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Author(s):

M. Baziar, E. Momtahan and S. Safaeeyan

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A MODULE THEORETIC APPROACH TO ZERO-DIVISOR GRAPH WITH RESPECT TO (FIRST) DUAL

M. BAZIAR, E. MOMTAHAN* AND S. SAFAEEYAN

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Abstract. Let M be an R-module and $0 \neq f \in M^* = \mathrm{Hom}(M, R)$. We associate an undirected graph $\Gamma_f(M)$ to M in which non-zero elements x and y of M are adjacent provided that $xf(y) = 0$ or $yf(x) = 0$. We observe that over a commutative ring R, $\Gamma_f(M)$ is connected and $\mathrm{diam}(\Gamma_f(M)) \leq 3$. Moreover, if $\Gamma_f(M)$ contains a cycle, then $\mathrm{gr}(\Gamma_f(M)) \leq 4$. Furthermore, if $|\Gamma_f(M)| \geq 1$, then $\Gamma_f(M)$ is finite if and only if M is finite. Also if $\Gamma_f(M) = \emptyset$, then f is monomorphism (the converse is true if R is a domain). If M is either a free module with $\mathrm{rank}(M) \geq 2$ or a non-finitely generated projective module, there exists $f \in M^*$ with $\mathrm{rad}(\Gamma_f(M)) = 1$ and $\mathrm{diam}(\Gamma_f(M)) \leq 2$. We prove that for a domain R, the chromatic number and the clique number of $\Gamma_f(M)$ are equal. Finally, we give answer to a question posed in [M. Baziar, E. Momtahan and S. Safaeeyan, A zero-divisor graph for modules with respect to their (first) dual, *J. Algebra Appl.* 12 (2013), no. 2, 11 pages].
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MSC(2010): Primary: 05C25; Secondary: 05C38, 05C40, 16D10, 16D40.

1. Introduction

The main idea of the zero-divisor graph of a ring R was first posed by Beck [8] in 1988. Then in [5], the authors continued the study of zero-divisor graphs. In their definition, all elements of R allowed to be vertices and two distinct elements x and y were adjacent if and only if $xy = 0$. Later on, in [4], another conception of zero-divisor graph has been introduced which became the accepted definition of the zero-divisor graph by many authors who wrote in this field of research in recent decades. They associated a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and two distinct x, y in $Z(R)^*$ are adjacent if and only if $xy = 0$. Hence $\Gamma(R)$ is the empty graph if and only if R is an integral domain. In this paper $\Gamma(R)$ is

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*Corresponding author.

called the *classic zero-divisor graph*. As we have already said, in recent decades, the zero-divisor graphs of commutative rings have been extensively studied by many authors and become a major field of research for its own, see for example [2–8, 13, 14] and [15]. S.P. Redmond replaced zero (ideal) in the definition of the classic zero divisor graph by an arbitrary ideal (see [17]) to get a generalization of the classic zero-divisor graph of a commutative ring. Some authors have also tried to extend the classic graph of zero-divisors for non-commutative rings see [1, 16]. In [10, 11], the classic graph of zero-divisors for commutative rings has been generalized to the annihilating-ideal graph of commutative rings (two ideals I and J are adjacent if $IJ = (0)$). It is also worth mentioning that DeMeyer et al. in [12] defined the zero-divisor graph of a commutative semigroup S with zero ($0x = 0$ for all $x \in S$) and quite recently in [21], the authors have defined zero-divisor graphs for partially ordered sets with a least element 0. The zero divisor graph for modules over a commutative ring, introduced in [9], was one of the first attempts to generalize the classic zero-divisor graphs in module theoretic context. According to [9], $m, n \in M$ are adjacent if and only if $(mR :_R M)(nR :_R M)M = 0$ which is a direct generalization of the classic zero divisor graphs. The present authors have studied and examined other conceptions of the classic zero-divisor graph for modules in [6, 18] (see also [7] for an application of zero-divisor graph of modules introduced in [18] to the category of $\mathbb{Z}$-modules). In this article we give a new interpretation of zero-divisor graph for modules, which in some cases, coincides with the classic zero-divisor graph of commutative rings. In Example 2.2 we will observe that our definition and those introduced in [6, 9, 18] are quite different.

