M(P_1) \times \mathcal{N}_c M(P_2) \times \dots \times \mathcal{N}_c M(P_n).$$

Using this property and similar arguments, we deduce the similar corollary for an $\mathcal{N}_c$ -covering group of a torsion nilpotent group as follows.

Corollary 4.3. *Let G be a torsion nilpotent group and let $\{S_i\}_{i \in I}$ be the set of its Sylow subgroups P_i , for $i \in I$. If P_i^* is an $\mathcal{N}_c$ -covering group of P_i for all $i \in I$, then the group $G^* = \prod_{i \in I}^{\times} P_i^*$ is an $\mathcal{N}_c$ -covering*

group for G .

We next establish the structure of a covering group for the direct and regular products of arbitrary many of groups which are generalizations of results of Wiegold [24] and Haebich [4].

Corollary 4.4. *Let $\{A_i\}_{i \in I}$ be an arbitrary family of finite groups and suppose that A_i^* is a covering group of A_i for any $i \in I$. Then the second nilpotent product of A_i 's is a covering group of $\prod_{i \in I}^\times A_i$.*

Proof. First, we recall that the second nilpotent product of a family of groups as $\{A_i^*\}_{i \in I}$, is defined to be

$$\prod_{i \in I}^2 A_i^* = \frac{\prod_{i \in I}^* A_i^*}{\gamma_3(\prod_{i \in I}^* A_i^*) \cap [A_i^*]^*},$$

where the subgroup $[A_i^*]^*$ is the kernel of the natural epimorphism $\pi : \prod_{i \in I}^* A_i^* \rightarrow \prod_{i \in I}^\times A_i^*$, which has also the following structure

$$[A_i^*]^* = \langle [A_i, A_j] ; i \neq j \rangle \prod_{i \in I}^* A_i^*.$$

Now similar to the proof of Corollary 4.2, we consider the directed system $\{\prod_{j \in J}^\times A_j ; \lambda_J^K\}$ of finite direct products and the inclusion maps, with the group $A = \prod_{i \in I}^\times A_i$ as the direct limit of this system.

By a result of Wiegold [24], the group $\prod_{i \in J}^2 A_i^*$ is a covering group of $\prod_{j \in J}^\times A_j$, where J is finite. Consider $\{\prod_{i \in J}^* A_i^* ; \tilde{\lambda}_J^K\}$ as a directed system, where the morphism $\tilde{\lambda}_J^K$ is clearly the inclusion map, $\tilde{\lambda}_J^K : \prod_{j \in J}^2 A_j^* \hookrightarrow \prod_{k \in K}^2 A_k^*$. Clearly $A^* = \prod_{i \in I}^2 A_i^*$ is the direct limit of the last induced directed system and so using Theorem 3.5 is a covering group for $\prod_{i \in I}^\times A_i$. $\square$

Notation and Corollary 4.5. *Let G be a regular product of its finite subgroups A_i , $i \in I$, where I is considered as an ordered set. For each $i \in I$, L_i denotes a fixed covering group for A_i and consider the exact sequence $1 \rightarrow M_i \rightarrow L_i \xrightarrow{\nu_i} A_i \rightarrow 1$ such that*

$$M_i \subseteq Z(L_i) \cap L_i' \text{ and } M_i \cong M(A_i),$$

where M_i is a normal subgroup of L_i , and $M(A_i)$ is the Schur multiplier of A_i . Assume that $A = \prod_{i \in I}^* A_i$ and $L = \prod_{i \in I}^* L_i$ are free products of A_i 's and L_i 's, respectively. We denote by ν the natural homomorphism from L onto A induced by the ν_i 's. Also, if ψ is the natural homomorphism from A onto G induced by the identity map on each A_i ,

$$L = \prod_{i \in I}^* L_i \xrightarrow{\nu} A = \prod_{i \in I}^* A_i \xrightarrow{\psi} G \rightarrow 1 ,$$

then we denote by H the kernel of ψ and set

$$J = \nu^{-1}(H) \cap [L_i^L], \quad N = \prod_{\substack{i,j=1 \\ i \neq j}}^n [M_i, L_j]^L, \quad M = \left(\prod_{i=1}^n M_i \right) J.$$

Finally, $\bar{L}$ and $\bar{M}$ denote the images of L and M under the natural homomorphism $L \rightarrow L/N[J, L]$, respectively. Then there is an exact sequence

$$1 \rightarrow \bar{M} \rightarrow \bar{L} \rightarrow G \rightarrow 1,$$

such that $\bar{M} \subseteq Z(\bar{L}) \cap [\bar{L}, \bar{L}]$ and $\bar{M} \cong M(G)$. In particular, $\bar{L}$ is a covering group of G .