We say that G is *connected* if there is a path between any two distinct vertices. For distinct vertices x and y in G, the *distance* between x and y, denoted by $d(x, y)$, is the length of a shortest path connecting x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if no such path exists). The *diameter* of G is

$$\mathrm{diam}(G) = \sup\{d(x, y) \mid \ x \text{ and } y \text{ are vertices of } G\}.$$

A *cycle* of length n in G is a path of the form $x_1 - x_2 - x_3 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. We define the *girth* of G, denoted by $\mathrm{gr}(G)$, as the length of a shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{gr}(G) = \infty$. A graph is *complete* if any two distinct vertices are adjacent. By a complete subgraph we mean a subgraph which is complete as a graph. In this article, all subgraphs are induced subgraphs, where a subgraph G' of a graph G is an *induced subgraph* of G if two vertices of G' are adjacent in G' if and only if they are adjacent in G. A complete subgraph of G is called a *clique*. The *clique number* of G, denoted by $\mathrm{cl(G)} = \sup\{|\mathrm{G'}| : \text{ where } G' \text{ is a complete subgraph of } G\}$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum (cardinal) number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Clearly $\mathrm{cl(G)} \leq \chi(\mathrm{G})$.

All rings in this paper are commutative with identity and modules assumed to be unitary right modules. A ring R is said to be *self-injective* if every R-homomorphism from an ideal I to R can be extended to an R-homomorphism from R to R. By M^* we mean $M^* = \mathrm{Hom}_R(M, R)$, i.e., its first dual of M. The reader is referred to [19, 20] for undefined terms and concepts.

2. Zero-divisor graphs of modules

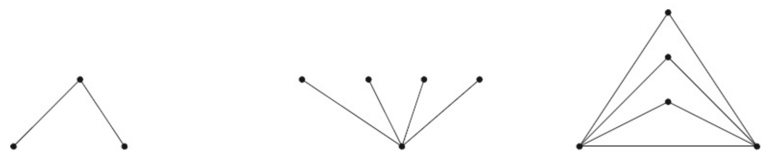
We begin with the definition of the zero-divisor graph of modules and then give some clarifications of the difference between this definition and those appeared in the literature.

Definition 2.1. Let M be an R-module and $f \in M^* = \mathrm{Hom}_R(M, R)$. We define $Z_f(M)$ to be the set of all $x \in M$ with the property that there exists a non-zero $y \in M$ such that $xf(y) = 0$ or $yf(x) = 0$.

Let M be an R-module and $f \in M^*$. We associate a simple graph $\Gamma_f(M)$ to M with vertices $Z_f(M)^* = Z_f(M) \setminus \{0\}$ such that for distinct $x, y \in Z_f(M)^*$ the vertices x and y are adjacent if and only if $xf(y) = 0$ or $yf(x) = 0$. Let $f, g \in M^*$ be two monomorphisms. If $x, y \in M$ with $xf(y) = 0$, then it can be easily seen that $yg(x) = 0$. Hence for any two monomorphisms $f, g \in M^*$, we have $\Gamma_f(M) = \Gamma_g(M)$. Now put $M = R$, and $f = id_R$, where by id_R we mean the identity map of R, then the classic zero-divisor graph is just $\Gamma_{id_R}(R)$. In fact for any monomorphism $g \in Hom(R, R)$, we have $\Gamma_g(R) = \Gamma_{id_R}(R) = \Gamma(R)$. For those modules M, with $M^* = 0$, the graph $\Gamma_f(M)$ is an empty graph; however, the converse is not true. To see this, let R be a domain and $M = R$. Then $M^* \cong R \neq 0$ but $\Gamma_f(M)$ is an empty graph, for all $f \in M^*$.

The next example and Figure 1, show that there is a sharp difference between the zero-divisor graph of R (as a right R-module) and the classic zero-divisor graph of R (as a ring). Furthermore this example shows that our graph is quite different from the zero-divisor graph for modules, introduced in [9].

Example 2.2. (1) The graph of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module is an empty graph because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) = 0$.
(2) The above figures are some examples of the zero-divisor graph of modules and the classic zero-divisor graphs.



$\mathbb{Z}_6$ *as ring* $\quad$ $\mathbb{Z}_6$ *as* $\mathbb{Z}_6$*-module and* $f(x) = 2x$ $\quad$ $\mathbb{Z}_6$ *as* $\mathbb{Z}_6$*-module and* $f(x) = 3x$

FIGURE 1.

The next lemma will be used frequently in the sequel.

Lemma 2.3. *Let M be an R-module and x, y be non-zero elements in M. Let $f \in M^*$, then the followings hold.*
(1) If $xf(y) = 0$ and $yf(x) \neq 0$, then $yf(x)$ is adjacent to every non-zero $m \in M$.
(2) If x is adjacent to y in $\Gamma_f(M)$, then $xr \neq 0$ is adjacent to $ys \neq 0$, $r, s \in R$.
(3) If $x \neq 0$ is in $\ker(f)$, then each non-zero element of M is adjacent to x.
(4) If x is the only element of $Z_f(M)^*$ adjacent to every element of $\Gamma_f(M)$, then either $\ker(f) = \{0, x\}$ or f is a monomorphism.

Proof. (1) Suppose $x, y \in M \setminus \{0\}$ such that $xf(y) = 0$ and $yf(x) \neq 0$. Since $f(yf(x)) = f(y)f(x) = f(x)f(y) = f(xf(y)) = f(0) = 0$, for every $m \in M$ we have $mf(yf(x)) = 0$. Thus $yf(x)$ is adjacent to every $m \in M \setminus \{0\}$.
(2) Let $xf(y) = 0$. Then for all $r, s \in R$, $xrf(ys) = 0$.
(3) It is clear.
(4) Let $\ker(f) \neq 0$, then there exists a non-zero element $y \in \ker(f)$. By Part (3), each element of $\Gamma_f(M)$ is adjacent to y. Therefore $y = x$ and hence $\ker(f) = \{0, x\}$. □

The next two results are generalizations of Theorem 2.3 and Theorem 2.4 in [4], respectively.

Theorem 2.4. *Let R be a ring, M an R-module and $f \in M^*$. Then $\Gamma_f(M)$ is connected and $\operatorname{diam}(\Gamma_f(M)) \leq 3$.*

Proof. If $\Gamma_f(M) = \emptyset$, then there is nothing to say. Hence assume that $|\Gamma_f(M)| \geqslant 1$. Let x and y be two distinct elements in $Z_f(M)^*$. If x is adjacent to y then $d(x, y) = 1$. Assume that x is not adjacent to y. Then there exist $a, b \in Z_f(M)^*$ such that x and y are adjacent to a and b, respectively. If $a = b$, then $x - a - y$ is a path between x and y of length 2. If $a \neq b$ and a is adjacent to b, then $x - a - b - y$ is a path between x and y of length 3. Now suppose that a and b are distinct vertices that are not adjacent. If $xf(a) = 0$ but $af(x) \neq 0$, then by Lemma 2.3, $af(x) \in Z_f(M)^* \setminus \{x, y, a, b\}$ is adjacent to each element

of $Z_f(M)^*$. Therefore, $x - af(x) - y$ is a path of length 2. If $bf(y) = 0$ and $yf(b) \neq 0$, then similarly we can find a path of length 2 between x and y.
If the above cases do not appear, then $af(x)$, $xf(a)$, $bf(y)$ and $yf(b)$ are zero. If $xf(b) = 0$, then $x - b - y$ is a path between x and y of length 2. Suppose that $xf(b) \neq 0$. Therefore,

$$af(xf(b)) = a(f(x)f(b)) = (af(x))f(b) = 0$$

and

$$(xf(b))f(y) = x(f(y)f(b)) = x(f(yf(b))) = xf(0) = 0.$$

These imply that $xf(b)$ is adjacent to both a and y. Therefore, $x - a - xf(b) - y$ is a path of length less than or equal to 3. □

Theorem 2.5. *Let R be a ring, M an R-module and $f \in M^*$. If $\Gamma_f(M)$ contains a cycle, then* $\mathrm{gr}(\Gamma_f(M)) \leq 4$.

Proof. Let $x_1 - x_2 - ... - x_n - x_1$ be a cycle of length $n \geq 5$. Put $x_{n+1} = x_1$. If for some $1 \leq i \leq n$, $x_i f(x_{i+1}) = 0$ and $x_{i+1} f(x_i) \neq 0$, then by Lemma 2.3, $x_{i+1} f(x_i)$ is adjacent to each element of $Z_f(M)^*$. Since $n \geq 5$ there exists $1 \leq j \leq n$ such that $x_{i+1} f(x_i)$ is different from both x_j and x_{j+1}. Therefore $x_j - x_{i+1} f(x_i) - x_{j+1} - x_j$ is a cycle of length 3. We may suppose that $x_i f(x_{i+1}) = x_{i+1} f(x_i) = 0$ for every $1 \leq i \leq n$. If $x_1 f(x_3) = 0$, then $x_1 - x_2 - x_3 - x_1$ is a cycle of length 3. Now, assume that $x_1 f(x_3) \neq 0$. Therefore,

$$x_2(f(x_1 f(x_3))) = (x_2 f(x_1))f(x_3) = 0,$$
$$x_1 f(x_3) f(x_4) = x_1 f(x_3 f(x_4)) = 0, \text{ and}$$
$$x_1 f(x_3) f(x_n) = x_1 f(x_n) f(x_3) = 0$$