Proof. First of all, we note that the group G is called the regular product of its subgroups A_i 's, with the ordered set I , whenever the following two conditions hold

$$G = \langle A_i ; i \in I \rangle, \quad A_i \cap \hat{A}_i = 1 \quad (\forall i \in I),$$

where $\hat{A}_i$ is defined to be the group

$$\hat{A}_i = \prod_{j \in I, j \neq i} A_j^G.$$

Now similar to the previous notes and using the definition, we clearly consider the directed system on finite regular products of A_i 's, with inclusion maps. As we saw before, it induces a directed system on their covering groups, with induced homomorphisms which are also inclusion.

By a result of Haebich [4], the corollary holds for any finite index set I . Now, using Theorem 3.5, it is easy to check that the group $\bar{L}$ as a direct limit of the induced system is a covering group of the regular products of A_i 's which is considered as a direct limit of the first system. $\square$

Note that the above theorem is in fact a generalization of the Haebich's formula for a covering group of any finite regular product of finite groups [4]. However the main proof of Haebich could be easily generalized to the regular product of infinitely many of finite groups, but our proof as an application of the main result is another proof to this generalization.

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REFERENCES

- [1] R. Baer, Representations of groups as quotient groups I,II,III, *Trans. Amer. Math. Soc.* **58** (1945), 295-419.
- [2] R. Baer, Endlichkeitskriterien für kommutatorgruppen, *Math. Ann.* **124** (1952), 161-177.
- [3] G. Ellis, On groups with a finite nilpotent upper central quotient, *Arch. Math.* **70** (1998), 89-96.
- [4] W. Haebich, The multiplier of a regular product of groups, *Bull. Aust. Math. Soc.* **7** (1972), 279-296.
- [5] W. Haebich, The multiplier of a splitting extension, *J. Algebra* **44** (1977), 420-433.
- [6] N. S. Hekster, Varieties of groups and isologisms, *J. Aust. Math. Soc. Ser. A* **46** (1989), 22-60.
- [7] G. Karpilovsky, *The Schur Multiplier*, London Math. Soc. Monographs, New Series 2, Oxford University Press, Oxford, 1987.
- [8] C. R. Leedham-Green, S. McKay, Baer invariant, isologism, varietal laws and homology, *Acta Math.* **137** (1976), 99-150.
- [9] B. Mashayekhy, A remark on generalized covering groups, *Indian J. Pure Appl. Math.* **29** (7) (1998), 711-713.
- [10] B. Mashayekhy, On the existence of $\mathcal{V}$ -covering groups, *Proceedings of 31st Iranian Mathematics Conference* (2000), 227-232.
- [11] B. Mashayekhy, A. Khaksar, On $\mathcal{N}_c$ -covering groups of a nilpotent product of cyclic groups, *International J. Math., Game Theory Algebra* **13** (2) (2003), 129-132.
- [12] B. Mashayekhy, M. A. Sanati, On polynilpotent covering groups of a polynilpotent group, *International J. Math., Game Theory Algebra* **15** (4) (2006), 381-386.
- [13] C. Miller, The second homology group of a group: relations among commutators, *Proc. Amer. Math. Soc.* **3** (1952), 588-595.
- [14] M. R. R. Moghaddam, The Baer invariant and the direct limit, *Monatsh. Math.* **90** (1980), 37-43.

- [15] M. R. R. Moghaddam, A. R. Salemkar, Characterization of varietal covering and Stem groups, *Comm. Algebra* **27** (11) (1999), 5575-5586.
- [16] M. R. R. Moghaddam, A. R. Salemkar, Varietal isologism and covering groups, *Arch. Math.* **75** (2000), 8-15.
- [17] H. Neumann, *Varieties of Groups*, Springer-Verlag, Berlin, 1967.
- [18] P. M. Neumann, J. Wiegold, Schreier varieties of groups, *Math. Z.* **85** (1964), 392-400.
- [19] D. J. S. Robinson, *A Course in the Theory of Groups*, Grand. Text. Math. 80, Springer-Verlag, Berlin, 1982.
- [20] J. J. Rotman, *An Introduction to Homological Algebra*, Pure and Applied Mathematics vol. 85, Academic Press, New York, 1979.
- [21] I. Schur, Über die darstellung der endlichen gruppen durch gebrochene lineare substitutionen, *J. Für Math.* **127** (1904), 20-50.
- [22] I. Schur, Untersuchungen über die darstellung der endlichen gruppen durch gebrochene lineare substitutionen, *J. Für Math.* **132** (1907), 85-137.
- [23] I. Schur, Über die darstellung der symmetrischen und der alternierenden gruppen durch gebrochene lineare substitutionen, *J. Für Math.* **139** (1911), 155-250.
- [24] J. Wiegold, The multiplier of a direct product, *Quart. J. Math. (Oxford)* (2) **22** (1971), 103-105.
- [25] J. Wiegold, The Schur multiplier: an elementary approach. Groups-St. Andrews 1981, London Math. Soc. Lecture Note Ser. **71** (1982), 137-154.

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