imply that $x_1 f(x_3)$ is adjacent to x_2, x_4 and x_n. One of the following cases may hold.
(Case 1) If $x_1 f(x_3) = x_2$, then $x_2 - x_3 - x_4 - x_2 = x_1 f(x_3)$ is a cycle of length 3.
(Case 2) If $x_1 f(x_3) = x_4$, then $x_4 - x_3 - x_2 - x_4 = x_1 f(x_3)$ is a cycle of length 3.
(Case 3) If $x_1 f(x_3) = x_3$, then $x_n - x_1 - x_2 - x_3 = x_1 f(x_3) - x_n$ is a cycle of length 4.
(Case 4) If $x_1 f(x_3) \notin \{x_2, x_3, x_4\}$, then $x_2 - x_3 - x_4 - x_1 f(x_3) - x_2$ is a cycle of length 4. □

The next theorem shows a connection between the cardinality of M and the cardinality of $Z_f(M)^*$. It generalizes Theorem 2.2 in [4].

Theorem 2.6. *Let R be a ring, M an R-module and $f \in M^*$.*
(1) If $|Z_f(M)^*| \geq 1$, then $\Gamma_f(M)$ is finite if and only if M is finite.
(2) $\Gamma_f(M) = \emptyset$ if and only if f is a monomorphism, $\mathrm{ann}(M)$ *is a prime ideal and $Z_f(M)^* \neq M \setminus \{0\}$.*

Proof. (1) Suppose that $Z_f(M)^*$ is finite and nonempty. Then there are nonzero $x, y \in M$ with $xf(y) = 0$. Put $I = \text{ann}_M(f(y))$. Then $I \subseteq Z_f(M)$ is finite and $xr \in I$ for all $r \in R$. If M is infinite, then there exists an $i \in I$ with

$$J_i = \{m \in M \mid xf(m) = i\}$$

infinite since $M = \bigcup_{i \in I} J_i$. For any $m, n \in J_i$,

$$xf(m - n) = xf(m) - xf(n) = 0.$$

Therefore x is adjacent to $(m - n)$ for each $m, n \in J_i$. This is a contradiction, thus M must be finite. The converse is trivial.
(2) Let $\Gamma_f(M) = \emptyset$ and $x \in \ker f$. Since $mf(x) = 0$ for each $m \in M$, hence $Z_f(M)^* = M \setminus \{0\}$ which is a contradiction. Assume that $a, b \in R$ such that $ab \in \text{ann}(M)$ but $a \notin \text{ann}(M)$ and $b \notin \text{ann}(M)$. There exist $m, n \in M$ such that $ma \neq 0$ and $nb \neq$. We know that

$$maf(nb) = mf(nb)a = mf(nba) = mf(0) = 0.$$

Hence $ma, nb \in Z_f(M)^*$, a contradiction. Conversely, let $m \in Z_f(M)^*$. Then there exists $n \in M$ such that either $mf(n) = 0$ or $nf(m) = 0$. Suppose that $mf(n) = 0$. This implies that $f(m)f(n) = 0 \in \text{ann}(M)$. Therefore $0 \neq f(m) \in \text{ann}(M)$ or $0 \neq f(n) \in \text{ann}(M)$ which implies that either m or n is adjacent to any nonzero element of M, a contradiction. □

The following corollary shows that for every simple faithful R-module M, either $\Gamma_f(M)$ is infinite or an empty graph.

Corollary 2.7. *Let M be an R-module and $f \in M^*$ with $1 \leq |\Gamma_f(M)| < \infty$. Then M is finite and not a simple faithful R-module.*

Proof. The finiteness of M is an immediate consequence of Theorem 2.6. Now, suppose that M is a simple faithful R-module. Since $|\Gamma_f(M)| \geq 1$, there exist $m, n \in Z_f(M)^*$ such that $mf(n) = 0$. It is easy to see that

$$N = \{x \in M \mid xf(n) = 0\}$$

is a non-zero submodule of M, therefore $N = M$. Since M is faithful, $Mf(n) = 0$ implies that $f(n) = 0$ and hence $n = 0$ because M is simple. this is a contradiction. □

Proposition 2.8. *Let M be a simple module and $f \in M^*$, then the followings hold:*

1. *If $|Z_f(M)^*| \geq 1$, then $\Gamma_f(M)$ is a complete graph with $Z_f(M)^* = M \setminus \{0\}$.*
2. *If R is semiprime, then $\Gamma_f(M) = \emptyset$ for every $0 \neq f \in M^*$.*
3. *If R is a local ring which is not semiprime, then $\Gamma_f(M)$ is a complete graph with $Z_f(M)^* = M \setminus \{0\}$ for every $0 \neq f \in M^*$.*

Proof. (1) Since $|\Gamma_f(M)| \geq 1$, there exist $m, n \in Z_f(M)^*$ such that $mf(n) = 0$. Hence $(mr)f(ns) = 0$ for each $r, s \in R$. The simplicity of M implies that any two nonzero elements of M are adjacent in $\Gamma_f(M)$.
(2) Suppose that $\Gamma_f(M) \neq \emptyset$. By (1), $f(M)f(M) = 0$ and hence $f(M) = 0$ due to R is semiprime, a contradiction.
(3) Suppose that J is the unique maximal ideal of R. It is clear that $J = \text{ann}(M)$. Since R is a local ring and $f(M) \neq R$ (because M is simple and R is not a field), we have $f(M) \subseteq \text{ann}(M)$ which implies that $xf(y) = 0$ for every $x, y \in M \setminus \{0\}$. □

It is worth mentioning that when R is an integral domain, M is an R-module and f is a nonzero monomorphism in M^*, then $\Gamma_f(M) = \emptyset$.

Corollary 2.9. *For any right R-module M and $f \in M^*$, put $K_f = \ker f \setminus \{0\}$.*

Let M be an R-module. By Lemma 2.3 it is clear that every element of K_f is adjacent to all non-zero element of M.

Proposition 2.10. *Let M be a right R-module, $f \in M^*$ and G be a maximal complete subgraph of $\Gamma_f(M)$. Then $K_f \subseteq V(G)$, where $V(G)$ is the set of all vertices of G.*

Proof. By contrary, assume that $x \in K_f \setminus V(G)$, i.e., $f(x) = 0$. Hence x is adjacent to any vertex of $\Gamma_f(M)$, in particular it is adjacent to any vertex of G. Then the induced subgraph $G \cup \{x\}$ is a complete subgraph of $\Gamma_f(M)$ properly containing G, a contradiction. □

Corollary 2.11. *Let M be a right R-module, $f \in M^*$ and $k = |K_f|$. Then we have the followings:*
i) If $2 \leq k < |Z_f(M)^*|$, then $\text{gr}(\Gamma_f(M)) = 3$.
ii) If $k = 1$ and $\Gamma_f(M)$ contains a cycle, then $\text{gr}(\Gamma_f(M)) = 3$.

Proof. (i) According to our hypothesis there exist at least $x, y \in K_f$ and $z \in Z_f(M)^* \setminus K_f$. Now x and y are adjacent and both of them are adjacent to z.
(ii) Since $\Gamma_f(M)$ contains a cycle, there exist at least $x, y \in Z_f(M)^* \setminus K_f$ which are adjacent. On the other hand, the only member of K_f is adjacent to both x and y. Therefore we have a cycle of length 3. □

Proposition 2.12. *Let R be a ring. If M is either a free R-module with $\text{rank}(M) \geq 2$ or a non-finitely generated projective R-module, then for each non-zero $x \in M$ there exists $f \in M^*$ such that $Z_f(M)^* = M \setminus \{0\}$, $\text{rad}(\Gamma_f(M)) = 1$ and $\text{diam}(\Gamma_f(M)) \leq 2$.*

Proof. Let $M = \bigoplus_{i \in I} R$ with $|I| \geq 2$ and $\{x_i\}_{i \in I}$ be a non-zero element of M. Then there exists $i_0 \in I$ such that $x_{i_0} \neq 0$. Fix an $i_0 \neq j \in I$. Now we define the map

$$f : M \longrightarrow R \text{ with } f(\{a_k\}_{k \in I}) = a_j x_{i_0} - a_{i_0} x_j.$$

It is easy to see that f is a non-zero $R-$homomorphism with the property that $f(\{x_i\}_{i\in I}) = 0$. Therefore $\{x_i\}_{i\in I} \in K_f$. By Lemma 2.3, $\{x_i\}_{i\in I}$ is adjacent to any element of $M \setminus \{0\}$. Consequently, $\mathrm{rad}(\Gamma_f(M)) = 1$ and $\mathrm{diam}(\Gamma_f(M)) \leq 2$.

Now suppose that M is a non-finitely generated projective R-module. By the Dual Basis Lemma there exists an infinite set of elements $\{a_i\}_{i\in I} \subseteq M$ and an infinite set of elements $\{f_i\}_{i\in I} \subseteq M^*$ such that for each $a \in M$, $f_i(a) = 0$ for almost all $i \in I$ and $a = \sum_{i\in I} a_i f_i(a)$. Now for each non-zero $a \in M$, there exists at least $j \in I$ such that $f_j(a) = 0$. Similarly, by Lemma 2.3, a is adjacent to any element of $M \setminus \{0\}$. Putting $f = f_j$ one conclude that $\mathrm{rad}(\Gamma_f(M)) = 1$ and $\mathrm{diam}(\Gamma_f(M)) \leq 2$. □

The above proposition is not true anymore if we replace "non-finitely generated projective" by "finitely generated projective". Let R be a domain and $M = R$. Since for every $f \in M^*$, $\Gamma_f(M)$ is an empty graph we have $\mathrm{diam}(\Gamma_f(M)) = \mathrm{rad}(\Gamma_f(M)) = \infty$.

The following proposition is motivated by [3, Theorem 2.5].

Proposition 2.13. *Let R be a domain which is not a field, S be a simple R-module, $M = S \oplus R$ and $f \in M^*$. Then the followings hold.*
(1) *$K_f = S \oplus 0 \setminus \{(0,0)\}$ and $\mathrm{diam}(\Gamma_f(M)) = 2$.*
(2) *If $|S| \geq 3$, then $\mathrm{gr}(\Gamma_f(M)) = 3$.*
(3) *The graph $\Gamma_f(M)$ contains a maximal complete subgraph of order $|K_f| + 1$. In particular, $\mathrm{cl}(\Gamma_f(M)) = |K_f| + 1$.*

Proof. (1) Let $0 \neq x \in S$ and f be a non-zero element of M^*. Since R is a domain which is not a field, $\mathrm{ann}(x) \neq 0$. Therefore $f((x,0))\mathrm{ann}(x) = 0$ and hence $f((x,0)) = 0$. Now, assume that $(a,b) \in K_f$. Then

$$0 = f((a,b)) = f((a,0)) + f((0,b)) = f((0,b)) = f((0,1))b.$$

Since $f \neq 0$, $b = 0$, thus $K_f = S \oplus 0 \setminus \{(0,0)\}$. For the second part, assume that $m, n \in Z_f(M)^*$. If m or n belong to K_f, then they are adjacent. Otherwise, for each $x \in K_f$, both m and n are adjacent to x. Thus there exists a path of length 2 between m and n.
(2) By hypothesis, there exist two non-zero elements x, y in K_f. Therefore clearly $x - (0,1) - y - x$ is a cycle in $\Gamma_f(M)$.
(3) R being a domain, we have for every $m, n \in Z_f(M)^*$, m is adjacent to n if and only if either $m \in K_f$ or $n \in K_f$. Hence $K_f \cup \{(0,1)\}$ are the vertices of a maximal complete subgraph of $\Gamma_f(M)$.

□

The following theorem shows the importance of cardinal number of K_f in determination of the clique number of $\Gamma_f(M)$.

Theorem 2.14. *Let R be a domain and M be an R-module. Then for each non-zero element $f \in M^*$, $\mathrm{cl}(\Gamma_f(M))$ is either k or $k+1$, where $K_f = \ker f \setminus \{0\}$ and $k = |K_f|$.*

Proof. We have three cases to discuss:
(Case 1) If $Z_f(M)^* = K_f$, then $\Gamma_f(M)$ is a complete graph because every two vertices in K_f are adjacent and hence $\mathrm{cl}(\Gamma_f(M)) = k$.
(Case 2) Now suppose that $Z_f(M)^* \neq K_f$ and K_f is finite. In this case we claim that $\mathrm{cl}(\Gamma_f(M)) = k+1$. Since $Z_f(M)^* \neq K_f$, there exists $x \in Z_f(M)^* \setminus K_f$ and hence $K_f \bigcup \{x\}$ is a complete subgraph of $\Gamma_f(M)$, which implies that $\mathrm{cl}(\Gamma_f(M)) \geq k+1$. Now we show that $\mathrm{cl}(\Gamma_f(M)) \leq k+1$, otherwise there exists a maximal complete subgraph $\mathcal{B}$ such that $|\mathrm{V}(\mathcal{B})| \geq k+2$. Since $K_f \subseteq \mathrm{V}(\mathcal{B})$, there exist $x, y \in \mathrm{V}(\mathcal{B}) \setminus K_f$. But x and y are adjacent, i.e., $xf(y) = 0$ or $yf(x) = 0$. Then $f(x)f(y) = 0$ and this implies that either $x \in K_f$ or $y \in K_f$, a contradiction.
(Case 3) Suppose that K_f is infinite and $Z_f(M)^* \neq K_f$. Then we observe that $\mathrm{cl}(\Gamma_f(M)) = k$. We must show that

$$\sup\{|\mathrm{V}(\mathcal{B})| : \mathcal{B} \text{ is a complete subgraph}\} = k$$

We know that $K_f \subseteq \mathrm{V}(\mathcal{B})$, i.e., $k \leq |\mathrm{V}(\mathcal{B})|$. On the other hand if $\mathcal{B}$ is a maximal complete subgraph, then $\mathrm{V}(\mathcal{B})$ can have at most one element more than K_f, i.e., $|\mathrm{V}(\mathcal{B})| = k+1$. But k is an infinite cardinal, hence $k = |\mathrm{V}(\mathcal{B})|$. Since $\mathcal{B}$ is an arbitrary complete subgraph, we have $\mathrm{cl}(\Gamma_f(M)) = k$. □

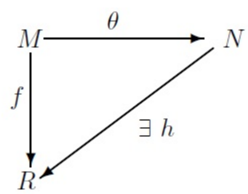
I. Beck in [8] conjectured that the clique number and chromatic number of the graph of zero-divisors are equal for a commutative ring. This conjecture has been refuted by Anderson and Naseer in [5]. However, along this line, some positive results has been obtained by Z. Xue and S. Liu in the zero-divisor graph of partially ordered sets (see [21]). The following corollary is a counter-part of the corresponding result in [21].

Corollary 2.15. *Let R be a domain, M be an R-module and $f \in M^*$. Then $\chi(\Gamma_f(M)) = \mathrm{cl}(\Gamma_f(M))$.*

Proof. When $Z_f(M)^* = K_f$, it is evident that $\mathrm{cl}(\Gamma_f(M))$ is equal to $\chi(\Gamma_f(M))$. In other cases, all vertices in K_f should be colored differently and those which are not in K_f need to be colored with only one color, because they are not adjacent due to R is a domain. □

Proposition 2.16. *Let $M, N \in R-\mathrm{MOD}$ and R be a self-injective ring. If M is embedded in N, then for each $f \in M^*$, there exists $h \in N^*$ such that $\Gamma_f(M)$ is embedded in $\Gamma_h(N)$.*

Proof. Let $\theta : M \longrightarrow N$ be an R-monomorphism. If m_1 is adjacent to m_2 with respect to $f \in M^*$, then $m_1 f(m_2) = 0$. Now, we have the following diagram



such that $h \circ \theta = f$. Now we observe that $\theta(m_1)h(\theta(m_2)) = 0$, for

$$\theta(m_1)h(\theta(m_2)) = \theta(m_1)h \circ \theta(m_2) = \theta(m_1)f(m_2) = \theta(m_1 f(m_2)) = \theta(0) = 0.$$

This implies that $\Gamma_f(M)$ is embedded in $\Gamma_h(N)$. □

3. An answer to a question

Let R be a commutative ring and M an R-module. In [6], the authors associated a graph to the module M which can be considered as a generalization of the classic zero-graph as well. For each $x, y \in M$ we say that $x * y = 0$ provided that $xf(y) = 0$ for some non-zero R-homomorphism $f \in M^* = \text{Hom}(M, R)$. For an R-module M, by $Z(M)$ we mean the set of all $x \in M$ such that there exists a non-zero $y \in M$ such that $x * y = 0$. Put $Z(M)^* = Z(M) \setminus \{0\}$. We associate an undirected graph $\Gamma(M)$ to M with vertices $Z(M)^*$ such that for distinct $x, y \in Z(M)^*$ the vertices x and y are adjacent if and only if either $x * y = 0$ or $y * x = 0$. In [6] some algebraic aspects of $\Gamma(M)$ have been studied and the following open question was asked. Inasmuch as $\Gamma(M)$ is very related to those graphs we studied in this paper, here we provide an answer to the aforementioned open problem.

Open Question: Is there an R-module M with $\text{diam}(\Gamma(M)) = 3$? Is there an R-module M with $\text{gr}(\Gamma(M)) = 4$?

As we see in the sequel, the answer is negative.

Answer. Let M be an R-module with $\text{gr}(\Gamma(M)) = 4$ and $a - b - c - d - a$ be a cycle in $\Gamma(M)$. If there exists $x \in \bigcup_{0 \neq f \in M^*} \text{Ker} f$, then $x - a - b - x$ is a cycle, a contradiction. So suppose that $\bigcup_{0 \neq f \in M^*} \text{Ker} f = \{0\}$ (every $f \in M^*$ is a monomorphism). There exists $f \in M^*$ such that $af(b) = 0$, so that $f(a)f(b) = 0$, $f(a) \neq 0$ and $f(b) \neq 0$. We define a homomorphism $g : M \longrightarrow R$ via $g(m) = f(m)f(b)$. It is easy to observe that g is in M^* and

$$a \in \text{Ker} g \subseteq \bigcup_{0 \neq f \in M^*} \text{Ker} f = \{0\}.$$

Thus $g = 0$ which implies that $Mf(b) = 0$ and hence $b - c - d - b$ is a cycle in $\Gamma(M)$, a contradiction. The proof for $\text{diam}(\Gamma(M)) = 3$ is similar to the above case.

If we ask the same question about $\Gamma_f(M)$, the answer is positive as we see in the next example.

Example 3.1. Consider $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}_{12}$-module and $f := id_{\mathbb{Z}_{12}}$. We may show that $\text{diam}(\Gamma_f(M)) = 3$ and $\text{gr}(\Gamma_f(M)) = 4$.

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References

[1] S. Akbari and A. Mohammadian, On zero-divisor graphs of finite rings, *J. Algebra* **314** (2007), no. 1, 168–184.
[2] D. F. Anderson, M. C. Axtell and J. A. Stickles, Zero-divisor graphs in commutative rings, Commutative Algebra, Noetherian and Non-Noetherian Perspective, 23–45, Springer, New York, 2011.
[3] D. F. Anderson, A. D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, The zero-divisor graph of a commutative ring, II, 61–72, Lecture Notes in Pure and Appl. Math., 220, Dekker, New York, 2001.
[4] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra.* **217** (1999), no. 2, 434–447.
[5] D. D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra.* **159** (1993), no. 2, 500–514.
[6] M. Baziar, E. Momtahan and S. Safaeeyan, A zero-divisor graph for modules with respect to their (first) dual, *J. Algebra Appl.* **12** (2013), no. 2, 11 pages.
[7] M. Baziar, E. Momtahan and S. Safaeeyan, Zero-divisor graph of abelian groups, *J. Algebra Appl.* **13** (2014), no. 6, 13 pages.
[8] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), no. 1, 208–226.
[9] M. Behboodi, Zero divisor graphs for modules over commutative rings, *J. Commut. Algebra* **4** (2012), no. 2, 175–197.
[10] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.* **10** (2011), no. 4, 727–739.
[11] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings II, *J. Algebra Appl.* **10** (2011), no. 4, 741–753.
[12] F. DeMeyer, T. McKenzie and K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum* **65** (2002), no. 2, 206–214.
[13] T. G. Lucas, The diameter of a zero divisor graph, *J. Algebra* **301** (2006), no. 1, 3533–3558.
[14] H. R. Maimani, M. R. Pournaki, A. Tehranian and S. Yassemi, Graphs attached to rings revisited, *Arab. J. Sci. Eng.* **36** (2011), no. 6, 997–1012.
[15] S. B. Mulay, Cycles and symmetries of zero-divisors, *Comm. Algebra* **30** (2002), no. 7, 3533–3558.
[16] S. P. Redmond, The zero-divisor graph of a non-commutative ring, *Internat. J. Commutative Rings* **1** (2002), no. 4, 203–211.
[17] S. P. Redmond, An ideal based zero-divisor graph of a commutative ring, *Comm. Algebra* **31** (2003), no. 9, 4425–4443.
[18] S. Safaeeyan, M. Baziar and E. Momtahan, A generalization of the zero-divisor graph for modules, *J. Korean Math. Soc.* **51** (2014), no. 1, 87–98.

[19] D. B. West, Introduction to Graph Theory, 2nd ed., Prentice Hall, Upper Saddle River, 2001.
[20] R. Wisbauer, Foundations of Modules and Rings Theory, Gordon and Breach Science Publishers, Philadelphia, 1991.
[21] Z. Xue and S. Liu, Zero-divisor graphs of partially ordered sets, *App. Math. Letters* **23** (2010), no. 4, 449–452.

(M. Baziar) Department of Mathematics, Yasouj University, Yasouj,75914, Iran.
E-mail address: mbaziar@yu.ac.ir

(E. Momtahan) Department of Mathematics, Yasouj University, Yasouj,75914, Iran.
E-mail address: e-momtahan@yu.ac.ir

(S. Safaeeyan) Department of Mathematics, Yasouj University, Yasouj,75914, Iran.
E-mail address: safaeeyan@yu.ac.ir